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Relative Equilibria and Stabilities of Spring-Connected Bodies in a Central Gravitational Field

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Abstract

This paper[1] discusses relative equilibria (or steady motions) and their stabilities for the motions of two spring-connected bodies in a central gravitational field. This two-body system can be regarded as a simplified model for the Tethered Satellite System (TSS). In the studies of TSS, typical assumptions include: (1) the center of mass and the center of gravity are both located at the massive one of the two end masses; (2) the center of mass moves on a great-circle orbit. In this paper, these assumptions are lifted to derive more exact models for analyses. In particular, for the simple system treated in this paper, it is proved that the nongreat-circle relative equilibria do exist, and hence the above assumption (2) is not always valid. Some fundamental concepts of the dynamics of an arbitrary assembly moving in a central gravitational field are discussed. The notion of radial relative equilibria, which is the familiar station-keeping mode for TSS, is introduced. Their stabilities are analyzed by adopting the reduced energy-momentum method. It is shown that with physically practical configuration, the system at radial relative equilibria is stable if certain conditions are satisfied.

1 Introduction

The system under consideration is composed of two end-point masses connected by an elastic spring, cf. Fig. 1, which will be termed as the *spring system*. It can be regarded as a simplified model for the Tethered Satellite System (TSS), which contains a satellite (or shuttle orbiter) connected to a subsatellite with a long tether. There have been many interesting discussions on this subject, especially after this idea of TSS was put forth by Colombo and Mario Rossi (1974) [7]. According to [4], the earliest report on such idea was described by Tsiolkovskii in 1895, where an "anchored tower" from the surface of the Earth to the altitude of geostationary orbit was conceived. The problems regarding dynamics and control of these large systems in orbit have been investigated by many researchers, cf. [8, 2, 3, 5, 6, 10, 11, 9, 14, 15], and the references therein. Many interesting discussions can be found in the survey paper of Misra and Modi[12]. In these literatures, either distributed model or lumped system were considered. While the analysis of the distributed system was observed to be quite difficult to deal with, many of previous discussions treated the tether as a massless rigid bar connecting two point masses. Furthermore, it is typically assumed that the center of mass of the system is located at the massive one of the two end-masses, and moves on a great-circle orbit, i.e. a circular orbit centering at the center of the field. However, even at steady motions, these assumptions may not be valid, when the masses of two end points are close to each other or the length of the tether is very large. In fact, for the simple system treated in this paper, it is proved that the nongreat-circle relative equilibria do exist; namely, the center of field and the circular orbit traced by the center of mass form a cone. Accordingly, the dynamical behavior of the *more* exact model without the classic assumptions becomes very interesting.

The assumptions made in this paper are as follows: (1) the attraction center is at rest in the inertial frame; (2) the spring is massless and undergoes extensive or compressive deformation along only one direction; (3) the gravitational attraction between the two end bodies is neglected. After constructing the kinetic energy and potential energy, it is observed that the system possess an $SO(3)$ symmetry. In geometric mechanics, cf. [1], such symmetry induces certain reduction of the dynamics and the notion of *relative equilibria* can be defined, which in fact corresponds to the notion of steady motions in the literature. The configuration of TSS at the station-keeping mode is essentially the configuration at relative equilibrium. With these observations, the techniques in dealing with symmetry, reduction, and stability analysis can be used. Here the Principle of Symmetric Criticality [13] is used to derive equations for relative equilibrium. Similar techniques can be applied to more complicated models, such as the one treated in [19].

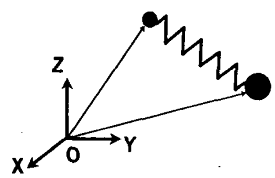

Figure 1: Two Spring-connected Bodies in a Central Gravitational Field

First of all, it is proved that with the model treated in this paper, nongreat-circle relative equilibria do exist. It is the *center of gravity* (instead of the center of mass) that traces a great-circle orbit at relative equilibria. These fundamental concepts of the dynamics of an arbitrary assembly moving in a central gravitational field are discussed in Section 2. The detailed analyses in Section 3 show that if the two end masses are not equal, there must exist a nongreat-circle relative equilibrium. Note that when the spring becomes more and more rigid, the spring system approaches a rigid bar system, which consists of two point masses connected by a massless rigid bar. The analyses show that there are also nongreat-circle relative equilibria for the rigid bar system.

Secondly, the classic notion of station-keeping mode for TSS is in fact the same as the radial relative equilibrium defined in Section 3. Even for the simple spring system, nonlinear stability of radial relative equilibrium is difficult to be established. Here, the reduced energy-momentum method, cf. [16, 17], is adopted to obtain the stability conditions for the radial relative equilibria of the spring system. It is proved that for physically realistic configuration, the systems at radial relative equilibria are *stable* (strictly speaking, *relatively stable*).

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2 Dynamics of an Arbitrary assembly in a Central Gravitational Field

Some fundamental concepts, regarding the dynamics of an assembly moving in a central gravitational field are discussed in this section. The assembly may be a rigid body, a coupled-body system, or a collection of point masses. Let M_{total} be the total mass of the assembly, i.e., $M_{total} = \int_{assembly} dm$, where dm represents the mass measure (discrete or continuous) on the assembly. Let O be both the center of the field, and the origin of the inertial frame. Denote the center of mass and the center of gravity of the assembly by C and G, respectively. The position vector (in the inertial frame) of G is defined by the vector $\boldsymbol{r}_g$ satisfying

$$\frac{\boldsymbol{r}_g}{|\boldsymbol{r}_g|^3} = \frac{1}{M_{total}} \int_{assembly} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} dm,$$

where $\boldsymbol{r}$ is the vector (in the inertial frame) of a mass point in the assembly.

Next the notion of relative equilibrium is introduced. Intuitively, at relative equilibrium (sometimes called steady motion), the assembly is stationary in a uniformly rotating frame located at the origin O. Accordingly, the assembly rotates with a constant angular velocity ω (a vector in the inertial frame) at these relative equilibria. In fact, ω is also the orbital angular velocity of the assembly. The mathematical definition of relative equilibrium (in geometric setting) can be found in [1], among others. If C moves on a great-circle orbit, i.e., $\boldsymbol{r}_c \cdot \omega = 0$, the relative equilibrium is called a great-circle relative equilibrium. On the other hand, if $\boldsymbol{r}_c \cdot \omega \neq 0$, then the relative equilibrium is nongreat-circle. Let the operator " $\widehat{\ }$ " denote an isomorphism between the space $\mathbb{R}^3$ and $so(3)$ (the space of 3×3 skew-symmetric matrices), and the isomorphism is defined by

$$\widehat{\boldsymbol{w}} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \text{where } \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3.$$

With this notation, a uniformly rotating frame with constant angular velocity ω can be represented by $\exp(\widehat{\omega} t) \in SO(3)$, where $SO(3)$ is the special orthogonal group.

The relative equilibria can be characterized by finding all $\boldsymbol{r}_o$'s (constant vectors in the inertial frame) such that $\boldsymbol{r} = \exp(\widehat{\omega}t)\boldsymbol{r}_o$ for all $\boldsymbol{r}$'s in the assembly, where ω is the constant angular velocity vector of the assembly. Analogously, there exist $\boldsymbol{r}_{co}$ and $\boldsymbol{r}_{go}$ (independent of time in the inertial frame) such that $\boldsymbol{r}_c = \exp(\widehat{\omega}t)\boldsymbol{r}_{co}$, and $\boldsymbol{r}_g = \exp(\widehat{\omega}t)\boldsymbol{r}_{go}$. From Newtonian mechanics, the following equation can be derived,

$$\frac{d^2\boldsymbol{r}_c(t)}{dt^2} + \frac{\mu \boldsymbol{r}_g(t)}{|\boldsymbol{r}_g(t)|^3} = 0. \tag{1}$$

At relative equilibrium, the above differential equation can be reduced to the following algebraic equation:

$$\widehat{\omega}\widehat{\omega}\boldsymbol{r}_{co} + \frac{\mu \boldsymbol{r}_{go}}{|\boldsymbol{r}_{go}|^3} = 0. \tag{2}$$

Taking inner product on (2) with ω, we find $\boldsymbol{r}_{go} \cdot \omega = 0$. Consequently, it is shown that *at relative equilibria, the center of gravity of an assembly moving in a central gravitational field traces a great circle.*

On the other hand, taking inner product on (2) with $\boldsymbol{r}_{go}$, we have $\mu = |\omega|^2 |\boldsymbol{r}_{go}| \boldsymbol{r}_{go} \cdot \boldsymbol{r}_{co}$. Note that $|\boldsymbol{r}_{go}| = |\boldsymbol{r}_g|$ and $\boldsymbol{r}_{go} \cdot \boldsymbol{r}_{co} = \boldsymbol{r}_g \cdot \boldsymbol{r}_c$. Thus the following modified Kepler's third law (applied to a circular orbit),

$$\mu = |\omega|^2 |\boldsymbol{r}_g| \boldsymbol{r}_g \cdot \boldsymbol{r}_c, \tag{3}$$

holds at relative equilibria. When the assembly is reduced to a point mass, the above formula is reduced to the classical Kepler's third law applied to a circular orbit.

3 Relative Equilibria of Two Spring-Connected Bodies

As depicted in Figure 1, the configuration space of the spring system can be modeled as $Q = \mathbb{R}^3 \times \mathbb{R}^3$. Let the two end masses be m_a and m_b, respectively. Denote the vectors from the attraction center O to m_a and m_b by $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. The potential energy of the system is given as

$$V(\boldsymbol{a},\boldsymbol{b}) = -\frac{\mu m_a}{|\boldsymbol{a}|} - \frac{\mu m_b}{|\boldsymbol{b}|} + W(|\boldsymbol{a}-\boldsymbol{b}|),$$

where the first two terms on the right-hand-side represent the gravitational potential energy, and the last term W represents the elastic potential energy. Here W is a real-valued function of the length of the spring; in particular, $W = \frac{1}{2}k(|\boldsymbol{a}-\boldsymbol{b}|-\ell_o)^2$ for a linearly elastic spring, with the spring constant k and the reference length ℓ_o (the initial length of the spring without experiencing any force). Let $\boldsymbol{\xi} \in \mathbb{R}^3$ be an arbitrary constant vector. The *augmented potential* is defined to be

$$V_\xi(\boldsymbol{a},\boldsymbol{b}) = V(\boldsymbol{a},\boldsymbol{b}) - \frac{m_a}{2} < \boldsymbol{\xi}\times\boldsymbol{a}, \boldsymbol{\xi}\times\boldsymbol{a} > - \frac{m_b}{2} < \boldsymbol{\xi}\times\boldsymbol{b}, \boldsymbol{\xi}\times\boldsymbol{b} > .$$

By the *Principle of Symmetric Criticality*, cf. [13, 16], relative equilibria can be characterized by the critical points of V_ξ for some $\boldsymbol{\xi}$. At such relative equilibria, the system rotates about the vector $\boldsymbol{\xi}$, with angular velocity magnitude $|\boldsymbol{\xi}|$.

The first derivative of the augmented potential is obtained as follows,

$$\begin{aligned} & DV_\xi(\boldsymbol{a},\boldsymbol{b}) \cdot (\delta \boldsymbol{a}, \delta \boldsymbol{b}) \\ = \ & \mu m_a \frac{\boldsymbol{a}}{|\boldsymbol{a}|^3} \cdot \delta \boldsymbol{a} + \mu m_b \frac{\boldsymbol{b}}{|\boldsymbol{b}|^3} \cdot \delta \boldsymbol{b} + m_a(\widehat{\boldsymbol{\xi}}\widehat{\boldsymbol{\xi}}\boldsymbol{a}) \cdot \delta \boldsymbol{a} \\ & + m_b(\widehat{\boldsymbol{\xi}}\widehat{\boldsymbol{\xi}}\boldsymbol{b}) \cdot \delta \boldsymbol{b} + W'(|\boldsymbol{a}-\boldsymbol{b}|) \frac{\boldsymbol{a}-\boldsymbol{b}}{|\boldsymbol{a}-\boldsymbol{b}|} \cdot (\delta \boldsymbol{a} - \delta \boldsymbol{b}), \end{aligned}$$

where the prime " $'$ "denotes the partial differentiation of W with respect to its argument. The conditions of relative equilibria are then,

$$\frac{\mu m_a \boldsymbol{a}_e}{a_e^3} + m_a \widehat{\boldsymbol{\xi}}\widehat{\boldsymbol{\xi}} \boldsymbol{a}_e + W' \frac{\boldsymbol{a}_e - \boldsymbol{b}_e}{|\boldsymbol{a}_e - \boldsymbol{b}_e|} = 0, \tag{4}$$

$$\frac{\mu m_b \boldsymbol{b}_e}{b_e^3} + m_b \widehat{\boldsymbol{\xi}}\widehat{\boldsymbol{\xi}} \boldsymbol{b}_e - W' \frac{\boldsymbol{a}_e - \boldsymbol{b}_e}{|\boldsymbol{a}_e - \boldsymbol{b}_e|} = 0, \tag{5}$$

where $a_e = |\boldsymbol{a}_e|$ and $b_e = |\boldsymbol{b}_e|$. The terms containing W' represent the elastic forces on the spring. When the spring become more and more rigid and the length $|\boldsymbol{a}_e - \boldsymbol{b}_e|$ at relative equilibrium approaches the reference length ℓ_o, the spring system becomes the rigid bar system.

It is easily checked that (4) and (5) are invariant under the transformation $\boldsymbol{R}_a = \boldsymbol{B}\boldsymbol{a}_e$, $\boldsymbol{R}_b = \boldsymbol{B}\boldsymbol{b}_e$, and $\boldsymbol{\Omega} = \boldsymbol{B}\boldsymbol{\xi}$, where $\boldsymbol{B} \in SO(3)$ is the transformation matrix from the inertial frame to the new frame and $\boldsymbol{R}_a$, $\boldsymbol{R}_b$, $\boldsymbol{\Omega}$ are vectors in the new frame. For simplicity, the notations $\boldsymbol{a}_e$, $\boldsymbol{b}_e$, $\boldsymbol{\xi}$ will be still used as vectors in the new frame in this paper. With this observation, a suitable frame is sought to make the problem tractable. The frame adopted here is shown to be more convenient, where the x-axis is parallel to the spring, the z-axis is perpendicular to both $\boldsymbol{a}_e$ and $\boldsymbol{b}_e$, and the y-axis completes the triad, cf. Fig. 2. As for the case that $\boldsymbol{a}_e$ and $\boldsymbol{b}_e$ are parallel, we may arbitrarily choose an z-axis without loss of generality.

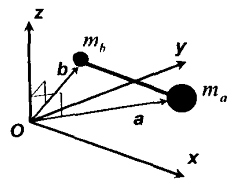

Figure 2: The Frame System

With respect to the chosen frame, the vectors can be expressed as $\boldsymbol{R}_a = (x_a, y_c, 0)^T$, $\boldsymbol{R}_b = (x_b, y_c, 0)^T$, and $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$. It is further assumed that x_a is greater than x_b. With this setting, (4) and (5) are rewritten as six equations,

$$\begin{aligned}
-(\Omega_2^2+\Omega_3^2)x_a+\Omega_1\Omega_2 y_c+\mu\frac{x_a}{R_a^3} &= -\frac{W'}{m_a}, && (6)\\
\Omega_1\Omega_2 x_a-(\Omega_1^2+\Omega_3^2)y_c+\mu\frac{y_c}{R_a^3} &= 0, && (7)\\
(\Omega_1 x_a+\Omega_2 y_c)\,\Omega_3 &= 0, && (8)\\
-(\Omega_2^2+\Omega_3^2)x_b+\Omega_1\Omega_2 y_c+\mu\frac{x_b}{R_b^3} &= \frac{W'}{m_b}, && (9)\\
\Omega_1\Omega_2 x_b-(\Omega_1^2+\Omega_3^2)y_c+\mu\frac{y_c}{R_b^3} &= 0, && (10)\\
(\Omega_1 x_b+\Omega_2 y_c)\,\Omega_3 &= 0, && (11)
\end{aligned}$$

where $R_a = \|\boldsymbol{R}_a\|$, and $R_b = \|\boldsymbol{R}_b\|$. The solutions of the above equations correspond to the relative equilibria of the spring system. Let $\ell = x_a - x_b > 0$. Define $\boldsymbol{R}_c = (x_c, y_c, 0)^T$ with $(m_a + m_b)x_c = m_a x_a + m_b x_b$. It can be easily derived that $x_a = x_c + m_b\ell/(m_a + m_b)$, $x_b = x_c - m_a\ell/(m_a + m_b)$. Given μ, m_a, m_b, ℓ, and $R_c = \|\boldsymbol{R}_c\|$, the process of solving the equations (6)–(11) can be divided into the following two cases. First, assume that $\Omega_3 \neq 0$. The equations of relative equilibria can be simplified as, with $\Omega_1 = 0$, $\Omega_2 y_c = 0$, $\Omega_3 \neq 0$,

$$\begin{aligned}
\left[-(\Omega_2^2+\Omega_3^2)+\frac{\mu}{R_a^3}\right]x_a &= -\frac{W'}{m_a}, && (12)\\
(-\Omega_3^2+\frac{\mu}{R_a^3})y_c &= 0, && (13)\\
\left[-(\Omega_2^2+\Omega_3^2)+\frac{\mu}{R_b^3}\right]x_b &= \frac{W'}{m_b}, && (14)\\
(-\Omega_3^2+\frac{\mu}{R_b^3})y_c &= 0. && (15)
\end{aligned}$$

Subtracting (13) from (15), we get $y_c(\frac{1}{R_a^3} - \frac{1}{R_b^3}) = 0$. The discussions may be further separated depending on whether $y_c = 0$ or not.

For $y_c \neq 0$, Ω_2 must be 0 and $R_a = R_b \equiv R$. The solution can be easily obtained from (12–15) as $\boldsymbol{R}_a = (\frac{\ell}{2}, y_c, 0)^T$, $\boldsymbol{R}_b = (-\frac{\ell}{2}, y_c, 0)^T$, $\boldsymbol{\Omega} = (0, 0, \Omega_3)^T$, with

$$y_c = \left[R_c^2 - \frac{\ell^2}{4}\left(\frac{m_a - m_b}{m_a + m_b}\right)^2\right]^{1/2}, \qquad \Omega_3^2 = \frac{\mu}{R^3}.$$

This is a great-circle relative equilibrium with $W' = 0$, which means that there is no force in the spring. At such relative equilibria, the two point masses move on the same circular orbit.

Next, under the condition of $y_c = 0$, since $x_b \leq 0 \leq x_a$ is not physically interesting, it is further assumed that $x_a > x_b > 0$. Thus, $\boldsymbol{R}_a = (x_a, 0, 0)^T$, $\boldsymbol{R}_b = (x_b, 0, 0)^T$, and $\boldsymbol{R}_c = (x_c, 0, 0)^T$. This is in fact the case of ***radial relative equilibrium***, i.e. the spring lying on a radial axis. The frame can be then selected such that $\Omega_2 = 0$. With these observations, the conditions for relative equilibria are further simplified as $\Omega_1 = 0$, $\Omega_2 = 0$, $\Omega_3 \neq 0$, $y_c = 0$, and

$$\begin{aligned}
\left(-\Omega_3^2+\frac{\mu}{x_a^3}\right)x_a &= -\frac{W'}{m_a}, && (16)\\
\left(-\Omega_3^2+\frac{\mu}{x_b^3}\right)x_b &= \frac{W'}{m_b}. && (17)
\end{aligned}$$

As a consequence, the configuration of a radial relative equilibrium is derived, with $\boldsymbol{R}_a = (x_a, 0, 0)^T$, $\boldsymbol{R}_b = (x_b, 0, 0)^T$, $\boldsymbol{\Omega} = (0, 0, \Omega_3)^T$, where

$$\Omega_3^2 = \frac{\mu}{(m_a+m_b)R_c}(\frac{m_a}{x_a^2}+\frac{m_b}{x_b^2}),\ W' = \frac{\mu m_a m_b(x_a^3 - x_b^3)}{(m_a+m_b)x_a^2 x_b^2 R_c} > 0.$$

This is a great-circle relative equilibrium with tensile elastic forces.

On the other hand, consider the case of $\Omega_3 = 0$. The equations of relative equilibria become:

$$\begin{aligned}
-\Omega_2^2 x_a+\Omega_1\Omega_2 y_c+\mu\frac{x_a}{R_a^3} &= -\frac{W'}{m_a}, && (18)\\
\Omega_1\Omega_2 x_a-\Omega_1^2 y_c+\mu\frac{y_c}{R_a^3} &= 0, && (19)\\
-\Omega_2^2 x_b+\Omega_1\Omega_2 y_c+\mu\frac{x_b}{R_b^3} &= \frac{W'}{m_b}, && (20)\\
\Omega_1\Omega_2 x_b-\Omega_1^2 y_c+\mu\frac{y_c}{R_b^3} &= 0. && (21)
\end{aligned}$$

For $y_c = 0$, the solution leads to the radial relative equilibrium as discussed previously, by interchanging Ω_2 and Ω_3. Therefore we restrict our attention to the case $y_c \neq 0$. Assume first that $R_a = R_b = R$, which implies $x_a = -x_b$. The above equations are rewritten as

$$\begin{aligned}
-\Omega_2^2 x_a+\Omega_1\Omega_2 y_c+\mu\frac{x_a}{R^3} &= -\frac{W'}{m_a}, && (22)\\
\Omega_1\Omega_2 x_a-\Omega_1^2 y_c+\mu\frac{y_c}{R^3} &= 0, && (23)\\
-\Omega_2^2 x_b+\Omega_1\Omega_2 y_c+\mu\frac{x_b}{R^3} &= \frac{W'}{m_b}, && (24)\\
\Omega_1\Omega_2 x_b-\Omega_1^2 y_c+\mu\frac{y_c}{R^3} &= 0. && (25)
\end{aligned}$$

From (23) and (25), it can be proved that $\mu = \Omega_1^2 R^3$, which implies further that $\Omega_2 = 0$. Thus, from (22), we must have $m_a = m_b$. Consequently, the solution for relative equilibrium is obtained as $\boldsymbol{R}_a = (\frac{\ell}{2}, y_c, 0)^T$, $\boldsymbol{R}_b = (-\frac{\ell}{2}, y_c, 0)^T$, $\boldsymbol{\Omega} = (\Omega_1, 0, 0)^T$, with $m_a = m_b$, $y_c = R_c$, $\Omega_1^2 = \frac{\mu}{R^3}$, and $W' = -\frac{\mu m_a \ell}{2R^3} < 0$. This relative equilibrium is also great-circle, at which the spring with compressive elastic force is perpendicular to the orbital plane and m_a and m_b are equidistant to the attraction center, cf. Fig. 3.

Next, for $R_a \neq R_b$, from (19) and (21), we obtain

$$(x_a - x_b)\Omega_1\Omega_2 = -\mu y_c(\frac{1}{R_a^3} - \frac{1}{R_b^3}) \neq 0, \qquad (26)$$

which implies $\Omega_1 \neq 0$ and $\Omega_2 \neq 0$. By adding $m_a\times$(19) and $m_b\times$(21), it is found that

$$-(m_a+m_b)(\Omega_2 x_c - \Omega_1 y_c)\Omega_1 = \mu\left(\frac{m_a}{R_a^3}+\frac{m_b}{R_b^3}\right)y_c \neq 0, \quad (27)$$

which implies $\Omega_2 x_c - \Omega_1 y_c \neq 0$. From (18) and (20), we have

$$(m_a+m_b)(\Omega_2 x_c - \Omega_1 y_c)\Omega_2 = \mu\left(\frac{m_a x_a}{R_a^3}+\frac{m_b x_b}{R_b^3}\right) \neq 0. \quad (28)$$

Define

$$f_x = \frac{m_a x_a}{R_a^3} + \frac{m_b x_b}{R_b^3} \neq 0, \qquad f_y = \left(\frac{m_a}{R_a^3} + \frac{m_b}{R_b^3}\right) y_c \neq 0.$$

From (27) and (28), it is observed that $\Omega_2/\Omega_1 = -f_x/f_y$. Since

$$f_x y_c - f_y x_c = \frac{m_a m_b}{m_a + m_b}\left(\frac{1}{R_a^3} - \frac{1}{R_b^3}\right)(x_a - x_b) y_c \neq 0,$$

we know that $x_c\Omega_1 + y_c\Omega_2 \neq 0$, which means there is no great-circle relative equilibrium for this case. It can be easily verified that the vectors $\boldsymbol{r}_g$ and $\boldsymbol{r}_c$ defined in Section 2 are not parallel to each other.

Eliminating Ω_1 and Ω_2 from (26) and (27), we obtain

$$f = f_x f_1 + f_y f_2 = 0, \tag{29}$$

where

$$f_1 = \frac{x_b}{R_a^3} - \frac{x_a}{R_b^3}, \qquad \text{and} \qquad f_2 = \left(\frac{1}{R_a^3} - \frac{1}{R_b^3}\right) y_c.$$

Now if $m_a = m_b = m$, then f becomes

$$m\left(\frac{1}{R_a^2} - \frac{1}{R_b^2}\right)\left[\frac{1}{2}(R_a^2 + R_b^2 - \ell^2)\left(\frac{1}{R_a^4} + \frac{1}{R_a^2 R_b^2} + \frac{1}{R_b^4}\right) + \frac{1}{R_a R_b}\right].$$

For natural configurations, $R_a^2 + R_b^2 > R_c^2 > \ell^2$, and (29) implies $R_a = R_b$. Consequently, the following theorem is concluded.

Theorem 1 *At the relative equilibrium where the constraint force is compressive, $R_a = R_b$ if and only if $m_a = m_b$.*

In particular, the case of $m_a \neq m_b$ leads toward the nongreat-circle relative equilibria, in which $\boldsymbol{R}_a = (x_a, y_c, 0)^T$, $\boldsymbol{R}_b = (x_b, y_c, 0)^T$, and $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, 0)^T$, which are all on the xy-plane. For such case, equation (29) needs to be satisfied, Let $x_c = R_c \cos\theta$ and $y_c = R_c \sin\theta$. Given μ, R_c, ℓ, m_a, and m_b, the formula (29) can be written as an equation of θ. Consequently, nongreat-circle relative equilibria can be obtained by solving $f(\theta) = 0$. Since $f(0) < 0$, $f(\pi) > 0$, and that $f(\theta)$ is a continuous function for $R_c > \ell$, there exists a solution for $f(\theta) = 0$. On the other hand, it can be shown that f is monotonically increasing for $\theta \in [0, \pi]$. In fact, the first derivative of f, $f'(\theta)$, can be found to be

$$\frac{\ell y_c}{m_a + m_b}\left\{m_a^2\left(\frac{3}{R_a R_b^5} + \frac{1}{R_a^6}\right) + m_b^2\left(\frac{3}{R_a^5 R_b} + \frac{1}{R_b^6}\right) + \right.$$
$$\left. m_a m_b\left[3(R_a^2 + R_b^2 - \ell^2)\left(\frac{1}{R_a^8} + \frac{1}{R_b^8}\right) - \left(\frac{1}{R_a^3} + \frac{1}{R_b^3}\right)^2\right]\right\},$$

which is always positive. As a consequence, it is proved that the equation $f(\theta) = 0$ has one and only one solution in the domain $\theta \in [0, \pi]$ for $m_a \neq m_b$ and $R_c > \ell$. With the value of θ, the variables Ω_1 and Ω_2 can be computed from $\Omega_1 = f_y\beta$, and $\Omega_2 = -f_x\beta$, respectively, where

$$\beta = \sqrt{\frac{\mu y_c}{\ell f_x f_y}\left(\frac{1}{R_a^3} - \frac{1}{R_b^3}\right)}.$$

The elastic force on the spring can be then obtained from either (18) or (20). This leads immediately to the following theorem.

Theorem 2 *For natural configurations ($R_c > \ell$), the dynamics of the spring system has exactly one nongreat-circle relative equilibrium if and only if $m_a \neq m_b$.*

The nongreat-circle relative equilibria are depicted in Fig. 4, where the spring also undergoes compressive forces.

While the nongreat-circle relative equilibrium for a rigid body was numerically obtained in [20], it is analytically verified in this paper. To justify the existence of nongreat-circle relative equilibria, some numerical computations were performed to solve $f(\theta) = 0$ for different R_c/ℓ and $m_a/(m_a + m_b)$ (it may be assumed that $m_a < m_b$ without loss of generality). The results are presented in Table 1.

θ (deg)	$R_c/\ell = 2$	$R_c/\ell = 20$	$R_c/\ell = 200$
$\frac{m_a}{m_a+m_b} = 0.01$	100.364898	91.052658	90.105280
$\frac{m_a}{m_a+m_b} = 0.1$	98.413532	90.859264	90.085943
$\frac{m_a}{m_a+m_b} = 0.2$	96.278697	90.644416	90.064457
$\frac{m_a}{m_a+m_b} = 0.3$	94.170709	90.429596	90.042971

Table 1: Solutions of $f(\theta) = 0$

The configurations for some nongreat-circle relative equilibria are next computed, as shown in Table 2. Here it is assumed that $R_c = 7,000$km, $\ell = 350$km, and $\mu = 4 \times 10^{14}$.

m_a (kg)	m_b (kg)	θ (deg)	ϕ (deg)	δ (deg)
1	9999	91.073934	1.073934	-0.0000002684
10	9990	91.071999	1.072002	-0.0000026767
100	9900	91.052659	1.052684	-0.000026048
1000	9000	90.858264	0.859458	-0.00019335

Table 2: Nongreat-circle Relative Equilibria

Note that $\delta \neq 0$ implies that the corresponding relative equilibrium is nongreat-circle. Although this angle is rather small, the deflection of the spring system from the verticle is significant. For longer tether at LEO, this attitude drift may reach several degrees.

4 Relative Stabilities of Radial Relative Equilibria

The radial relative equilibria discussed in the previous section actually correspond to the station-keeping mode for TSS. The stability of such mode is important during the operation of TSS. However, the classical energy method is not applicable for the system under consideration. Accordingly, the reduced energy-momentum method, cf. [19, 16, 18] is employed in this section to prove the stability of radial relative equilibria.

Let $\boldsymbol{a}_e = (a, 0, 0)^T$, $\boldsymbol{b}_e = (b, 0, 0)^T$, $a > b > 0$, and $\boldsymbol{\xi} = (0, 0, \omega)^T$. The conditions for radial relative equilibria become

$$\frac{\mu m_a}{a^2} - m_a\omega^2 a + W' = 0, \quad \frac{\mu m_b}{b^2} - m_b\omega^2 b - W' = 0,$$

from which we have

$$\mu = \frac{m_a a + m_b b}{m_a/a^2 + m_b/b^2}\omega^2, \qquad (30)$$

and

$$W' = \frac{(a^2 + ab + b^2)m_a m_b \omega^2}{a^2 b^2 (m_a/a^2 + m_b b^2)} \cdot (a - b) > 0. \qquad (31)$$

The block diagonalization technique described in [16, 18] cannot be directly applied to the system considered in this paper, since the locked inertia tensor is singular. Instead, the reduced energy-momentum method requires being used in its more general form. Adopt the notations used in the above-mentioned references and denote TQ and T^*Q as the tangent and cotangent spaces of Q, respectively. The momentum map $J : T^*Q \to so(3)^*$ is $J(p_a, p_b) \cdot \xi = < p_a, \xi \times a > + < p_b, \xi \times b >$. The *locked inertia tensor* $I_{lock}(a,b)$ is found by the following computations:

$$\begin{aligned} & \ll I_{lock}(a,b)(\widehat{\eta}), \widehat{\xi} \gg_{so(3)} \\ = \; & m_a < \eta \times a, \xi \times a >_{\mathbb{R}^3} + m_b < \eta \times b, \xi \times b >_{\mathbb{R}^3} \\ = \; & < \eta, I^0_{lock}(a,b) \cdot \xi >_{\mathbb{R}^3}, \end{aligned}$$

where η, $\xi \in \mathbb{R}^3$ and $I^0_{lock}(a,b) = -m_a \widehat{a}\widehat{a} - m_b \widehat{b}\widehat{b}$. The premomentum map $\tilde{J} : Q \times so(3) \to so(3)^*$ is

$$\tilde{J}(a, b, \xi) = I^0_{lock}(a,b)\xi.$$

Then,

$$\begin{aligned} & D\tilde{J}(a,b,\xi) \cdot (\delta a, \delta b, \delta\eta) \\ = \; & m_a(2\widehat{a}\widehat{\eta} - \widehat{\eta}\widehat{a})\delta a + m_b(2\widehat{b}\widehat{\eta} - \widehat{\eta}\widehat{b})\delta b \\ & -(m_a \widehat{a}\widehat{a} + m_b \widehat{b}\widehat{b})\delta\eta. \end{aligned}$$

At radial relative equilibrium, $a_e = ae_1$, $b_e = be_1$ and $\xi = \omega e_3$, where e_i's denote the unit vectors of the coordinate axes of the uniformly rotating frame. Thus,

$$I^0_{lock}(a_e, b_e) = I_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad I_e = m_a a^2 + m_b b^2.$$

Accordingly, the locked inertia tensor is singular at such radial relative equilibrium, and $\tilde{J}(a_e, b_e, \xi) = I_e \omega e_3 = \mu_e$. Next it is required to find the kernel space of $D\tilde{J}$ at the radial relative equilibria. Let $\delta a = (\delta a_1, \delta a_2, \delta a_3)^T$, $\delta b = (\delta b_1, \delta b_2, \delta b_3)^T$ and $\delta\eta = (\delta\eta_1, \delta\eta_2, \delta\eta_3)^T$. It can be checked that $D\tilde{J}(a_e, b_e, \xi) \cdot (\delta a, \delta b, \delta\eta)$ is

$$\begin{pmatrix} -\omega(m_a a \delta a_3 + m_b b \delta b_3) \\ I_e \delta\eta_2 \\ I_e \delta\eta_3 + 2\omega(m_a a \delta a_1 + m_b b \delta b_1) \end{pmatrix}.$$

As a result, the kernel of $D\tilde{J}(a_e, b_e, \xi)$ can be expressed as

$$\begin{aligned} & \{(\delta a, \delta b, \delta\eta) : m_a a \delta a_3 + m_b b \delta b_3 = 0, \\ & \delta\eta_2 = 0, \; \delta\eta_3 = -\frac{2\omega}{I_e}(m_a a \delta a_1 + m_b b \delta b_1)\}. \end{aligned} \qquad (32)$$

Now we need to find $T_{(a,b,\eta)}(G_\mu \cdot (a,b,\eta))$, the tangent space on the group orbit, where $G_\mu = \{B \in SO(3) : B\mu = \mu\}$ is the isotropy subgroup. With $\mu_e = I_e \omega e_3$, we have $G_{\mu_e} = \{\exp(\widehat{q}) : q // e_3\}$. From the above observation, the tangent space is immediately obtained,

$$T_{(a,b,\eta)}(G_{\mu_e} \cdot (a,b,\eta)) \; = \; \{(\alpha a e_2, \alpha b e_2, o) : \alpha \in \mathbb{R}\} \qquad (33)$$

Obviously, $T_{(a,b,\eta)}(G_{\mu_e} \cdot (a,b,\eta))$ is a subspace of ker $(D\tilde{J}(a_e, b_e, \xi))$. In light of the symmetry, the augmented Hamiltonian is invariant on the orbit generated by the isotropy subgroup. Therefore it is only required to check the second variation of the augmented Hamiltonian on a subspace S which satisfies

$$\ker(D\tilde{J}(a_e, b_e, \xi)) = S \oplus T_{(a_e,b_e,\xi)}(G_{\mu_e} \cdot (a_e, b_e, \xi)).$$

From (32) and (33), the space S can be written as

$$\begin{aligned} S \; = \; & \{(\delta a, \delta b, \delta\eta) : \; \delta\eta_3 = -\frac{2\omega}{I_e}(m_a a \delta a_1 + m_b b \delta b_1), \\ & \delta a_2 = \delta b_2 = \delta\eta_2 = 0, \; m_a a \delta a_3 + m_b b \delta b_3 = 0 \}. \end{aligned} \qquad (34)$$

The *augmented Hamiltonian* function is composed of the Hamiltonian function and the momentum map as

$$H_\xi(a, b, p_a, p_b) = H(a, b, p_a, p_b) + J(p_a, p_b) \cdot \xi.$$

Denote the induced energy-momentum map on S by $\widetilde{H}_\xi$. The second variation of $\widetilde{H}_\xi$ on S can be derived as

$$\begin{aligned} & D^2 \widetilde{H}_\xi(a_e, b_e, \xi) \cdot (\delta a, \delta b, \delta\eta) \cdot (\delta a, \delta b, \delta\eta) \\ = \; & \frac{4\omega^2}{I_e}(m_a a \delta a_1 + m_b b \delta b_1)^2 \\ & + m_a \left[-(\frac{2\mu}{a^3} + \omega^2)\delta a_1^2 + \frac{\mu}{a^3}\delta a_3^2 \right] \\ & + m_b \left[-(\frac{2\mu}{b^3} + \omega^2)\delta b_1^2 + \frac{\mu}{b^3}\delta b_3^2 \right] \\ & + W'' \cdot (\delta a_1 - \delta b_1)^2 + W' \cdot (\delta a_3 - \delta b_3)^2/(a-b). \end{aligned} \qquad (35)$$

Since $W' > 0$ and $a > b > 0$, it is easily checked that the terms containing δa_3 and δb_3 in the above expression are always positive. Hence for determining the positiveness of $D^2 \widetilde{H}_\xi$, we only need to consider the following terms:

$$\begin{aligned} F \; = \; & \left[\frac{4\omega^2}{I_e} m_a^2 a^2 - (\frac{2\mu}{a^3} + \omega^2) m_a + W'' \right] \delta a_1^2 \\ & + \left[\frac{4\omega^2}{I_e} m_b^2 b^2 - (\frac{2\mu}{b^3} + \omega^2) m_b + W'' \right] \delta b_1^2 \\ & + 2(\frac{4\omega^2}{I_e} m_a m_b ab - W'')\delta a_1 \delta b_1. \end{aligned} \qquad (36)$$

Define

$$u = \frac{4 m_a a^2}{I_e} - \frac{2\mu}{a^3\omega^2} - 1, \qquad v = \frac{4 m_b b^2}{I_e} - \frac{2\mu}{b^3\omega^2} - 1.$$

Equation (36) is simplified to be

$$\begin{aligned} F \; = \; & (m_a \omega^2 u + W'')\delta a_1^2 + (m_b \omega^2 v + W'')\delta b_1^2 \\ & + 2(4\omega^2 m_a m_b ab / I_e - W'')\delta a_1 \delta b_1 \\ = \; & F_{11}\delta a_1^2 + F_{22}\delta b_1^2 + 2F_{12}\delta a_1 \delta b_1, \end{aligned}$$

which is in fact a quadratic form. The necessary and sufficient condition for F being positive definite is that

$$F_{11} = m_a \omega^2 u + W'' > 0, \qquad (37)$$

and

$$\begin{aligned} F_{11}F_{22} - F_{12}^2 \; = \; & (m_a u + m_b v + 8 m_a m_b ab / I_e)\omega^2 W'' \\ + \; & (uv - 16 m_a m_b a^2 b^2 / I_e^2) m_a m_b \omega^4 > 0 \end{aligned} \qquad (38)$$

The results are summarized in the following Theorem.

Theorem 3 *For a spring system whose spring characteristics are governed by the elastic potential energy function W (possibly nonlinear), the radial relative equilibrium is stable if (37) and (38) are satisfied.*

5 Conclusions

In this paper, we discussed the dynamical behavior of the system of two spring-connected point masses moving in a central gravitational field. In particular, the center of mass and the center of gravity of an arbitrary assembly were defined. The notion of relative equilibrium was introduced, and it was shown that it is the center of gravity (instead of the center of mass) that must trace a great circle at relative equilibria. This leads to the notion of nongreat-circle relative equilibria, in which the center of mass traces a nongreat circle. It was proved that for the natural spring system under consideration, there exist such nongreat-circle relative equilibria if and only if the two end masses are unequal. To analyze the stability of the radial relative equilibria, the reduced energy-momentum method leading successfully to conditions for stability was used. It was shown that for general (linear or nonlinear) springs, the stability conditions are in terms of the second derivatives of the elastic potential energy functions. The derived conditions may be helpful in the design of future large tether systems.

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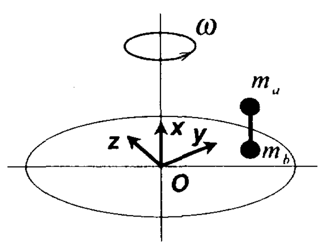

Figure 3: Relative Equilibrium with Compressive Elastic Force and $m_a = m_b$.

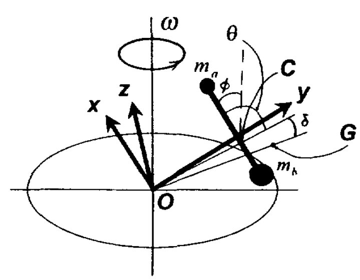

Figure 4: Nongreat-circle Relative Equilibrium