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Energy Velocity and Group Velocity in General Lossless Structures

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The velocity of energy propagation is proved to be identical with group velocity of the wave packet propagated along lossless, anisotropic, dispersive, and inhomogeneous guide structures.

I. INTRODUCTION

THE energy flow velocity has been shown to be identical with the group velocity for waves propagated in common lossless guides(1),(2). And the similar problem has been treated recently by Collin(3), Bertoni and Hessel(4). In the papers by Bertoni and Hessel the group velocity are related to power flow for surface waves propagated in plane-stratified anisotropic medium. However in these previous works energy velocity has been loosely defined and used.

The purpose of this paper is to provide a more satisfactory theory of the energy propagation in general lossless cylindrical structures. The guides are taken as infinitely long with uniform cross section of arbitrary shape, and the sources are located at infinity. The fields will be discussed in frequency domain through the factor $e^{j\omega t}$, and the mks rationalized units are employed throughout. The velocity of energy propagation is defined and discussed with great care in this paper, and emphasis has been put on the guided waves.

II. FORMULATION OF THE PROBLEM

A. Guide

P is any point on the boundary, $\vec{n}$, $\vec{i}_t$, and $\vec{i}_z$

S is the whole transverse plane. The media both inside and

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(1) J. D. Jackson, "*Classical Electrodynamics*", p. 248 (John Wiley and Sons, New York, 1962)
(2) R. B. Adler, Proc. IRE, **40**, 339 (1952)
(3) R. E. Collin, "*Foundations for Microwave Engineering*", p. 379 (McGraw-Hill, New York, 1966)
(4) H. C. Rertoni and A. Hessel, IEEE Trans. AP-14, 334 and 352, (1966)

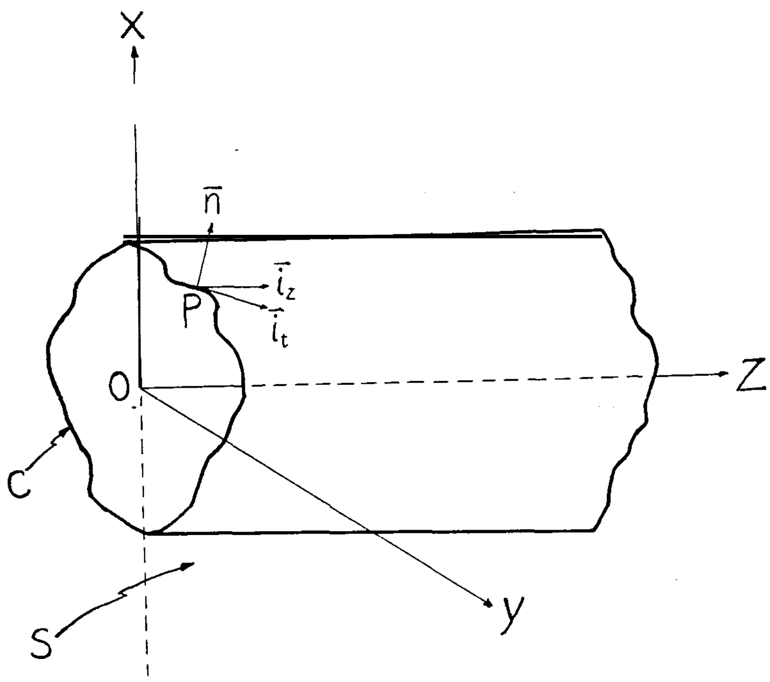

Fig. 1. Cylindrical Structure

outside of the guide are lossless, anisotropic, dispersive, and inhomogeneous over the transverse plane. Hence the permittivity $\bar{\varepsilon}(x, y, \omega)$ and permeability $\bar{\mu}(x, y, \omega)$ are Hermitean tensor functions[(5)] of the sinusoidal frequency ω and the transverse coordinates (x, y). They may suffer discontinuities across the boundary C.

In such general guide structure the waves propagated along the longitudinal direction are assumed to be

$$\vec{E}=\vec{E}_0(x, y, \omega)e^{-j\beta z} \tag{1}$$

$$\vec{H}=\vec{H}_0(x, y, \omega)e^{-j\beta z}, \tag{2}$$

where $\vec{E}$ and H are complex field vectors in frequency domain,

E_0 and $\vec{H}_0$ are complex vector functions of (x, y, ω),

β, the propagation constant, is a real function of ω only.

And the asterisk(*) is used to represent the complex conjugate.

B. ***Energy Velocity***

The z-component of the real part (denoted by Re) of the complex Poynting vector

$$\vec{S}=\tfrac{1}{2}\vec{E}\times\vec{H}^{*} \tag{3}$$

can be interpreted as the time-average power flow per unit area across any trans-

(5) L. D. Landau and E. M. Lifshitz, "*Electrodynamics of Continuous Media*", Chap. IX and XI. (Addison-Wesley, 1960)

verse plane. And the time-average stored energy densities in non-absorbing media are[5]

$$\langle W_e \rangle = \frac{1}{4}\left(\frac{\partial \omega \bar{\varepsilon}}{\partial \omega}\right)\vec{E}\cdot\vec{E}^*, \tag{4}$$

$$\langle W_m \rangle = \frac{1}{4}\left(\frac{\partial \omega \bar{\mu}}{\partial \omega}\right)\vec{H}\cdot\vec{H}^*, \tag{5}$$

w h e $\langle W_e \rangle$ is the time-average electric stored energy density,

$\langle W_m \rangle$ is the time-average magnetic stored energy density.

lossless guide can be given "intuitively

by

$$V_e = \lim_{\Delta\omega\to 0} \frac{\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}\int_S \mathrm{Re}\,\frac{1}{2}\vec{E}\times\vec{H}^*\cdot\vec{i}_z\,da\,d\omega}{\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega}\int_S (\langle W_e \rangle + \langle W_m \rangle)\,da\,d\omega} \tag{6}$$

where a very narrow band of frequencies about ω_0 is considered, and the surface integral extends over the whole transverse plane. The significance of (6) can be seen intuitively: V_e is the total real power flow of wave packet along the guide divided by the total time-average energy stored per unit length of guide. Hence it would be reasonable to think V_e as the velocity of energy flow along the guide.

C. Group *Velocity*

The group velocity along the guide for wave packet in a narrow frequency band about ω_0 is given by

$$V_g = \left(\frac{d\beta}{d\omega}\right)^{-1}_{\omega=\omega_0} \tag{7}$$

III. PROOF

In general lossless medium the Maxwell's equations in frequency domain are

$$\nabla\times\vec{E} + j\omega\bar{\mu}\vec{H} = 0, \tag{8}$$

$$\nabla\times\vec{H} - j\omega\bar{\varepsilon}\vec{E} = 0. \tag{9}$$

By straightforward calculation, we obtain

$$\nabla\cdot\left(\vec{E}^*\times\frac{\partial\vec{H}}{\partial\omega} + \frac{\partial\vec{E}}{\partial\omega}\times\vec{H}^*\right) = -4j(\langle W_e \rangle + \langle W_m \rangle). \tag{10}$$

where $\langle W_e \rangle$ and $\langle W_m \rangle$ are given in (4) and (5) respectively. In view of the fields assumed in (1) and (2),(10) can be reduced to

$$\nabla\cdot\left(\vec{E}_0^*\times\frac{\partial\vec{H}_0}{\partial\omega} + \frac{\partial\vec{E}_0}{\partial\omega}\times\vec{H}_0^*\right) = 4j\left\{\frac{d\beta}{d\omega}\,\mathrm{Re}\,\frac{1}{2}\vec{E}\times\vec{H}^*\cdot i_z - (\langle W_e \rangle + \langle W_m \rangle)\right\} \tag{11}$$

Integrate (11) over the whole transeverse plane S, then left-hand side of (11) can be shown to be zero by virtue of two-dimensional divergence theorem, continuity conditions across the guide boundary, and the fact that the fields must vanish at infinity on the transverse plane. Hence

$$\int_S \mathrm{Re}\,\frac{1}{2}\vec{E}\times\vec{H}^*\cdot\vec{i}_z da=\left(\frac{d\beta}{d\omega}\right)^{-1}\int_S(\langle W_e\rangle+\langle W_m\rangle)da. \tag{12}$$

Applying (12) togather with the mean value theorem for integrals to the definition (6), we obtain

$$V_e=\left(\frac{d\beta}{d\omega}\right)^{-1}_{\omega=\omega_0}, \tag{13}$$

or

$$V_e=V_g. \tag{14}$$

Therefore the velocity of energy propagation is identical with the group velocity of wave packet propagated along general lossless structures.

IV. CONCLUSIONS

The energy velocity is defined in (6) by considering a very narrow band of frequencies with the help of the concept of limit. It has been shown that the energy velocity is equal to the group velocity for general lossless structures. Thus the idea of energy velocity is indeed a physically meaningful concept. Meanwhile it has also given an additional interpretation for group velocity of the propagated wave packet as the velocity of energy transport along these structures.

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