Dielectrophoretic force and torque on an ellipsoid in an arbitrary time varying electric field

C. Y. Yang
Department of Automatic Control Engineering, Chungchou Institute of Technology, Yuan Lin, Changhua, 51003 Taiwan, Republic of China

U. Lei[a)]
Institute of Applied Mechanics, National Taiwan University, Taipei, 10764 Taiwan, Republic of China

(Received 12 February 2007; accepted 9 March 2007; published online 10 April 2007)

The dielectrophoretic force and torque on an ellipsoid were derived in an arbitrary time varying electric field using the effective moment method. The result depends on the local instantaneous electric field felt by the particle and the polarization history experienced by the particle. In addition to the crossing trajectory effect for a moving particle in a nonuniform field in literatures, the polarization history effect is also significant for a particle when it rotates at a speed of the same order or greater than the electric frequency. The result may find applications to the ac electrokinetic manipulation of particles. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2721398]

Dielectrophoresis is an effective tool in the manipulation of particles,[1] and its fundamentals can be found from Jones.[2] As a particle moves during the manipulation, it cuts across the spatial nonuniform field with varying velocities, and is thus exposed to an arbitrary time varying electric field and experiences unsteady viscous drag and resistive torque from the surrounding fluid, in general. Recently, Yang and Lei[3] have derived the dielectrophoretic force and torque on a sphere in an arbitrary time varying electric field. The present work is to extend the theory of Yang and Lei[3] to ellipsoidal particles, with particular emphasis paid to the effect of particle rotation, which exists also in spherical particle and was not discussed.

Consider a homogeneous ellipsoidal particle with semi-axes a, b, and c, such that $a \geqslant b \geqslant c$. Cartesian coordinates (ξ_1, ξ_2, ξ_3), with unit vectors $(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)$, are fixed at the center of the ellipsoid, with ξ_1, ξ_2, and ξ_3 along the semiaxes a, b, and c, respectively. Consider first the case when the ellipsoid is subjected to an arbitrary time (t) varying electric field along the ξ_1 direction,

$$\mathbf{E}(t) = E_1(t)\hat{\xi}_1. \tag{1a}$$

In terms of the Fourier integral, the magnitude can be expressed as

$$E_1(t) = \int_{-\infty}^{\infty} E_{1\omega}(\omega) e^{j\omega t} d\omega, \tag{1b}$$

where $j=\sqrt{-1}$. For each Fourier component $E_{1\omega}e^{j\omega t}$, the induced complex effective dipole moment can be found from Ref. 2 as $m_{1\omega}e^{j\omega t}$, where

$$m_{1\omega} = 4\pi\varepsilon_m abc E_{1\omega} K_1^* = 4\pi\varepsilon_m abc E_{1\omega} \frac{\varepsilon_p^* - \varepsilon_m^*}{3[\varepsilon_m^* + (\varepsilon_p^* - \varepsilon_m^*)L_1]}, \tag{2a}$$

with K_1^* the complex Clausius-Mossotti factor along the ξ_1 direction, L_1 the depolarization factor along the ξ_1 direction as defined on Ref. 2 and $\varepsilon_m^* = \varepsilon_m + \sigma_m/(j\omega)$ and $\varepsilon_p^* = \varepsilon_p + \sigma_p/(j\omega)$. Here $\varepsilon_m(\sigma_m)$ and $\varepsilon_p(\sigma_p)$ are the permittivities (conductivities) of the surrounding medium and the particle, respectively. After some manipulations, Eq. (2a) becomes

$$m_{1\omega} = 4\pi\varepsilon_m abc E_{1\omega} K_1 \left[1 + \frac{\Delta_1}{\tau_{\mathrm{MW1}}^{-1} + j\omega}\right], \tag{2b}$$

where

$$K_1 = \frac{\varepsilon_p - \varepsilon_m}{3L_1\varepsilon_p + (3-3L_1)\varepsilon_m}, \quad \Delta_1 = \frac{1}{\tau_o} - \frac{1}{\tau_{\mathrm{MW1}}},$$

$$\tau_0 = \frac{\varepsilon_p - \varepsilon_m}{\sigma_p - \sigma_m}, \quad \tau_{\mathrm{MW1}} = \frac{L_1\varepsilon_p + (1-L_1)\varepsilon_m}{L_1\sigma_p + (1-L_1)\sigma_m}. \tag{3}$$

Here τ_{MW1} is the relaxation time for Maxwell-Wagner surface polarization along the ξ_1 direction. With Eqs. (2b) and (1b), and $E_{1\omega}(\omega) = 1/2\pi\int_{-\infty}^{\infty} E_1(\tau)e^{-j\omega\tau}d\tau$, the inverse Fourier transform, the induced effective dipole moment for the arbitrary electric field in Eq. (1a) can be obtained via superposition in a way similar to that in Lei *et al.*[4] The result is

$$\begin{aligned} m_1(t) &= \int_{-\infty}^{\infty} m_{1\omega} e^{j\omega t} d\omega = 4\pi\varepsilon_m abc K_1 \left\{ E_1(t) + \frac{\Delta_1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \frac{e^{j\omega(t-\tau)}}{\tau_{\mathrm{MW1}}^{-1} + j\omega} d\omega\right] E_1(\tau) d\tau \right\} \\ &= 4\pi\varepsilon_m abc K_1 \left\{ \{E_1(t) + \frac{\Delta_1}{2\pi}\left[\int_{-\infty}^{t}\left(\int_{-\infty}^{\infty} \frac{e^{j\omega(t-\tau)}}{\tau_{\mathrm{MW1}}^{-1} + j\omega} d\omega\right) E_1(\tau) d\tau + \int_{t}^{\infty}\left(\int_{-\infty}^{\infty} \frac{e^{-j\omega(\tau-t)}}{\tau_{\mathrm{MW1}}^{-1} + j\omega} d\omega\right) E_1(\tau) d\tau\right]\right\}, \end{aligned} \tag{4a}$$

$$m_1(t) = 4\pi\varepsilon_m abc K_1 \left\{ E_1(t) + \Delta_1 \int_{-\infty}^{t} Q_1 E_1(\tau) d\tau \right\}, \tag{4b}$$

where $Q_1 = \exp[-(t-\tau)/\tau_{\mathrm{MW1}}]$. There are two definite integrals over ω involving complex values in Eq. (4a). Only the real part of the first integral is nonzero, which leads to Eq.

[a)] Author to whom correspondence should be addressed; electronic mail: leiu@spring.iam.ntu.edu.tw

0003-6951/2007/90(15)/153901/3/$23.00

(4b). The imaginary parts of the first integral and the second integral are identically zero. For a three dimensional arbitrary time varying field $\mathbf{E}(t)=\sum_{n=1}^{3} E_n(t)\hat{\xi}_n$, instead of Eq. (1a), the corresponding induced effective dipole moment for the ellipsoid can be found via superposition as

$$\mathbf{m}(t)=4\pi\varepsilon_m abc\sum_{n=1}^{3} K_n\left[E_n(t)+\Delta_n\int_{-\infty}^{t} Q_n E_n(\tau)d\tau\right]\hat{\xi}_n, \tag{5}$$

where K_n, Δ_n, and Q_n are the same as K_1, Δ_1, and Q_1, with the subscript 1 replaced by n.

As in Yang and Lei,[3] there are two origins for the time variation of the field felt by a moving particle: one is the temporal variation of the field itself and the other is caused by the translation of the particle in a spatially nonuniform field and was named as the "crossing trajectory effect." The time variation of $\mathbf{m}(t)$ is thus more appropriate to be represented as $\mathbf{m}(\mathbf{Y}(t),t)$ to emphasize the crossing trajectory effect, where $\mathbf{Y}(t)$ is the displacement of the center of the particle at time t. When the ellipsoidal particle is moving through a time and spatial varying electric field, it rotates, in general, as it translates, which implies that the directions of the unit vectors $(\hat{\xi}_1,\hat{\xi}_2,\hat{\xi}_3)$ also change with time. Even when a particle rotates constantly in a dc field, it experiences a time varying field in a frame fixed with respect to it. Thus the particle rotation itself is another source of time varying field experienced by the particle, which was not discussed in Yang and Lei.[3] Therefore, for a clear illustration of the physics, we rewrite Eq. (5) as

$$\mathbf{m}(\mathbf{Y}(t),t)=4\pi\varepsilon_m abc\sum_{n=1}^{3} K_n\Bigg[E_n(\mathbf{Y}(t),t)+\Delta_n\int_{-\infty}^{t} Q_n\mathbf{E}(\mathbf{Y}(\tau),\tau)\cdot\hat{\xi}_n(\tau)d\tau\Bigg]\hat{\xi}_n(t). \tag{6}$$

The n component of the field, $E_n(\tau)$, felt by the particle at time τ when it is at $\mathbf{Y}(\tau)$ in Eq. (5) is written in Eq. (6) as $E_n(\tau)=E_n(\mathbf{Y}(\tau),\tau)=\mathbf{E}(\mathbf{Y}(\tau),\tau)\cdot\hat{\xi}_n(\tau)$ to emphasize that the direction of the unit vector $\hat{\xi}_n(\tau)$ at a previous time τ is different from $\hat{\xi}_n(t)$ at the current time t. Here $\{\mathbf{Y}(\tau),-\infty<\tau\le t\}$ and $\{\hat{\xi}_1(\tau),\hat{\xi}_2(\tau),\hat{\xi}_3(\tau),-\infty<\tau\le t\}$ describe the history of the displacement and orientation of the particle, respectively. Six scalar variables, three components of $\mathbf{Y}(\tau)$, and three Euler angles for depicting the orientation of the particle axes are required for a full description of the motion of the particle, which are governed by the conservation of linear and angular momentum according to Newton's second law. Once these six variables are known, one may evaluate the induced effective dipole moment according to Eq. (6), and then the dielectrophoretic force and torque using the effective moment method.[2] After some manipulation, we found

$$\begin{aligned}\mathbf{F}(\mathbf{Y}(t),t)&=\mathbf{m}(\mathbf{Y}(t),t)\cdot\nabla\mathbf{E}(\mathbf{Y}(t),t)\\&=4\pi\varepsilon_m abc\Bigg\{\frac{1}{2}\nabla\sum_{n=1}^{3}K_nE_n^2(\mathbf{Y}(t),t)\\&\quad+\sum_{n=1}^{3}K_n\Delta_n\int_{-\infty}^{t}Q_nE_n(\mathbf{Y}(\tau),\tau)\frac{\partial\mathbf{E}(\mathbf{Y}(t),t)}{\partial\xi_n}d\tau\Bigg\}\end{aligned} \tag{7a}$$

and

$$\mathbf{T}(\mathbf{Y}(t),t)=\mathbf{m}(\mathbf{Y}(t),t)\times\mathbf{E}(\mathbf{Y}(t),t)=\sum_{n=1}^{3}T_n(\mathbf{Y}(t),t)\hat{\xi}_n(t), \tag{7b}$$

where

$$\begin{aligned}T_n(\mathbf{Y}(t),t)&=4\pi\varepsilon_m abc\Bigg\{(K_p-K_q)E_p(\mathbf{Y}(t),t)E_q(\mathbf{Y}(t),t)\\&\quad+\int_{-\infty}^{t}[K_p\Delta_pQ_pE_p(\mathbf{Y}(\tau),\tau)E_q(\mathbf{Y}(t),t)\\&\quad-K_q\Delta_qQ_qE_q(\mathbf{Y}(\tau),\tau)E_p(\mathbf{Y}(t),t)]d\tau\Bigg\}.\end{aligned} \tag{7c}$$

The index notation of Eq. (7c) is as follows. When $n=1$, $p=2$, and $q=3$; when $n=2$, $p=3$, and $q=1$; when $n=3$, $p=1$, and $q=2$.

Equations (6) and (7) can be validated by checking against the special cases in literatures. Consider the particle in a unidirectional spatially uniform ac field, $\mathbf{E}(t)=E_0\cos(\omega t)\hat{\xi}_1$, where E_0 and ω are constants. The particle remains stationary at its original position, and Eq. (6) reduces to Eq. (5.26) of Ref. 2. If the particle is further a sphere and is allowed to rotate with a constant angular velocity $\mathbf{\Omega}$ perpendicular to the field, we calculated the torque using Eqs. (7b) and (7c) and found the same result as Eq. (4.16) of Ref. 2, which was studied previously by Turcu.[5]

For a spherical particle with radius R, we have $a=b=c=R$, $L_n=1/3$, $K_n=K=(\varepsilon_p-\varepsilon_m)/(\varepsilon_p+2\varepsilon_m)$, $\tau_{\mathrm{MW}n}=\tau_{\mathrm{MW}}=(\varepsilon_p+2\varepsilon_m)/(\sigma_p+2\sigma_m)$, $\Delta_n=\Delta=1/\tau_0-1/\tau_{\mathrm{MW}}$, and $Q_n=Q=\exp[-(t-\tau)/\tau_{\mathrm{MW}}]$, for $n=1$, 2, and 3. Equation (6) becomes

$$\mathbf{m}(\mathbf{Y}(t),t)=4\pi\varepsilon_m R^3K\Bigg\{\mathbf{E}(\mathbf{Y}(t),t)+\Delta\int_{-\infty}^{t}Q\sum_{n=1}^{3}[(\mathbf{E}(\mathbf{Y}(\tau),\tau)\cdot\hat{\xi}_n(\tau))\hat{\xi}_n(t)]d\tau\Bigg\}. \tag{8}$$

If the particle does not rotate as it translates, $\hat{\xi}_n$ $(n=1,2,3)$ are fixed vectors for all the time, and Eq. (8) becomes

$$\mathbf{m}(\mathbf{Y}(t),t)=4\pi\varepsilon_m R^3K\Bigg\{\mathbf{E}(\mathbf{Y}(t),t)+\Delta\int_{-\infty}^{t}Q\mathbf{E}(\mathbf{Y}(\tau),\tau)d\tau\Bigg\}, \tag{9}$$

which is the same as that in Yang and Lei.[3]

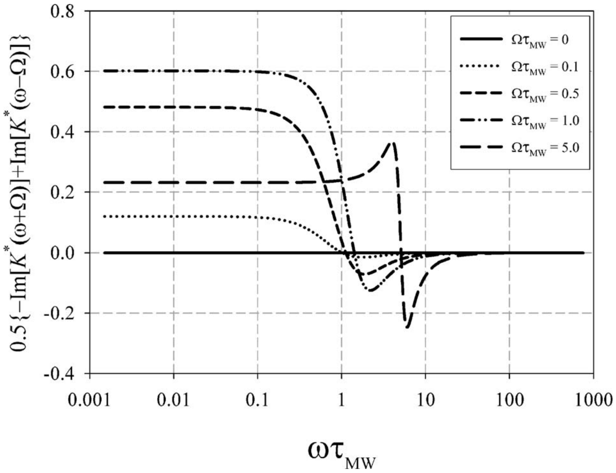

FIG. 1. Variation of the factor for the effect of particle spinning on the torque in Eq. (11), $0.5\{-\mathrm{Im}[K^*(\omega+\Omega)]+\mathrm{Im}[K^*(\omega-\Omega)]\}$, with various dimensionless rotation speeds and electric frequencies.

As shown in Eqs. (6) and (7), the induced effective dipole moment, the dielectrophoretic force, and the dielectrophoretic torque depend on the instant field felt by the particle and the polarization history experienced by the particle. Such results are similar to those for the sphere[3] except there are geometric dependences in the present results for ellipsoids as reflected by the parameters K_n and τ_{MWn}, with $n=1$, 2, or 3. Yang and Lei[3] found that the polarization history effect in the equations (the integral terms) is significant for a moving spherical particle when it is traveling with a sufficiently high speed, when it is traveling across a nonuniform field with a sufficiently small length scale, or when the applied electric frequency is relatively low. Such findings hold qualitatively for the present ellipsoid particles by carrying out a similar analysis as that in Ref. 3. The only physics (other than the geometric effect) that is new in the present result is the effect of particle rotation. Consider an ellipsoidal particle which is rotating with a constant speed Ω about the $\hat{\xi}_3$ axis in a general two dimensional quasistatic ac field with frequency ω,[6]

$$\mathbf{E}(x,y,t) = E_{0x}\cos(\omega t + \phi_x)\hat{x} + E_{0y}\cos(\omega t + \phi_y)\hat{y}. \quad (10)$$

Here the field amplitudes E_{0x} and E_{0y} and the phase components ϕ_x and ϕ_y are functions of the Cartesian coordinates (x,y) for describing the electric field, with $(\hat{x},\hat{y})$ the corresponding unit vectors. The ξ_3 axis is perpendicular to the xy plane. The unit vectors of the body fixed coordinates can be expressed as $\hat{\xi}_1(\tau)=\hat{x}\cos\theta+\hat{y}\sin\theta$ and $\hat{\xi}_2(\tau)=-\hat{x}\sin\theta+\hat{y}\cos\theta$, where $\theta=\Omega\tau$ and $-\infty\leqslant\tau\leqslant t$. The induced effective dipole moment components, $m_1(\mathbf{Y}(t),t)$ and $m_2(\mathbf{Y}(t),t)$, in Eq. (6) and the torque in Eq. (7b) can then be calculated by carrying out the integration. For an axial symmetric ellipsoid, such as an oblate spheroid $(a=b)$ or a sphere, $L_1=L_2$, the torque of the particle with its center at (x,y) at time t is found to be $\mathbf{T}(x,y,t)=T_3(x,y,t)\hat{\xi}_3$, with the time averaged torque magnitude,

$$\begin{aligned}\langle T_3\rangle = &-2\pi\varepsilon_m a^2 c\{0.5(E_{0x}^2+E_{0y}^2)(-\mathrm{Im}[K_1^*(\omega+\Omega)]\\&+\mathrm{Im}[K_1^*(\omega-\Omega)])+E_{0x}E_{0y}(\mathrm{Im}[K_1^*(\omega+\Omega)]\\&+\mathrm{Im}[K_1^*(\omega-\Omega)])\sin(\phi_x-\phi_y)\}. \quad (11)\end{aligned}$$

Here the complex Clausius-Mossotti factor $K_1^*(\omega\pm\Omega)$ is the same as that defined in Eq. (2a) except that the frequency ω is replaced by the reduced frequency, $\omega\pm\Omega$, and "$\mathrm{Im}[K_1^*(\omega\pm\Omega)]$" is the imaginary part of "$K_1^*(\omega\pm\Omega)$." The first term on the right hand side of Eq. (11) involving the sum of the square of the field magnitudes is contributed by the self-spinning of the particle, and is identically zero if $\Omega=0$. The second term on the right hand side of Eq. (11) involving the phase difference of the field is associated with the electric rotation of the ac field for $\phi_x\neq\phi_y$. For a purely rotational ac field, we have $E_{0x}=E_{0y}=E_0=$constant and $\phi_x-\phi_y=\pi/2$, Eq. (11) reduces to $\langle T_3\rangle=-4\pi\varepsilon_m a^2 c E_0^2\,\mathrm{Im}[K_1^*(\omega-\Omega)]$, as in Ref. 2. For a sphere with radius R rotating constantly in a nonrotational ac field with $\phi_x=\phi_y$, $E_{0x}=E_0$, and $E_{0y}=0$, Eq. (11) reduces to $\langle T_3\rangle=-\pi\varepsilon_m R^3E_0^2(-\mathrm{Im}[K^*(\omega+\Omega)]+\mathrm{Im}[K^*(\omega-\Omega)])$, which is the same as that in Turcu.[5] Here we write $K_1^*(\omega\pm\Omega)$ as $K^*(\omega\pm\Omega)$ for spherical particles. To illustrate the effect of particle rotation on the torque in Eq. (11), we plotted $0.5\{-\mathrm{Im}[K^*(\omega+\Omega)]+\mathrm{Im}[K^*(\omega-\Omega)]\}$ and $0.5\{\mathrm{Im}[K^*(\omega+\Omega)]+\mathrm{Im}[K^*(\omega-\Omega)]\}$ for spherical particles with $\varepsilon_p/\varepsilon_0=5$ and $\sigma_p=1\times10^{-6}$ S/m in the medium (water) with $\varepsilon_m/\varepsilon_0=78.5$ and $\sigma_m=1\times10^{-7}$ S/m in Figs. 1 and 2, respectively, for different values of $\omega\tau_{MW}$ and $\Omega\tau_{MW}$. Here ε_0 is the permittivity of free space. The curve for $\Omega\tau_{MW}=0$ in Fig. 2 corresponds to the result without particle rotation. By comparing different curves in Figs. 1 and 2 with the curve for $\Omega\tau_{MW}=0$ in Fig. 2, together with the form in Eq. (11), it is found that the effect of particle rotation is significant when the electric frequency ω is of the same order or less than the angular speed of the particle, Ω.

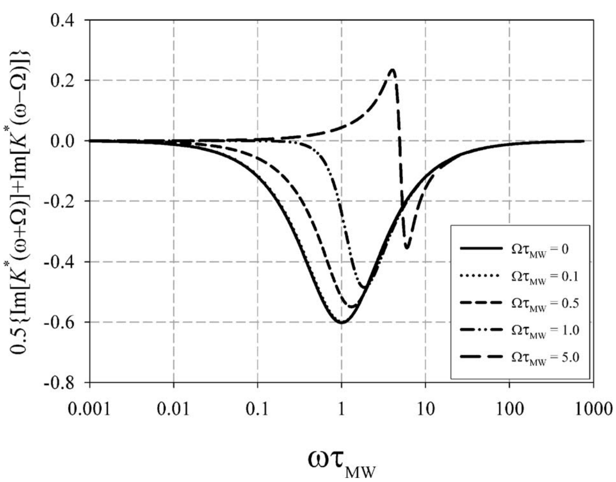

FIG. 2. Variation of the factor for the effect of electric rotation on the torque in Eq. (11), $0.5\{\mathrm{Im}[K^*(\omega+\Omega)]+\mathrm{Im}[K^*(\omega-\Omega)]\}$, with various dimensionless rotation speed and electric frequencies.

This work is supported partially by the National Science Council through Grant No. NSC 95-2120-M-002-006 and the Department of Economic affair through Grant No. 95-EC-17-A-05-S1-017 of Taiwan, Republic of China.

[1]M. P. Hughes, *Nanoelectromechanics in Engineering and Biology* (CRC, Boca Raton, FL, 2003).
[2]T. B. Jones, *Electromechanics of Particles* (Cambridge University Press, Cambridge, 1995).
[3]C. Y. Yang and U. Lei, Appl. Phys. Lett. **89**, 163902 (2006).
[4]U. Lei, C. Y. Yang, and K. C. Wu, Appl. Phys. Lett. **89**, 181908 (2006).
[5]I. Turcu, J. Phys. A **20**, 3301 (1987).
[6]X.-B. Wang, Y. Huang, F. F. Becker, and P. R. C. Gascoyne, J. Phys. D **27**, 1571 (1994).