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以重整群分析紊流模式(III)
RNG Analysis of Turbulence Models (III)

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本研究旨在以重整群分析方法研究三維磁化流的流場特性, 在大尺度渦漩與小尺度渦漩爲統計 之獨立的假設下, 我們可以建立一個遞迴重整群的程序, 而建構出重整群轉換, 在此轉換下得 到一具有換尺不變性的 MHD 方程, 此轉換的固定點在數學上可被等價成渦漩黏滯力在 Fourier 空 間中的積微分方程式, 藉由此積微分方程式的求解, 證明速度場和磁感應場的能量光譜在慣性 子區都遵循 Kolmogorov 光譜定律-4/3次方呈正比的關係, 此結果與實驗室量測和天文物理上的 觀測結果一致, 此解亦可進而推導出 Smagorinsky 模型, 並且決定 Smagorinsky 常數與隔點大小以及流場特徵波數的關係。

In this study, we continue with a recursive renormalization group (RG) analysis of incompressible turbulence, aiming at investigating various turbulent properties of three-dimensional magneto-hydrodynamics (MHD). In particular, we are able to locate the fixed point (i.e. the invariant effective eddy viscosity) of the RG transformation under the following conditions. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and are approximately equal in magnitude. It is found under these conditions that re-normalization does not incur an increment of the magnetic resistivity, while the coupling effect tends to reduce the invariant effective eddy viscosity. Both the velocity and magnetic energy spectra are shown to follow the Kolmogorov $k^{-5/3}$ in the inertial subrange; this is consistent with some laboratory measurements and observations in astronomical physics. By assuming further that the velocity and magnetic induction share the same specified form of energy spectrum, we are able to determine the dependence of the (magnetic) Kolmogorov constant C_K (C_M) and the model constant C_S of the Smagorinsky model for large-eddy simulation on some characteristic wavenumbers.

Key words: Renormalization group analysis, magneto-hydrodynamic turbulence, Alfvén effect, effective eddy viscosity, magnetic resistivity

I. INTRODUCTION

Recently, the authors [1],[2] carried out a recursive renormalization group (RG) analysis of incompressible turbulence for flow turbulence and thermal turbulent transport. In this study, we continue with this previous RG analysis for magneto-hydrodynamic (MHD) turbulence, aiming at investigating various transport properties, in particular, the coupling effects between the flow and magnetic induction fields on the kinetic energy spectrum and the effective eddy viscosity.

The plasma science is widely applied to many areas from laser skill, thin film produce, nuclear rocket, even to astronomical physics (for example, solar wind, solar flares and coronal structure). Like in ordinary Newtonian fluids, MHD turbulence is expected to arise in plasma or magnetized fluids as the Reynolds number is increased beyond some critical value. In spite of the already scarce literature, the interest of MHD turbulence may further be divided into two-dimensional and three-dimensional turbulence. Kim and Yang [3] studied the scaling behavior of the randomly stirred MHD plasma in two dimensions and were able to show existence of the scaling solution at the fixed point of the RG transformation and derive the dependence of the power exponent of the energy spectrum on the driving Gaussian noise. Liang and Diamond [4] also presented their study for two-dimensional MHD turbulence by introducing the velocity stream function and the magnetic flux function in MHD equations. However, the latter authors showed no existence of a fixed point of the RG transformation and especially suggested that the applicability of RG method to turbulent system is intrinsically limited, especially in the case of systems with dual-direction energy transfer.

In contrast to flow in two dimensions, the effect of dual-direction energy transfer becomes weak in three dimensions (cf. McComb [5]). It would therefore be legitimate to employ the RG analysis for MHD turbulence in three dimensions. In the literature, there are some measured evidences about the validity of the Kolmogorov spectrum for the three-dimensional MHD turbulence. Alemany et al. [6] designed an equipment in the laboratory which produced turbulence by passing magnetized fluid to a mesh under an additional magnetic induction. In the area of astronomical physics, Matthaeus et al. [7] measured the magnetic energy spectrum of the solar wind, while Leamon et al. [8] measured the MHD turbulence within the coronal mass ejection. Both of their results suggested the Kolmogorov power law for the energy spectrum. Besides, Biskamp [9] mentioned that the Kolmogorov constant depends on the precise definition of the average magnetic induction, and hence on the geometry of the large scale eddies. On the theoretical side, Hatori [10] obtained the Kolmogorov spectrum for the three-dimensional MHD turbulence, but suggested that the Kolmogorov constant is universal. Verma [11] constructed a recursive renormalization group procedure for MHD turbulence and also found that the energy spectrum for the velocity obeys the Kolmogorov spectrum. It is the purpose of the present study to provide a recursive renormalization group analysis for MHD turbulence in three dimensions with the specific points of interest as follows. We will obtain the energy spectra for both of the velocity and magnetic induction fields, look for the invariant effective eddy viscosity and determine the dependence of the (magnetic) Kolmogorov constant C_K (C_M) and the model constant C_S for the Smagorinsky model for large-eddy simulation (LES).

Let us give a brief description of the present work. MHD is governed by a coupling set of equations, meanwhile, the MHD turbulence considered is further assumed to be isotropic, homogeneous and stationary. It is found convenient to introduce the Elsässer variables to write the equations for the velocity and magnetic induction fields in a symmetric form. In Section 3, a recursive RG analysis is carried out for the MHD equations in the wavenumber domain and a recursive relationship for the effective eddy viscosity $\nu_n(k)$ between two successive steps is established. The resulting expression is complicated enough and is apparently not amenable to further RG analysis. Instead, we restrict ourselves to the case when the following conditions hold. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. As a matter of fact, the two conditions imply a negligible effect of the subgrid cross helicity between the velocity and magnetic fields. In spite of these restrictions, the present RG analysis still warrants a sufficient interest as we investigate several observations in the area of astronomical physics. In Section 4, the energy spectra of the velocity and the magnetic fields are determined through use of the RG transformation, i.e. the recursive relation. Both spectra are found to follow the Kolmogorov $k^{-5/3}$ law in the inertial subrange. The results are consistent with the experimental results of Alemany et al. [6], and the observational results of Matthaeus et al. [7]. From a different approach, Chen and Montgomery [12] obtained the same power law in the inertial subrange by using some multiple-scale self-consistent calculations of turbulent MHD transport coefficients. In Section 5, the fixed point of the RG equation is located to give the invariant effective eddy viscosity $\nu(k)$ and magnetic resistivity $\tau(k)$. By assuming further a combination form of the energy spectra proposed respectively by Pao [13] and Quarini and Leslie [14], the invariant effective eddy viscosity is then employed in Section 6 to determine the dependence of the (magnetic) Kolmogorov constant C_K (C_M) and the Smagorinsky constant C_S on the cutoff wavenumber k_c, the wavenumber k_s of the largest eddies and the wavenumber k_p that peaks in the energy spectrum.

II. RENORMALIZATION GROUP ANALYSIS FOR MHD TURBULENCE

The basic idea of recursive RG analysis is to divide the wavenumber space $(0, k_0)$, where k_0 is Kolmogorov's scale, to a supergrid region $(0, k_c)$ and a subgrid region (k_c, k_0). The subgrid modes are then removed shell by shell by taking the subgrid average over a spherical shell (k_{n+1}, k_n).

The purpose of this study is to look for the invariant effective eddy viscosity by pursuing a differential version of the recursive relationship. Recall that the basic idea underlining the recursive RG analysis is to divide the wavenumber space $(0, k_0)$ to a supergrid region $(0, k_c)$ and a subgrid region (k_c, k_0); the subgrid modes are then removed piece by piece by taking subgrid averaging over a spherical shell (k_{n+1}, k_n). The result will certainly depend on the cutoff ratio $\Lambda = k_{n+1}/k_n$; and thus the invariant (limiting) effective eddy viscosity should be sought by taking the limiting operation $\Lambda \to 1$. By applying the energy disspation equation

$$\begin{aligned}&\int_{k_s}^{k_c} 2k^2\nu(k)E(k)dk \\ &= 2C_K^{\frac{3}{2}}\varepsilon_v \int_{k_s}^{k_c} F(k)k^{\frac{1}{3}}\phi(k/k_p)dk \\ &= \varepsilon_v,\end{aligned}$$

where k_s denotes the wavenumber of the largest eddy existing in the flow. Canceling out ϵ on both sides, we obtain the Kolmogorov constant C_K in terms of the three characteristic wavenumbers k_c, k_p and k_s:

$$C_K = \left[2\int_{k_s}^{k_c} F(k)k^{\frac{1}{3}}\phi(k/k_p)dk\right]^{\frac{-2}{3}}. \tag{1}$$

On the other hand,

$$C_M = \left[2k_c^{\frac{4}{3}}\hat{\tau}(k_c/k_p)\int_{k_s}^{k_c} k^{-1}\varphi(k/k_p)dk\right]^{\frac{-2}{3}}. \tag{2}$$

So far, we have not given a precise form for ϕ and φ. Let us assume further that both the velocity and magnetic induction fields share the same form of the energy spectrum which is a combination form of the scaling laws proposed respectively by Pao [13], Leslie and Quarini [14], that is,

$$\phi(k/k_p) = A_p\left(\frac{k}{k_p}\right)\exp\left(\frac{-3}{2}C_K^{\frac{-1}{2}}\varepsilon_v^{\frac{-1}{3}}\nu(k)k^{\frac{4}{3}}\right),$$

and

$$\varphi(k/k_p) = A_p\left(\frac{k}{k_p}\right)\exp\left(\frac{-3}{2}C_M^{\frac{-1}{2}}\varepsilon_M^{\frac{-1}{3}}\tau(k)k^{\frac{4}{3}}\right).$$

It follows immediately from Eq. (40) that

$$E^v(k) = A_p\left(\frac{k}{k_p}\right)C_K\varepsilon_v^{2/3}k^{-5/3} \exp\left(-\frac{3}{2}C_K^{\frac{-1}{2}}\varepsilon_v^{-1/3}\nu(k)k^{4/3}\right), \quad (3)$$

and

$$E^M(k) = A_p\left(\frac{k}{k_p}\right)C_M\varepsilon_M^{2/3}k^{-5/3} \exp\left(-\frac{3}{2}C_M^{\frac{-1}{2}}\varepsilon_M^{-1/3}\tau(k)k^{4/3}\right). \quad (4)$$

In these formulas, we have the factor

$$A_p(x) = \frac{x^{s+5/3}}{1+x^{s+5/3}}, \quad \text{where}$$

to take care of energy-containing eddies, where s is a parameter for flow. We may rewrite Eq. (1) in a more precise form as follows:

$$C_K = \left\{\frac{2\mathcal{M}}{s+5/3}e^{-1.5\mathcal{M}}\log\left[\frac{(k_c/k_p)^{s+5/3}+1}{(k_s/k_p)^{s+5/3}+1}\right]\right\}^{\frac{-2}{3}}, \quad (5)$$

$$C_M = \left\{\frac{2k_c^{\frac{3}{4}}\hat{\tau}\left(\frac{k_c}{k_p}\right)\exp\left[-1.5k_c^{\frac{3}{4}}\hat{\tau}\left(\frac{k_c}{k_p}\right)\right]}{s+5/3} \log\left[\frac{(k_c/k_p)^{s+5/3}+1}{(k_s/k_p)^{s+5/3}+1}\right]\right\}^{\frac{-2}{3}}. \quad (6)$$

As a matter of fact, Biskamp [9] indicated that the Kolmogorov constant depends on the precise definition of the averge magnetic induction, and hence on the geometry of the large scale eddies. Here, we have provided two relationships which show how the large-scale eddies can influence not only the Kolmogorov constant C_K but also the magnetic Kolmogorov constant C_M, as shown in (5) and (6). The large-scale eddies with the wavenumbers k_p and k_s indeed play an important role in deciding both of C_K and C_M. Here we recall that, k_p denotes the $1/($*geometric size*$)$ of the energy-containing eddies and k_s denotes the $1/($*geometric size*$)$ of the largest eddies in fluid.

In order to carry out large eddy simulation (LES) for MHD turbulence, we need to evaluate the Smagorinsky constant for MHD turbulence. First of all, we suppose no matter whenever we perform the RG analysis, the cutoff k_c is always very close to the Kolmogorov scale k_0. In doing so, we evaluate the effective eddy viscosity at $k = k_c$ which is far from k_0. We express ε_v in the resolvable velocity,

$$\varepsilon_v = \frac{\nu(k_c)}{2}\left(\frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i}\right)^2.$$

Solving the above algebraic equation for $\nu(k_c)$ and replacing k_c by $2\pi/\Delta$ where Δ denotes the cutoff size, we obtain

$$\nu(k_c) \equiv C_S\Delta^2\left|\frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i}\right|, \quad (7)$$

where

$$C_S = \frac{1}{4\sqrt{2}\pi^2} \left\{\hat{\nu}_0(k_0/k_p) - \frac{135}{364}\left[\frac{\phi(k_0/k_p)}{4\mathcal{H}_1} + \frac{\chi\varphi(k_0/k_p)}{4\mathcal{H}_2}\right]\right\}^{\frac{3}{2}} C_K^{\frac{3}{4}}. \quad (8)$$

This is the Smagorinsky constant, where we have left two undetermined parameters $\mathcal{H}_1$ and $\mathcal{H}_2$ which require two additional conditions to be fully determined.

In summary, the closed-form solutions for $\nu(k)$ and $\tau(k)$ have enabled derivation of the functional dependence of C_K, C_M and C_S. In other words, these numbers C_K, C_M and C_S are not genuine constants but dependent upon the characteristic wavenumbers k_p and k_s of the energy-containing eddies. Namely, the theory requires an input of the large-eddy wavenumbers k_p and k_s from observations and/or experiments. The value of k_s is approximately that of k_p. This was done in our early study for incompressible flow turbulence as well as in thermal-fluid turbulence; the range of variation of the relevant Kolmogorov's and Batchelor's constants were found in close agreement with experiments (cf. Chang et al. [1] and Lin et al. [2]).

III. CONCLUDING REMARKS

In this study, we have extended our previous RG analysis of incompressible flow turbulence to incompressible MHD turbulence.

The Elsässer variables are introduced to write the MHD equations for the velocity and magnetic induction fields in a symmetric form. RG analysis is then performed in the wavenumber domain. Taking subgrid averaging of the equation governing the supergrid modes yields a renormalizable form of the MHD equations. To proceed further with the RG transformation, we have to the following two assumptions. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfv*é*n effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. That these conditions still warrant sufficient interest are illustrated by some available data from observations in astronomical physics. Under these conditions, renormalization does not incur an increment of the magnetic resistivity τ, while the coupling effect tends to reduce the invariant effective eddy viscosity $\nu(k)$. Both the velocity and magnetic energy spectra are shown to follow the Kolmogorov $k^{-5/3}$ in the inertial subrange; this is consistent with some available laboratory measurements and observations in astronomical physics. Furthermore, by assuming that the velocity and magnetic induction fields share the same combined form of the energy spectra proposed respectively by Pao and by Leslie and Quarini, we are able to determine the dependence of the Kolmogorov constant C_K and the magnetic Kolmogorov constant C_M on the characteristic wavenumbers k_c, k_p and k_s. The results are applied to obtain the dependence of the Smagorinsky constant C_S for large-eddy simulation, which however contains two undetermined constants to be resolved.

In spite of the present success, it must be stressed upon that the imposed conditions (i) and (ii) imply a negligible effect of the subgrid cross helicity between the velocity and magnetic fields. There are cases where the effect is important and which may lead to quite different energy spectrum. In an early study, Kraichnan [22] derived a $k^{-3/2}$ energy spectrum of the inertial subrange when the magnetic energy in the sub-inertial wavenumbers exceeds the total energy in the inertial subrange. Pouquet et al. [23] had an intensive study on strong MHD helical turbulence and the nonlinear dynamo effect. Recently, Nakayama [24], [25] obtained also the $k^{-3/2}$ energy spectrum in the inertial subrange by constructing a spectral theory of strong shear Alfvén turbulence anisotropized by the presence of a uniform mean magnetic field. Of particular interest, we refer to Yoshizawa et al. [26] for reviewing the importance of the cross-helicity effect, and more generally for an extensive review of turbulence theories and modeling of fluids and plasmas.

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