速度—渦度之不可壓縮黏性流分析(I)

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中文摘要

本文係以速度--渦度為主要變數求解穩態不可壓縮黏性流之 Navier-Stokes 方程式。方法以求解包含速度之二階微分方程式及渦度之 convection-diffusion 方程式，並建立其適當之渦度邊界條件。對於本文中所精巧設計建立之渦度積分邊界條件，將詳細說明其數值求解之困難性及其優越性。
關鍵詞：有限差分方法、不可壓縮、Navier-Stokes、速度—渦度

Abstract

We consider in this progress report for solving the steady-state Navier-Stokes equations for incompressible fluid flows using velocities and vorticity as working variables. The method involves solving a second-order differential equation for the velocity and a convection-diffusion equation for the vorticity. The key to the success of the numerical simulation of this class of flow equations depends largely on proper simulation of vorticity transport equation subject to proper vorticity boundary condition. While the derivation of the proposed integral vorticity boundary condition is more elaborate and is more difficult to solve than conventional local approaches, the present approach offers significant advantages.
Keywords: finite difference, incompressible, Navier-Stokes, velocity-vorticity

1. Introduction

The traditional approach to the numerical solution for incompressible Navier-Stokes equations has been to solve working equations in velocity-pressure variables. A serious problem which was encountered while performing the primitive variable formulation is owing to the absence of pressure in the continuity equation. In addition, discretization of pressure gradients in the incompressible equations on curvilinear grids presents considerable difficulties owing to the fact that the approximation of pressure gradient operator should be irrotational [1]. While this difficulty can be effectively resolved on staggered grids [2], special care is needed when grids are non-uniformly and non-orthogonally laid on the flow [1]. It is the added grid complexity that complicates further the incompressible flow analysis. Another popular approach to numerical solution of the Navier-Stokes equations is the velocity-vorticity approach. This formulation is the most appropriate choice for solving the vortex dominated flow. The reason lies in the fact that the advection of vorticity is the most important process determining the flow dynamics. Additionally, it appears that studying incompressible Navier-Stokes equations in terms of vorticity and velocity is closer to physical reality [3]. For the present spatial discretization on collocated grids, we abandon the DC problem and confine ourselves to the second-order Poisson equations to solve for velocity components. Another second-order differential equation for the vorticity scalar must be solved subject to proper boundary conditions, which are the subject of the present study. An accurate prediction of the transport of vorticity is another consideration. We will address this issue in the use of an exponential compact scheme for the flux discretization.

2. Mathematical model

The traditional approach to the numerical solution of incompressible Navier-Stokes equations has been the primitive-variable formulation. Using the kinematic definition of the vorticity $\underline{w} = \nabla \times \underline{u}$, the resulting transport equation is derived as

$$\underline{u} \cdot \nabla \underline{w} - \underline{w} \cdot \nabla \underline{u} = \frac{1}{\text{Re}} \nabla^2 \underline{w} \quad (1)$$

The vorticity stretching term, $\underline{w} \cdot \nabla \underline{u}$, represents the generation or destruction of vorticity due to the stretching or compression of the vortex line. As the space dimension decreases by one, the vortex stretching term vanishes in two-dimensional cases, and the resulting vorticity transport equation is reduced to a scalar equation for the vorticity component which is normal to the planar motion of the flow :

$$\underline{u} \cdot \nabla \underline{w} = \frac{1}{\text{Re}} \nabla^2 \underline{w} \quad (2)$$

The working equations for the velocity components can also be obtained by taking the curl of the definition $\underline{w} = \nabla \times \underline{u}$ and by using the continuity equation. The resulting second-order Poisson equations for velocity components u and v are derived, respectively, as

$$\begin{aligned} \nabla^2 \underline{u} &= -w_y \\ \nabla^2 v &= w_x \end{aligned} \quad (3)$$

The theoretical equivalence between this classical second-order velocity-vorticity formulation and the velocity-pressure formulation has been given. For the details we refer to the paper by Daube et al. [4].

3. Vorticity integral condition

The key element in the vorticity-velocity formulation is to obtain the a priori unknown boundary values of the vorticity for the second-order transport equation (1). The theory behind our derivation of the vorticity boundary condition is the Green's identity, which relates two scalar potentials φ and ψ as follows:

$$\int_\Omega f \nabla^2 y - y \nabla^2 f \, dA = \oint \left(f \frac{\partial y}{\partial n} - y \frac{\partial f}{\partial n}\right) ds \quad (4)$$

Provided that the scalar potential ψ is assigned as the stream function, the following two equations ensure satisfaction of mass conservation :

$$\begin{aligned} u &= \frac{\partial y}{\partial y} \\ v &= -\frac{\partial y}{\partial x} \end{aligned} \quad (5)$$

Now, let φ be the scalar potential which satisfies the Laplace equation. The boundary value of φ is enforced to be zero everywhere except at one point where the value is one :

$$\begin{aligned} \nabla^2 f &= 0 \quad \text{in } \Omega \\ f_i &= d_{ij} \quad \text{on } \Omega \end{aligned} \quad (6)$$

We can get

$$\int_\Omega f w \, dA = \oint_{\partial\Omega} f \, u_t \, ds + \int_\Omega \left(v \frac{\partial f}{\partial x} - u \frac{\partial f}{\partial y}\right) dA$$

where $u_t = -\frac{\partial y}{\partial n} |_{\partial\Omega}$

This completes the derivation of the vorticity integral equation for the transport equation (1). It is worth noting that the assignment of φ=1 leads to

$$\int_\Omega w \, dA = \oint_{\partial\Omega} u_t \, ds$$

4. Numerical results

4.1 Advection-diffusion scheme for the vorticity transport equation

The test problem is that of the skew advection-diffusion problem schematically shown in Fig. 1. The velocity vector of magnitude 1 remains unchanged in the flow and is parallel to the dividing line. In this study, the square domain is uniformly discretized, resulting in a grid with h=0.05. The fluid remains under investigation has a viscosity of $\nu=10^{-4}$. As Fig. 2 reveals, oscillation-free solutions are observed in regions close to as well as away from the dividing line. Results computed from the first-order upwind scheme and the compact scheme of Dennis and Hudson [5, 6] are also plotted for the comparison purposes.

4.2 Lid-driven cavity flow problem

We present a two-dimensional simulation for the fluid flow in a square cavity defined by B:D=1:1. The Reynolds numbers chosen for this study was 1000, which were computed based on the lid speed, the width of the cavity, and the kinematic viscosity of the fluid. In this study, the solutions were computed on uniform grids of 131x131 for Re=1000. For comparison purposes, the velocity profiles of Ghia et al. [7] are also plotted in Fig. 3.

5. Concluding remarks

The goal for the present study was to simulate incompressible viscous flows by means of the velocity-vorticity formulation. In order for the solutions to be accurately predicted, it is important to develop a theoretically rigorous framework which can provide us with boundary vorticity without using field variables outside of the physical domain. The equation governing the boundary vorticity is derived in integral form. Thus, boundary vorticities are simultaneously solved from the matrix equation. The solution algorithm involves a scalar transport equation for the vorticity variable and two Poisson equations for velocity components. Specific to our flux discretization scheme is that the coefficient matrix of the compact nine-point stencil scheme is classified as an irreducibly diagonal dominant M-matrix. To better understand the compact finite difference scheme developed here, we have conducted computational exercises. In the Navier-Stokes flow analyses, we have considered the lid-driven cavity problems. The results demonstrate that the integral approach designed to provide the boundary vorticity is applicable to simulation of fluid flows which are vortical in nature.

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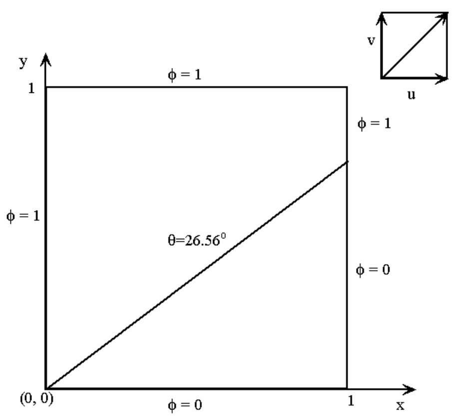

Fig. 1. A schematic of the skew advection-diffusion problem.

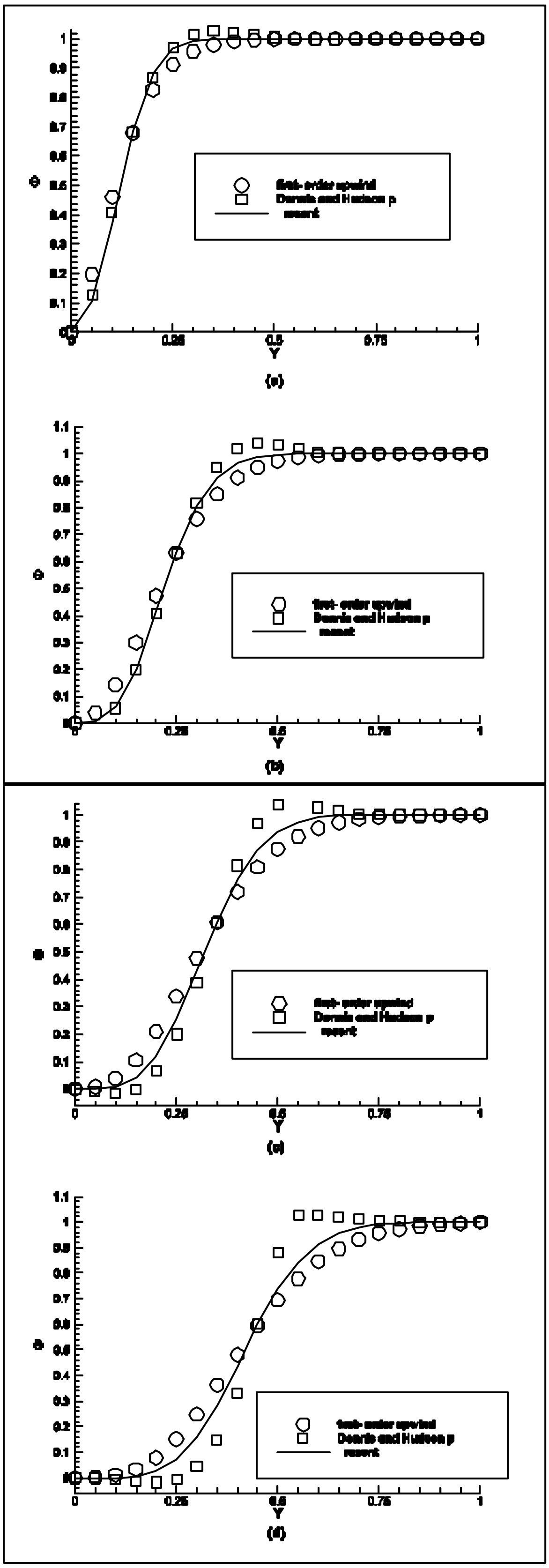

Fig. 2. The plot of ϕ at different x for showing the oscillation-free solution profiles: (a) x=0.2; (b) x=0.4; (c) x=0.6; (d) x=0.8.

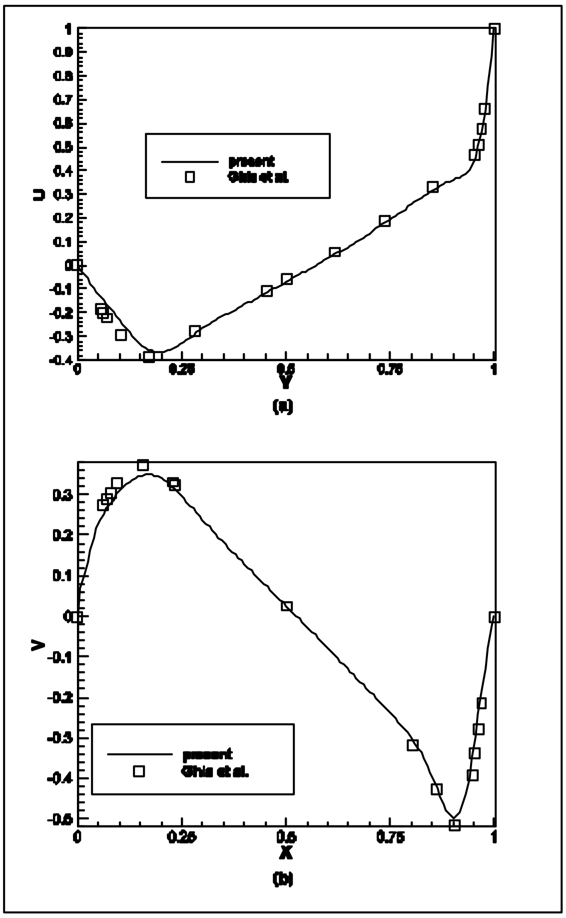

Fig. 3 Velocity profiles plotted on the centerlines for the case Re=1000 (a) u-y plot at x=0.5; (b) v-x plot at y=0.5.