行政院國家科學委員會補助專題研究計畫成果報告

（計畫名稱）黏性渦漩方法之分析比較與研發

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計畫主持人：黃美嬌
共同主持人：
計畫參與人員： 謝仁祥、陳伯毅、黃國壽、李彥旻

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行政院國家科學委員會專題研究計畫成果報告

黏性渦漩方法之分析比較與研發

Investigation and Comparison of Viscous Vortex Methods

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主持人：黃美嬌　　執行機構及單位名稱：臺灣大學機械系

1. 中文摘要

本計畫開發出一新型的渦漩數值方法來模擬二維黏性流場。此新型的渦漩數值方法採用所謂的「擴散速度」以降低渦漩粒子特徵半徑擴大的速度，並令環度在「擴散質面積」上要守恆以使模擬結果更符合力學原理。計劃先進行理論分析以預測誤差，再選擇兩個軸對稱流場進行模擬測試，以了解新方法的種種特性暨準確性與計算效率。計劃也使用傳統的 Leonard 法及其他也運用「擴散速度」的數個渦漩數值方法進行模擬比較，結果顯示本計劃所提出的新方法確實較其他方法準確(Leonard 法除外），且新方法渦漩粒子的特徵半徑成長速度較 Leonard 法慢。後者將有助於非軸對稱流場的對流誤差控制。然而新方法的數值穩定性並不好，尤其是環度守恆的「擴散質面積」愈大時愈差。如何改善穩定性與簡化計算將是未來研究重點之一。

關鍵詞：渦漩方法、擴散速度、環度守恆

Abstract

In this project, a new viscous vortex method in use of the so-called diffusing velocity was developed and proposed for a simulation of two-dimensional viscous flows. The truth that the circulation must be conserved on any diffusing material surfaces was taken care of in the new method. Relevant properties associated with this method were explored theoretically and the convection as well as the diffusion error was predicted also. Two axisymmetric flows were selected for a test of the accuracy and efficiency of the new method. Traditional Leonard's method and several versions of the diffusion vortex methods were also implemented for comparisons. The simulation results show that the present circulation-conserved scheme is more accurate than all the other methods except Leonard's, and it works better than Leonard's in the sense that the core sizes of the vortex particles grow slower. The latter will be very helpful in controlling the convection errors when non-axisymmetric flows are simulated. The stability of the proposed scheme however is poor, unless the material surface, on which the circulation is forced to be conserved, is taken to be small enough. The improvement of the stability and the simplification of the computations will be the future work.

Keywords: Vortex method, Diffusion velocity, Circulation conservation

2. INTRODUCTION

The discrete vortex method has been developed as a numerical simulator for two-dimension incompressible inviscid flows. In this method, the convection of packets of vorticity is tracked. The method can be thus implemented grid-free. The compact of vorticity field compared to those of primitive variables also makes the vortex method advantageous, in addition to its exact satisfaction of boundary conditions at infinity for external flows.

Viscous effects however are important but difficult to be added in such a Lagrangian approach because of poor evaluation of the Laplacian operator due to scattered, unpredictable distributions of the vortex particles. The most straightforward viscous treatment simply uses grid-based finite difference methods by mapping data between the Eulerian and Lagrangian grids, such as Chang et al (1991) [4]. Interpolations are therefore needed in order to evaluate the Laplacian and perform the mapping, which possibly results in excessive numerical diffusion. Several purely Lagrangian schemes have also been proposed. The core expansion technique was introduced by Leonard (1980) [3] and extended by Rossi (1996,1997) [5,6]. The core size of vortex particles is allowed to expand in time to simulate diffusion in such a way that the heat diffusion equation is solved exactly. A localized re-gridding is required nonetheless for a correct convergence to the Navier-Stokes equations [7]. The random walk approach proposed by Chorin [8] added a pseudo-random velocity to the particle velocity. The scheme was shown to converge slowly and provide low-resolution, although being stable. The particle strength exchange scheme (Degond et al [9]) redistributes the strength (circulation) among vortex particles to account for diffusion by formulating a kernel which can evolve with the vorticity field. Difficulties are encountered when the flow becomes strained however.

The interest of the present work, however, does not lie in those methods mentioned above but a newly proposed scheme by Ogami and Akamatsu called the diffusion vortex method in 1991 [1], in which a diffusion velocity is defined and employed to model the diffusion process. In spite of the inefficiency and the possible difficulty in evaluating the diffusion velocity, this scheme has a stronger physical basis and the circulation on any arbitrary "diffusion material surface" (a surface which is convected at both the fluid velocity and the diffusion velocity) turns out to be conserved. Unfortunately, the scheme was not correctly introduced at the first time due to the neglect of the nonzero-divergence of the diffusion velocity. The modified versions (Shintani and Akamatsu [2]) basically accepted the belief that the core size expands at a rate as if it were a material diffusion surface, which however is wrong again. In fact, the expanding rate of vortex core size must be determined according to the constraint of circulation conservation; while the divergence of the diffusion velocity gives the area change rate per unit area of a material diffusion surface. These two rates are different physically and should not be confused. And this forms the motivation of the present work, namely, to derive a truly physically correct diffusion vortex method.

3. RESULTS AND DISCUSSIONS

3.1 THEORECTICAL ANALYSIS

Considered are Gaussian vortex particles constituting the flow. That is, the vorticity field $\omega(\vec{x},t)$ is discretized as

$$\omega(\vec{x},t)=\sum_{j=1}^{N}\frac{\Gamma_j}{\pi\sigma_j^2}\exp\left(-\frac{\left|\vec{x}-\vec{x}_j\right|^2}{\sigma_j^2}\right) \tag{1}$$

where $\vec{x}_j$, Γ_j and σ_j are the location, strength and core size of the j^{th} particle. In the Leonard's core expansion scheme, the vortex particles are convected at the fluid velocities and the expanding rate of the core size is determined, on the condition of no diffusion errors, that is,

$$\frac{d\vec{x}_j}{dt}=\vec{u}\left(\vec{x}_j,t\right) \tag{2}$$

$$\frac{1}{\sigma_j^2}\frac{d\sigma_j^2}{dt}=\frac{4\nu}{\sigma_j^2} \quad (3)$$

with ν being the fluid viscosity. On the other hand, the diffusion vortex methods including the circulation-conserved scheme have the particle convected at the sum of the fluid velocity and the viscous diffusion velocity (VDM) $\vec{u}_d(\vec{x}_j,t)$ as follows:

$$\frac{d\vec{x}_j}{dt}=\vec{u}(\vec{x}_j,t)+\vec{u}_d(\vec{x}_j,t) \quad (4)$$

$$\vec{u}_d \equiv -\frac{\nu}{\omega}\nabla\omega \quad (5)$$

Several versions of the diffusion vortex method are studied herein. The first one labeled as "VDM1" chooses the expanding rate of the core size σ_j equal to the average of the divergence of the diffusion velocity on the disk centered at $\vec{x}_j$ of a radius σ_j, namely

$$\frac{1}{\sigma_j^2}\frac{d\sigma_j^2}{dt}=\overline{\nabla\cdot\vec{u}_d}(\vec{x}_j) \quad (6)$$

The second one labeled as "VDM2" replaces the average value by the central value on the disk for an approximation:

$$\frac{1}{\sigma_j^2}\frac{d\sigma_j^2}{dt}=\nabla\cdot\vec{u}_d(\vec{x}_j) \quad (7)$$

The last one labeled as "VDM3" further views the relative motion of two nearby vortex particles as an indication of the expanding/contraction of a diffusion material surface. Consequently, the VDM3 employs:

$$\frac{1}{\sigma_j^2}\frac{d\sigma_j^2}{dt}=\frac{1}{r_{1j}^2}\frac{dr_{1j}^2}{dt} \quad (8)$$

where r_{1j} is the distance between the jth particle and its nearest neighbor.

None of the above diffusion vortex methods considers the truth or the constraint that the circulation must be conserved on the diffusion material surfaces. The circulation conserved scheme computes the expanding rates instead by enforcing the conservation of the circulation on the diffusion material surface centered at $\vec{x}_j$ of a radius $\delta_j=\alpha\sigma_j$, where a diffusion material surface is defined as a surface moving at the sum of the fluid velocity and the diffusion velocity. The parameter α is a free parameter to adjust the size of the diffusion material surface. When the mutual contributions between vortex particles are taken into consideration, the expansion rates of the vortex cores are determined by

$$\sum_{j=1}^{N}A_{ij}\frac{1}{\sigma_j^2}\frac{d\sigma_j^2}{dt}=\frac{1}{\delta_i^2}\frac{d\delta_i^2}{dt}\sum_{j=1}^{N}D_{ij}-\sum_{j=1}^{N}B_{ij}\frac{1}{r_{ij}^2}\frac{dr_{ij}^2}{dt} \quad (9)$$

where elements A_{ij}, D_{ij}, and B_{ij} count for the effects on σ_i of a varying σ_j, an expanding/contracting surface area (δ_i), and a moving-away neighboring particle *j*, respectively. Noticed is when the mutual interactions are ignored, the scheme is then reduced to VDM1 or Equation (6), saying that the core size must expand as fast as the diffusion material surface in order to maintain a same amount of circulation contributed from itself. It is thus not surprised to find that $A_{ii}=D_{ii}$ and $B_{ii}=0$ (no summation). Detailed expressions of these matrices are given below:

$$A_{ij}=\frac{\Gamma_j}{\pi}\frac{\delta_i^2}{\sigma_j^2}F\left(y_{ij},\frac{\delta_i^2}{\sigma_j^2}\right) \quad (10)$$

$$B_{ij}=\frac{\Gamma_j}{\pi}\frac{\delta_i^2}{\sigma_j^2}G\left(y_{ij},\frac{\delta_i^2}{\sigma_j^2}\right) \quad (11)$$

$$D_{ij}=\Gamma_j\frac{\delta_i^2}{\sigma_j^2}I_0\left(2y_{ij}\frac{\delta_i^2}{\sigma_j^2}\right)\exp\left(-\frac{\delta_i^2}{\sigma_j^2}(1+y_{ij}^2)\right) \quad (12)$$

with

$$F=\int_0^1 ydy\int_0^{2\pi}d\theta\,(1-K(y,\theta))\exp(-K(y,\theta)) \quad (13)$$

$$G=\int_0^1 ydy\int_0^{2\pi}d\theta\left(y_{ij}^2-yy_{ij}\cos\theta\right)\exp(-K(y,\theta)) \quad (14)$$

$$K(y,\theta)=\frac{\delta_i^2}{\sigma_j^2}\left(y^2+y_{ij}^2-2yy_{ij}\cos\theta\right) \quad (15)$$

$$y_{ij} \equiv r_{ij}/\delta_i \tag{16}$$

and I_0 being the modified Bessel function of order zero.

Noticed is that none of the above schemes simulates the convection term correctly (resulting in convection errors) and no convergence to the Navier-Stokes equations can be expected unless the core sizes of vortex particles remain small in time. The slower the vortex cores expand, the smaller the convection error is. The convection error at the location $\vec{x}_i$ can be shown analytically as follows:

$$\varepsilon_{conv.}(\vec{x}_i) = \sum_{j=1}^{N}\sum_{k=1}^{N}\left(\vec{u}_k(\vec{x}_j) - \vec{u}_k(\vec{x}_i)\right)\cdot\nabla\omega_j(\vec{x}_i) \tag{17}$$

where $\vec{u}_k(\vec{x}_i)$ and $\omega_k(\vec{x}_i)$ are the velocity and vorticity induced by the k^{th} vortex particle at the location $\vec{x}_i$. When the flow remains axisymmetric, the convection errors disappear as long as the vortex particles are distributed in the space axisymmetrically too. Finally, the diffusion error can also be shown to be equal to

$$\varepsilon_{diffu.}(\vec{x}_i) = \sum_{j=1}^{N}\omega_j(\vec{x}_i)\left\{-\nabla\cdot\vec{u}_d(\vec{x}_i) + \left(1 - \frac{|\vec{x}_i - \vec{x}_j|^2}{\sigma_j^2}\right)\frac{1}{\sigma_j^2}\frac{d\sigma_j^2}{dt}\right\} \tag{18}$$

which magnitude obviously depends on the way how the core expanding rates are evaluated.

3.2 SIMULATIONS

In the following section, axisymmetric flows are chosen as test flow fields, in order to avoid the convection errors and highlight the diffusion errors. The diffusion errors through the different choices of the expanding rates of the vortex cores are then compared with. Preferred is a scheme that generates smaller diffusion errors and slower expanding vortex cores.

Gaussian Vortex

An Gaussain vorticity field with an initial core radius 0.01 and viscosity ν=1 is simulated herein. The analytical solution is known to be

$$\omega(\vec{x},t) = \frac{1}{(4t+0.01)\pi}\exp\left(-\frac{|\vec{x}|^2}{4t+0.01}\right) \tag{19}$$

The flow is initially discretized into 18 vortex particles having a same initial core size σ = 0.1. The relative errors, averaged over the locations of these 18 vortex particles of all the vortex methods mentioned above are shown together in Fig.1. It is not surprising to see that Leonard's scheme performs best although its core size expands fastest as seen in Fig.2. The results also show that a use of the viscous diffusion velocity does reduce the expanding rates of the core size. When non-axisymmetric flows are simulated, a better performance may be expected if the diffusion velocity is employed. As far as the circulation conserved scheme is concerned, the expanding rates of the core size in the circulation conserved schemes are found to be only weakly dependent on the value of α. Finally, the stability of the circulation conserved scheme (CCS) is found to be increasingly poor as α or the number of vortex particles increases.

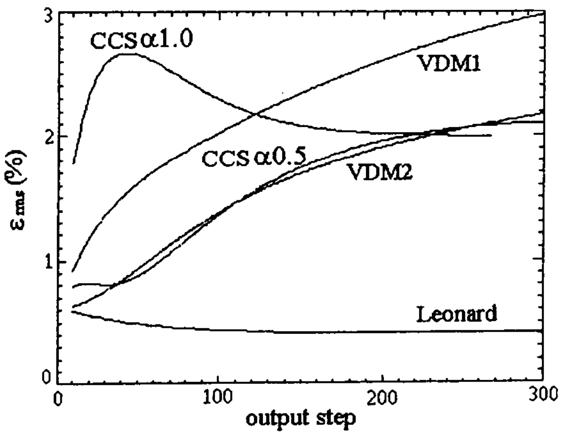

Fig.1 The root-mean-square errors ε_{rms} from the simulations of a Gaussian vortex discretized into 18 vortex particles.

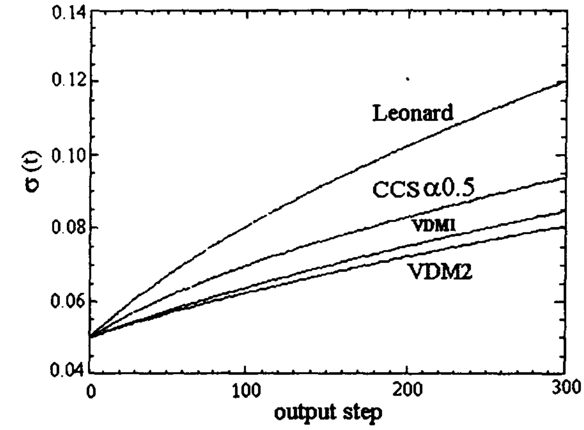

Fig.2 The growing vortex core size from the simulations of a Gaussian vortex discretized into 18 vortex particles.

Vorticity Disk

Considered now is a vorticity field initially uniformly distributed within a disk of radius r_0 =1.5. The total circulation Γ_0 is one and the viscosity ν= 1. The exact solution is known to be

$$\omega(\bar{x},t)=\frac{1}{2\nu t}\frac{\Gamma_0}{\pi r_0^2}\int_0^{r_0} I_0\left(\frac{r\xi}{2\nu t}\right)\exp\left(-\frac{r^2+\xi^2}{4\nu t}\right)\xi d\xi \quad (20)$$

where I_0 is the modified Bessel function of order zero.

A total of 665 vortex particles having an initial σ = 0.1 are employed and initially uniformly distributed within the disk. The root-mean-square error over the particles' locations are shown in Fig.3. The accuracy of CCS is found to be better than all the other diffusion vortex methods and close to that of the Leonard's. Because the vorticity gradient is small within the disk and becomes large near the edge of the disk, the growth of the core size of particles near the edge is expected faster in those methods in use of diffusion velocity. In Fig.4, the growth of the core size of one of such particles is plotted. As seen, the expanding rate of the circulation conserved scheme is between Leonard's and the other diffusion vortex methods. In Fig.5 to Fig.9, the exact vorticity distribution and the computed values together with the particles' radial locations are shown together at the last simulation time (least accurate). Because of the lack of the diffusion velocity, it is seen (Fig.5) that Leonard's particles always stay within the disk. In other words, the movement of Leonard's vortex particles do not respond to the diffusion of voriticty in the flow field. Regridding is thus needed for a long time simulation. Even so, a careful examination still shows that Leonard's method has a pretty good prediction of the vorticity magnitudes outside the disk. This is because the simulated flow is axisymmetric and therefore there exists no convection errors.

Particles in all the other methods diffuse outward gradually on the other hand, reasonably indicating the outward diffusion of vorticity. No re-distributing particles is needed therefore. The VDM1 has the worst result, probably because it has too many computations in computing the average of the divergence of the diffusion velocity. The VDM3 is not too good either, which is not surprising because it is the least physically supported one. The CCS performs pretty well, in spite of its huge amount of computations.

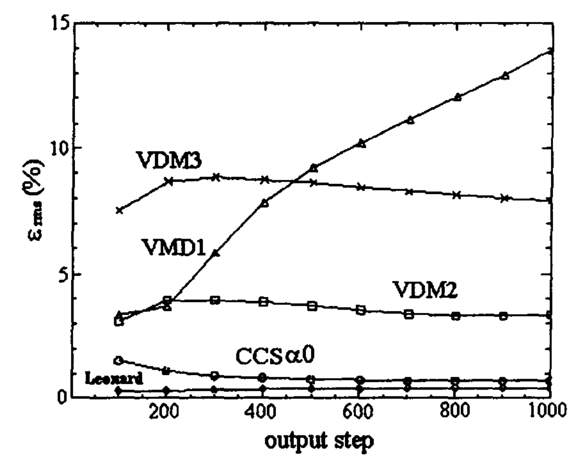

Fig.3 The root-mean-square errors ε_{rms} from the simulations of a vorticity disk.

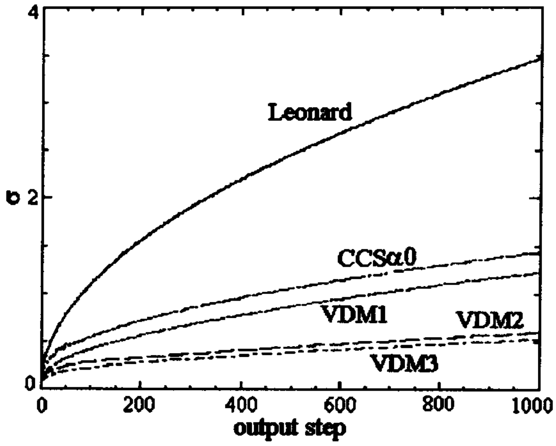

Fig.4 The growing vortex core size from the simulations of a vorticity disk.

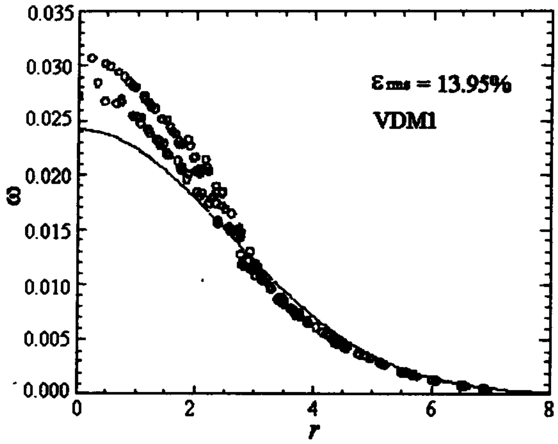

Fig.7 The vorticity distribution from the VDM1.

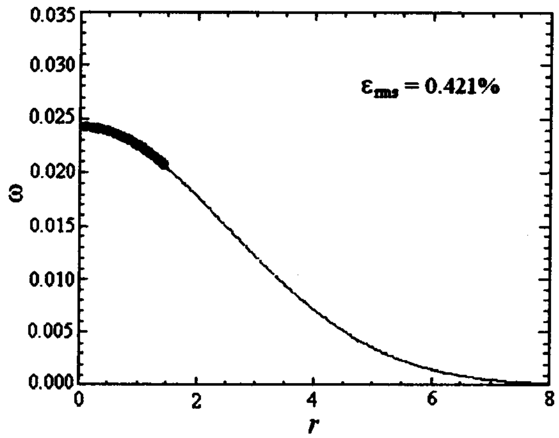

Fig.5 The vorticity distribution from Leonard's scheme. The solid curve is the analytical solution. Symbols indicate both the locations and the vorticity magnitudes of the vortex particles.

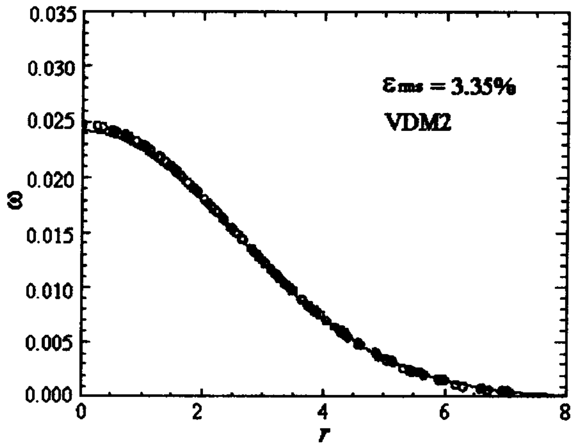

Fig.8 The vorticity distribution from the VDM2.

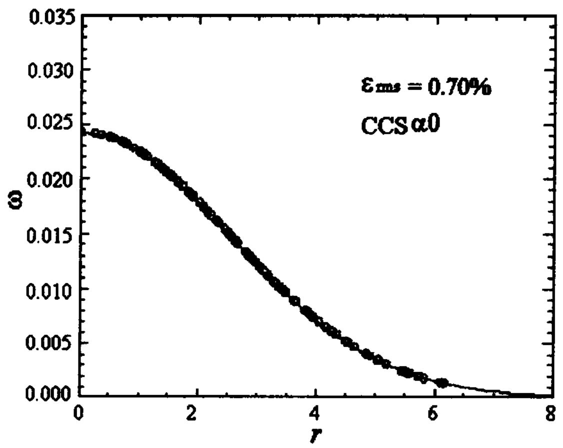

Fig.6 The vorticity distribution from the circulation-conserved diffusion vortex method.

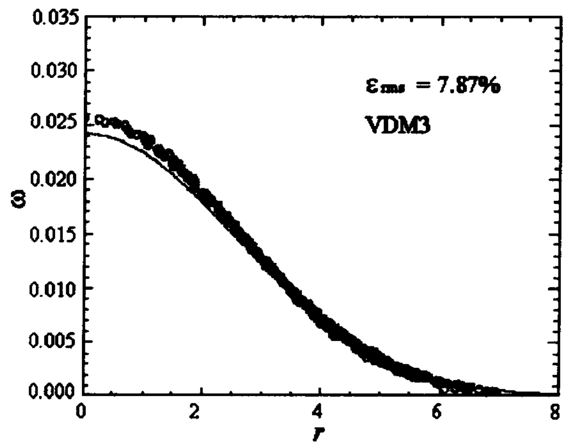

Fig.9 The vorticity distribution from the VDM3.

3.3 CONCLUSIONS

In use of the diffusion velocity, the growth rate of the vortex core size is indeed reduced, compared to that in the Leonard's core expansion scheme. A smaller convection error, if exists, can thus be expected. However, theoretically the circulation on any diffusion material surfaces must be conserved. Previous diffusion vortex methods mishandle this by equalizing the expanding rate of the vortex core size to that of a diffusion material surface. The circulation conserved scheme fixes this problem and improve the accuracy, in spite of resulting in a larger growth rate of the core size but still smaller than that in Leonard's method.

The numerical stability however is poor due to the complicated mutual interactions. The matrix A even becomes singular at some critical times. The amount of computations is also too much because of the calculation of the divergence of the diffusion velocity and the additional work of solving the matrix equation (5). Future investigations will be focused on the improvement in stability and the reduction of computational amount via possible simplifications without much degrading the accuracy. Simulations of non-axisymmetric flows will be of importance as well to ensure the reduction of convection errors through a use of the diffusion velocity.

四、計畫成果自評

The planed goals including the accuracy analysis of all the involved vortex methods, the investigation of the relevant properties of CCS, simulations of simple flows, comparisons among various methods and so on have all been successfully finished. A related journal paper has been published on the Transactions of the Aeronautical and Astronautical Society of the Republic of China Vol.35 (pp.65-72, 2003)[10], and a conference paper has also presented at the 10th National Conference of Computational Fluid Dynamics (Hua-Lien, Aug. 14-16, 2003)[11]. The latter has also won the honor of the conference **BEST PAPER AWARD**.

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