Slipping Stokes flow around a slightly deformed sphere

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When a fluid may slip at the surface of a particle, the conventional boundary condition must be modified to incorporate the tangential stress at the surface. Even for the simplest nontrivial shapes of the slip particle, the resulting Stokes problem could not be analytically solved. We present a first attempt to obtain analytical approximations for the resistance relations for a rigid, slightly deformed slip sphere in an unbounded Stokesian flow. To the first order in the small parameter characterizing the deformation, we derive expressions for the hydrodynamic force and torque exerted on the particle, which are found to be in very good agreement with the available numerical results, even in the case in which deformations are not small. The drag force on a spheroid is found to be an either decaying or growing function of the aspect ratio of the particle. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2337666]

The movement of a solid particle immersed in an arbitrary fluid flow at low Reynolds numbers is of much fundamental and practical interest. In the general formulation of the Stokes problem,[1] it is usually assumed that no slippage arises at the solid-fluid interface. Actually, this is an idealization of the transport processes involved. The phenomena that the adjacent fluid can slip frictionally over a solid surface, occurring for cases such as the low-density gas flow surrounding an aerosol particle[2] and the aqueous liquid flow near a hydrophobic surface,[3] have been confirmed. Presumably any such slipping would be proportional to the local velocity gradient next to the solid surface,[1,4] at least as long as this gradient is small. The constant of proportionality, β^{-1}, is called a "slip coefficient."

The slip coefficient has been determined experimentally for various cases and found to agree with the general kinetic theory of gases. For a viscous fluid with the dynamic viscosity μ, the slip coefficient can be evaluated from the relation $\beta^{-1}=C_m l/\mu$, where l is the mean free path of a gas molecule, and C_m is a dimensionless constant in the range 1.0–1.5.[5]

In general, the underlying Stokes problem could not be exactly solved for nonspherical slip particles. However, the consideration of a slightly deformed sphere[1] yields sufficient information about the physics of the processes involved.

Consider an incompressible Newtonian fluid of viscosity μ, extending to infinity, undergoing some arbitrary Stokes flow with velocity and pressure distributions $\mathbf{v}$ and p, respectively. Thus, the fluid motion at small Reynolds numbers is governed by the Stokes equations

$$\mu\Delta\mathbf{v}=\boldsymbol{\nabla}p, \quad \boldsymbol{\nabla}\cdot\mathbf{v}=0, \tag{1}$$

subject to a particular set of boundary conditions. The general solution of Eq. (1) is given by Ref. 1:

$$\mathbf{v}=\sum_n\left(\boldsymbol{\nabla}\varphi_n+\boldsymbol{\nabla}\times(\mathbf{r}\chi_n)+\frac{n+3}{2\mu(n+1)(2n+3)}r^2\boldsymbol{\nabla}p_n - \frac{n}{\mu(n+1)(2n+3)}\mathbf{r}p_n\right), \tag{2}$$

$$p=\sum_n p_n, \tag{3}$$

where φ_n, p_n and χ_n are solid spherical harmonics of degree n, $\mathbf{r}$ is a position vector, and $r=|\mathbf{r}|$.

The fluid is allowed to slip on the surface S of the particle, which leads to the boundary condition[1,4]

$$\mathbf{v}|_S=\beta^{-1}\mathbf{P}_\tau(\mathbf{v})|_S, \tag{4}$$

where $\mathbf{P}_\tau(\mathbf{v})$ is the tangential component of the stress vector $\mathbf{P}(\mathbf{v})=\boldsymbol{\sigma}(\mathbf{v})\cdot\mathbf{n}$, with $\boldsymbol{\sigma}(\mathbf{v})$ being a deviatoric stress tensor and $\mathbf{n}$ a unit normal vector on the surface of the particle. Note that the perfect slip, such as would occur for a gas bubble in a liquid, corresponds to $\beta\to 0$, whereas the standard no-slip boundary condition for a solid surface is recaptured in the limit $\beta\to\infty$.

In the fluid far away from the particle, the velocity field is prescribed as

$$\mathbf{v}|_{r\to\infty}=\mathbf{v}_\infty. \tag{5}$$

Evidently, $\mathbf{v}_\infty$ must itself be a solution of Eq. (1).

It is convenient to go to the comoving frame, $\mathbf{v}=\mathbf{v}_R+\mathbf{v}_\infty$, and then the boundary conditions acquire the form

$$\mathbf{v}_R|_S=[-\mathbf{v}_\infty+\beta^{-1}\mathbf{P}_\tau(\mathbf{v}_R)+\beta^{-1}\mathbf{P}_\tau(\mathbf{v}_\infty)]|_S, \tag{6}$$

$$\mathbf{v}_R|_{r\to\infty}=\mathbf{0}. \tag{7}$$

In what follows, we consider the cases of a plug flow $\mathbf{v}_\infty=\mathbf{U}$ and a rotating flow $\mathbf{v}_\infty=\boldsymbol{\omega}\times\mathbf{r}$ at infinity. These two cases can be treated separately due to the linear structure of Eq. (1) and boundary conditions (4) and (5).

We consider a slightly deformed spherical particle fixed in a prescribed velocity field $\mathbf{v}_\infty$. The shape of the particle in

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spherical coordinates (r,θ,ϕ) is given by $r=a[1+\varepsilon f(\theta,\phi)]$, where ε is a dimensionless parameter, $|\varepsilon|\ll 1$, and $f(\theta,\phi)$ is an arbitrary function of angular position, which is of order unity with respect to ε. The expansions for the velocity and pressure fields naturally acquire the structure

$$\mathbf{v}_R=\sum_{i=0}^{\infty}\varepsilon^i\mathbf{u}_i,\quad p=\sum_{i=0}^{\infty}\varepsilon^i p^{(i)}. \tag{8}$$

Obviously, each pair $(\mathbf{u}_i,p^{(i)})$ must itself satisfy the Stokes equations. Using the expression for the unit normal vector on the particle surface, $\mathbf{n}=\mathbf{r}/r-\varepsilon a\boldsymbol{\nabla} f$, we expand boundary condition (6) in successive powers of ε by a Taylor series expansion about the unperturbed geometry, which corresponds to the spherical surface of $r=a$, to obtain

$$(\mathbf{u}_0-\beta^{-1}\mathbf{P}_{\tau 0})|_{r=a}=-\mathbf{v}_\infty|_{r=a} \tag{9}$$

for the zeroth-order term and

$$(\mathbf{u}_1-\beta^{-1}\mathbf{P}_{\tau 1})|_{r=a}=\mathbf{B}_1+\mathbf{B}_2+\mathbf{B}_3 \tag{10}$$

for the first-order term, with

$$\mathbf{B}_1=af(\theta,\phi)\frac{\partial}{\partial r}(-\mathbf{u}_0-\mathbf{v}_\infty+\beta^{-1}\mathbf{P}_{\tau 0})|_{r=a}, \tag{11}$$

$$\mathbf{B}_2=-a\beta^{-1}(\boldsymbol{\sigma}_0\cdot\boldsymbol{\nabla} f)|_{r=a}, \tag{12}$$

$$\mathbf{B}_3=a\beta^{-1}\left.\left[2(\mathbf{P}_0\cdot\boldsymbol{\nabla} f)\frac{\mathbf{r}}{r}+\left(\mathbf{P}_0\cdot\frac{\mathbf{r}}{r}\right)\boldsymbol{\nabla} f\right]\right|_{r=a}, \tag{13}$$

where digital subscripts mark zeroth- and first-order terms in the expansions of the appropriate quantities.

Physically, $\mathbf{B}_1$ represents the correction due to the change in the size of the particle, while $\mathbf{B}_2$ and $\mathbf{B}_3$ take into account the changes in the shape of the particle, due to a local correction in the direction of the normal vector. The omission of the terms $\mathbf{B}_2$ and $\mathbf{B}_3$ in the analyses of an axisymmetrical slip flow, performed by several authors,[6,7] led to erroneous results for the drag force.

In the spherical coordinates with their origin at the center of the undeformed sphere, the perturbation of the surface of the particle may be expanded into an infinite series of surface spherical harmonics as

$$f(\theta,\phi)=\sum_{k=0}^{\infty}f_k(\theta,\phi). \tag{14}$$

We now consider the general solution of Eq. (1) supplied with slip boundary condition (4). As can be shown,[1] to get the solution of Eq. (1) with boundary conditions (6) and (7) for the case of a spherical particle of radius a, one has to calculate the following quantities:

$$\frac{1}{r}(\mathbf{r}\cdot\mathbf{A})|_{r=a}=\sum_n X_n(\theta,\phi), \tag{15}$$

$$\left.\left((\mathbf{r}\cdot\boldsymbol{\nabla})\frac{(\mathbf{r}\cdot\mathbf{A})}{r}-r\boldsymbol{\nabla}\cdot\mathbf{A}\right)\right|_{r=a}=\sum_n Y_n(\theta,\phi), \tag{16}$$

$$\mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{A})|_{r=a}=\sum_n Z_n(\theta,\phi), \tag{17}$$

where according to Eqs. (9) and (10), $\mathbf{A}=\mathbf{v}_R-\beta^{-1}\mathbf{P}_\tau$, while X_n, Y_n, and Z_n are surface spherical harmonics. This and similar techniques were widely used for a variety of different boundary conditions, e.g., those describing Stokesian flows around a submerged fluid drop or gas bubble, elastic particle, etc.[8]

Substituting the expression for the fluid velocity field in Eq. (2) and stress vector in Eq. (4) with $\mathbf{n}=\mathbf{r}/r$ into Eqs. (15)–(17), and simultaneously solving the resulting set of linear equations, we obtain (only those terms that define the flow in an unbounded fluid, i.e., with negative powers of r, are retained)

$$\chi_{-n-1}=\frac{\beta a Z_n}{n(n+1)[\beta a+\mu(n+2)]}\left(\frac{a}{r}\right)^{n+1},\quad n\geq 1, \tag{18}$$

the vortical term and

$$\varphi_{-n-1}=\frac{a\{\beta a Y_n+X_n[\beta a n+2\mu(n^2-1)]\}}{2(n+1)[\beta a+\mu(2n+1)]}\left(\frac{a}{r}\right)^{n+1}, \tag{19}$$

$$p_{-n-1}=\frac{\mu(2n-1)}{a(n+1)}\left(\frac{a}{r}\right)^{n+1}\times\frac{\beta a Y_n+X_n(n+2)(\beta a+2n\mu)}{\beta a+\mu(2n+1)},\quad n\geq 1, \tag{20}$$

the hydrodynamic potential and pressure terms. Thus, Eqs. (18)–(20) allow us to describe the structure of the fluid flow when the surface harmonics of Eqs. (15)–(17) are given. Note that, in the limit $\beta a\gg\mu$, expressions (18), (19), and (10) reproduce the well-known result for the conventional no-slip flow.[1]

Equations (18)–(20) allow us to obtain the zeroth-order terms in the expansion of the velocity field. Substituting $\mathbf{v}_\infty=\mathbf{U}$ into Eqs. (15)–(17) and using Eqs. (18)–(20), we get

$$\mathbf{v}_0=\mathbf{U}\left(1-\frac{B}{r}-\frac{D}{r^3}\right)+\frac{(\mathbf{U}\cdot\mathbf{r})\mathbf{r}}{r^2}\left(\frac{3D}{r^3}-\frac{B}{r}\right), \tag{21}$$

$$D=\frac{a^3}{4}\frac{\beta a}{\beta a+3\mu},\quad B=\frac{3a}{4}\frac{\beta a+2\mu}{\beta a+3\mu} \tag{22}$$

for a plug flow. Similarly, $\mathbf{v}_\infty=\boldsymbol{\omega}\times\mathbf{r}$ yields

$$\mathbf{v}_0=\left(r-\frac{4D}{r^2}\right)\frac{(\boldsymbol{\omega}\times\mathbf{r})}{r} \tag{23}$$

for a rotating flow. Both expressions (21) and (23) are in agreement with the results obtained previously.[1,4]

In the general situation, having expressed vectors $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ in Eq. (10) in terms of surface spherical harmonics, and utilizing the relations (18)–(20), one can construct the first-order corrections to the background velocity fields (21) and (23) as an infinite sum over the angular terms $f_k(\theta,\phi)$ in Eq. (14) relevant to a particular perturbation of the shape of a spherical particle. However, to obtain physically important characteristics of the fluid flows around the particle, such as the hydrodynamic resistance tensors, all sufficient information can be extracted from the finite number of terms in the infinite sum of Eq. (2), representing the velocity field.

To evaluate the hydrodynamic drag force and torque acting on a deformed spherical particle, we use the well-known formulas derived in terms of the solid spherical harmonics p_{-2} and χ_{-2},[1]

$$\mathbf{F} = -4\pi\boldsymbol{\nabla}(r^3 p_{-2}), \quad \mathbf{T} = -8\pi\mu\boldsymbol{\nabla}(r^3\chi_{-2}), \tag{24}$$

and recall that these general expressions for the force and torque are independent of the shape of the particle. Naturally introducing the expansions in powers of perturbation parameter ε,

$$\mathbf{F} = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{F}_i, \quad \mathbf{T} = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{T}_i, \tag{25}$$

we proceed with the appropriate terms in the expansion (2), viz., $p_{-2}^{(1)}$ and $\chi_{-2}^{(1)}$ [see Eqs. (18) and (20)] with respect to the first order of ε. Performing some analytical manipulations by means of MATHEMATICA 4.0, we get

$$p_{-2U}^{(1)} = -\frac{3\mu a r^{-3}}{2}\left(\frac{\beta^2 a^2 + 6\beta a\mu + 6\mu^2}{(\beta a + 3\mu)^2}(\mathbf{U}\cdot\mathbf{r}) f_0 - \frac{\beta^2 a^2 + 6\beta a\mu + 24\mu^2}{10(\beta a + 3\mu)^2}(\mathbf{U}\cdot\boldsymbol{\nabla})(r^2 f_2)\right), \tag{26}$$

$$\chi_{-2U}^{(1)} = a^2 r^{-2}\frac{3(\beta a + 2\mu)}{4(\beta a + 3\mu)}\mathbf{U}\cdot\boldsymbol{\nabla}\times(\mathbf{r} f_1) \tag{27}$$

for the case of a plug flow and

$$p_{-2\omega}^{(1)} = -\mu a^2 r^{-2}\frac{3(\beta a + 2\mu)}{2(\beta a + 3\mu)}\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\times(\mathbf{r} f_1), \tag{28}$$

$$\chi_{-2\omega}^{(1)} = -a^3 r^{-3}\frac{3\beta a(\beta a + 4\mu)}{(\beta a + 3\mu)^2}\left[(\boldsymbol{\omega}\cdot\mathbf{r}) f_0 - \frac{1}{10}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla}) \times (r^2 f_2)\right] \tag{29}$$

for a rotating flow.

Usefully introducing notations for the resistance tensors,

$$\begin{pmatrix}\mathbf{F}_i\\ \mathbf{T}_i\end{pmatrix} = \begin{pmatrix}6\pi\mu a\boldsymbol{\Phi}^{(i)} & 6\pi\mu a^2\mathbf{D}^{(i)}\\ 6\pi\mu a^2\mathbf{C}^{(i)} & 8\pi\mu a^3\boldsymbol{\Omega}^{(i)}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{U}\\ \boldsymbol{\omega}\end{pmatrix}, \tag{30}$$

we obtain

$$\Phi_{ij}^{(1)} = \frac{\beta^2 a^2 + 6\beta a\mu + 6\mu^2}{(\beta a + 3\mu)^2}\delta_{ij} f_0 - \frac{\beta^2 a^2 + 6\beta a\mu + 24\mu^2}{10(\beta a + 3\mu)^2}\nabla_i\nabla_j(r^2 f_2), \tag{31}$$

$$\Omega_{ij}^{(1)} = \frac{3\beta a(\beta a + 4\mu)}{(\beta a + 3\mu)^2}\left[\delta_{ij} f_0 - \frac{1}{10}\nabla_i\nabla_j(r^2 f_2)\right], \tag{32}$$

$$C_{ij}^{(1)} = -D_{ij}^{(1)} = -\frac{\beta a + 2\mu}{\beta a + 3\mu}\varepsilon_{ijk}\nabla_k(r f_1), \tag{33}$$

where δ_{ij} is a Kronecker delta, ε_{ijk} is a Levi-Civita permutation symbol, and components of the resistance tensors are given in a Cartesian frame. In the limit $\beta a \gg \mu$, the expressions of the resistance tensors in Eqs. (31)–(33) coincide with those for a no-slip deformed sphere.[1] Furthermore, the translational and rotational resistance tensors $\boldsymbol{\Phi}^{(1)}$ and $\boldsymbol{\Omega}^{(1)}$ are symmetric, while the coupling tensors are antisymmetric and conjugated, $(\mathbf{C}^{(1)})^T = \mathbf{D}^{(1)}$, which in turn follows from the general symmetry properties of the kinetic coefficients (the so-called *Onsager relations*).

As particular cases of interest, we consider the following form of a spheroidal particle in the Cartesian frame (x, y, z):

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{a^2(1-\varepsilon)^2} = 1. \tag{34}$$

For the case $\varepsilon > 0$, the spheroid is oblate; for the case $\varepsilon < 0$, the spheroid is prolate. To $O(\varepsilon)$, the shape of the particle can be written as

$$r = a\left[1 - \varepsilon\left(\frac{1}{3}P_0(\cos\theta) + \frac{2}{3}P_2(\cos\theta)\right)\right], \tag{35}$$

with the Legendre polynomials $P_0(\cos\theta)$ and $P_2(\cos\theta)$. Using Eqs. (31) and (32) for the translational and rotational resistance tensors in the spheroidal geometry, we get

$$(\Phi_{spheroid}^{(1)})_{ij} = -\frac{2}{5}\delta_{ij} + \frac{\beta^2 a^2 + 6\beta a\mu + 24\mu^2}{5(\beta a + 3\mu)^2}\delta_{i3}\delta_{j3}, \tag{36}$$

$$(\Omega_{spheroid}^{(1)})_{ij} = -\frac{3\beta a(\beta a + 4\mu)}{5(\beta a + 3\mu)^2}(2\delta_{ij} - \delta_{i3}\delta_{j3}) \tag{37}$$

while the coupling tensors obviously vanish.

Concerning the axisymmetric flow past a spheroid, i.e., $\mathbf{U} = U_\parallel\mathbf{e}_z$ and $\mathbf{F} = F_\parallel\mathbf{e}_z$, in which $\mathbf{e}_z$ is the unit vector in the z direction, using Eq. (36) we have the simple compact expression for the total drag force,

$$F_\parallel = 6\pi\mu a U_\parallel\left(\frac{\beta a + 2\mu}{\beta a + 3\mu} - \varepsilon\frac{\beta^2 a^2 + 6\beta a\mu - 6\mu^2}{5(\beta a + 3\mu)^2}\right). \tag{38}$$

Expression (38) for the drag force acting on a spheroid fixes the erroneous result published in Refs. 6 and 7. Interestingly, in the case in which the undisturbed fluid velocity is perpendicular to the axis of revolution of the spheroid, i.e., $\mathbf{U}\cdot\mathbf{e}_z = 0$, $\mathbf{F}\cdot\mathbf{e}_z = 0$, $|\mathbf{U}| = U_\perp$, and $|\mathbf{F}| = F_\perp$, the slip effect on the drag force disappears to the $O(\varepsilon)$ terms,

$$F_\perp = 6\pi\mu a U_\perp\left(\frac{\beta a + 2\mu}{\beta a + 3\mu} - \varepsilon\frac{2}{5}\right). \tag{39}$$

Similarly, for the case of the axisymmetrical rotation of the flow, i.e., $\boldsymbol{\omega} = \omega_\parallel\mathbf{e}_z$ and $\mathbf{T} = T_\parallel\mathbf{e}_z$, the hydrodynamic torque exerted on the spheroid about the axis of symmetry becomes

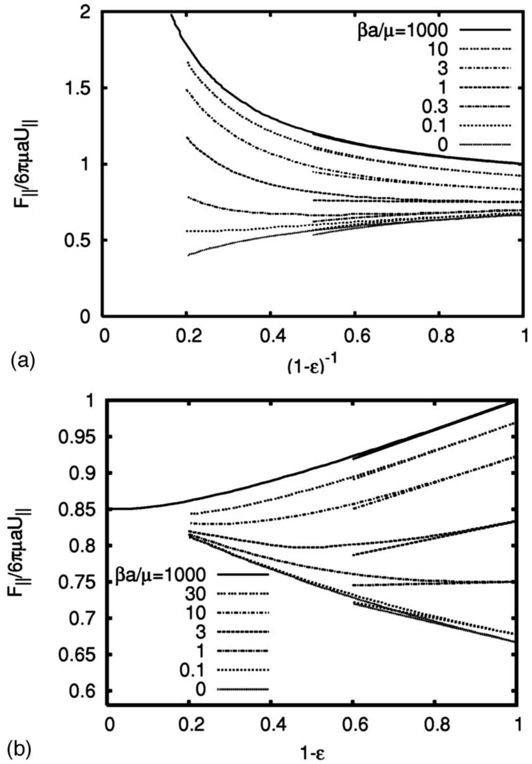

FIG. 1. The dimensionless drag force for the longitudinal motion of prolate and oblate spheroids as functions of the aspect ratio for various values of slip parameter $\beta a/\mu$. The short curves represent the analytical approximation given by Eq. (38) and the long curves denote the numerical solution obtained by Keh and Huang in Ref. 9.

$$T_{\|} = 8\pi\mu a^3 \omega_{\|}\left(\frac{\beta a}{\beta a + 3\mu} - \varepsilon\frac{3\beta a(\beta a + 4\mu)}{5(\beta a + 3\mu)^2}\right). \tag{40}$$

For the rotational flow around the spheroid with respect to an equatorial axis, i.e., $\boldsymbol{\omega}\cdot\mathbf{e}_z=0$, $\mathbf{T}\cdot\mathbf{e}_z=0$, $|\boldsymbol{\omega}|=\omega_\perp$, and $|\mathbf{T}| = T_\perp$, we get

$$T_{\perp} = 8\pi\mu a^3 \omega_{\perp}\left(\frac{\beta a}{\beta a + 3\mu} - \varepsilon\frac{6\beta a(\beta a + 4\mu)}{5(\beta a + 3\mu)^2}\right). \tag{41}$$

Recently, Keh and Huang[9] investigated the problem of slow axisymmetrical flow of a viscous incompressible fluid past a slip spheroid with an arbitrary aspect ratio using a method of internal singularity distribution combined with a boundary-collocation technique. The analytical approximation of the dimensionless form factor for the axisymmetrical motion of a spheroid given by Eq. (38) is plotted in Fig. 1 as functions of its aspect ratio (i.e., $1-\varepsilon$) against the values obtained numerically[9] for the prolate and oblate cases, respectively. It can be seen from the figures that the analytical solution closely follows the numerical results (up to the aspect ratio ≈ 2 in the prolate case and aspect ratio ≈ 0.6 in the oblate case).

The competition between the slip and surface area effects, characterized by one dimensionless parameter $\beta a/\mu$, allows for the existence of different behaviors of the form factor (i.e., dimensionless drag force) as a function of the aspect ratio of a spheroid. For the case of a plug flow past a spheroid with a no-slip surface or a slip surface having large values of $\beta a/\mu$, the value of the form factor increases monotonically [up to the $O(\varepsilon)$ term] with an increase of the aspect ratio. For a slip spheroid with a small value of $\beta a/\mu$, say, ≤ 0.1, however, the form factor is a monotonically decreasing function of the aspect ratio. When the aspect ratio is large, the major portion of the fluid slip at the particle surface occurs in the direction of the particle's movement. However, when the aspect ratio becomes small, the main component of the fluid slip at the surface of a spheroidal particle is in the direction normal to the motion of the spheroid. To conclude, we note that the form factor is a monotonically increasing function of $\beta a/\mu$ for a given value of the aspect ratio (see Fig. 1).

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