行政院國家科學委員會專題研究計畫　成果報告

以重整群分析耦合場紊流模式

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一、中文摘要

本研究旨在以重整群分析方法研究不可壓縮之耦合磁流場紊流模式，在(1)**平均磁導相較於平均流速為可忽略及**(2)Alfvén **效應成立的假設下，我們可以建立一個遞迴重整群的程序，而建構出重整群轉換，在此轉換下得到一具有換尺不變性的耦合磁流場方程組，此轉換的固定點在數學上可被等價成渦漩黏滯力在** Fourier **空間中的積微分方程組，藉由此積微分方程組的求解，發現流場能量光譜與波數的**-5/3 **次方呈正比關係。**

關鍵詞：重整群分析、磁流紊流場、Alfvén **效應、等效渦漩黏滯係數、磁阻係數**

Abstract

In this study, we continue with a recursive renormalization group (RG) analysis of incompressible turbulence, aiming at investigating various turbulent properties of three-dimensional magneto- hydrodynamics (MHD). In particular, we are able to locate the fixed point (i.e. the invariant effective eddy viscosity) of the RG transformation under the following conditions. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and are approximately equal in magnitude. It is found under these conditions that re-normalization does not incur an increment of the magnetic resistivity, while the coupling effect tends to reduce the invariant effective eddy viscosity. Both the velocity and magnetic energy spectra are shown to follow the Kolmogorov $k^{-5/3}$ in the inertial subrange; this is consistent with some laboratory measurements and observations in astronomical physics.

Keywords: Renormalization group analysis, magnetohydrodynamic turbulence, Alfvén effect, effective eddy viscosity, magnetic resistivity.

1. Introduction

Recently, the authors carried out a recursive renormalization group (RG) analysis of incompressible turbulence for flow turbulence and thermal turbulent transport. In this study, we continue with this previous RG analysis for magneto- hydrodynamic (MHD) turbulence, aiming at investigating various transport properties, in particular, the coupling effects between the flow and magnetic induction fields on the kinetic energy spectrum and the effective eddy viscosity.

The plasma science is widely applied to many areas from laser skill, thin film produce, nuclear rocket, even to astronomical physics (for example, solar wind, solar flares and coronal structure). Like in ordinary Newtonian fluids, MHD turbulence is expected to arise in plasma or magnetized fluids as the Reynolds number is increased beyond some critical value. In spite of the already scarce literature, the interest of MHD turbulence may further be divided into two-dimensional and three-dimensional turbulence. Kim and Yang studied the scaling behavior of the randomly stirred MHD plasma in two dimensions and were able to show existence of the scaling solution at the fixed point of the RG transformation and derive the dependence of the power exponent of the energy spectrum on the driving Gaussian noise. Liang and Diamond also presented their study for two-dimensional MHD turbulence by introducing the velocity stream function and the magnetic flux function in MHD equations. However, the latter authors showed no existence of a fixed point of the RG transformation and especially suggested that the applicability of RG method to turbulent system is intrinsically limited, especially in the case of systems with dual-direction energy transfer.

In contrast to flow in two dimensions, the effect of dual-direction energy transfer becomes weak in three dimensions (cf. McComb). It would therefore be legitimate to employ the RG analysis for MHD turbulence in three dimensions. There are some measured evidences about the validity of the Kolmogorov spectrum for the three-dimensional MHD turbulence. Alemany et al. designed an equipment in the laboratory which produced turbulence by passing magnetized fluid to a mesh under an additional magnetic induction. In the area of astronomical physics, Matthaeus et al. measured the magnetic energy spectrum of the solar wind, while Leamon et al. measured the MHD turbulence within the coronal mass ejection. Both of their results suggested the Kolmogorov power law for the energy spectrum. On the theoretical side, Hatori obtained the Kolmogorov spectrum for the three-dimensional MHD turbulence, but suggested that the Kolmogorov constant is universal.

The purpose of the present study is providing a recursive renormalization group analysis for MHD turbulence in three dimensions with the specific points of interest as follows. We will obtain the energy spectra for both of the velocity and magnetic induction fields, look for the invariant effective eddy viscosity.

Let us give a brief description of the present work. MHD is governed by a coupling set of equations, meanwhile, the MHD turbulence considered is further assumed to be isotropic, homogeneous and stationary. It is found convenient to introduce the Elsasser variables to write the equations for the velocity and magnetic induction fields in a symmetric form. A recursive RG analysis is carried out for the MHD equations in the wavenumber domain and a recursive relationship for the effective eddy viscosity $n_n(k)$ between two successive steps is established. The resulting expression is complicated enough and is apparently not amenable to further RG analysis. Instead, we restrict ourselves to the case when the following conditions hold. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. In spite of these restrictions, the present RG analysis still warrants a sufficient interest as we investigate several observations in the area of astronomical physics. In addition, the energy spectrum of the velocity is determined self-consistently through the use of the RG transformation, i.e. the recursive relation. And, the fixed point of the RG equation is located to give the invariant effective eddy viscosity $n(k)$.

2. RG Procedure for MHD equations

In considering magnetohydrodynamic(MHD) turbulence, the SI units are introduced and we can write down the following MHD equations.

$$\begin{cases} \partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + (\nabla\times\mathbf{B})\times\mathbf{B} + \nu_0\nabla^2\mathbf{v}; \\ \partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) + \tau_0\nabla^2\mathbf{B}, \end{cases}$$

with the solenoidal equations

$$\begin{cases} \nabla\cdot\mathbf{v} = 0; \\ \nabla\cdot\mathbf{B} = 0, \end{cases}$$

where the gravitation is incorporated into p. It is convenient to introduce the Elsässer variables for Eq. (1), defined by

$$\begin{cases} \Phi = \mathbf{v} + \mathbf{B}; \\ \Psi = \mathbf{v} - \mathbf{B}. \end{cases}$$

Eq. (1) can then be transformed to

$$\begin{cases} \partial\Phi/\partial t + (\Psi\cdot\nabla)\Phi = -\nabla p^* + \alpha_0\nabla^2\Phi + \beta_0\nabla^2\Psi; \\ \partial\Psi/\partial t + (\Phi\cdot\nabla)\Psi = -\nabla p^* + \alpha_0\nabla^2\Psi + \beta_0\nabla^2\Phi, \end{cases}$$

where $p^* = p + (\mathbf{B}\cdot\mathbf{B})/2$, and we have set

$$\begin{cases} \alpha_0 = (\nu_0 + \tau_0)/2; \\ \beta_0 = (\nu_0 - \tau_0)/2, \end{cases}$$

It is obvious from the definitions of Φ, Ψ that they are also solenoidal, i.e.

$$\nabla\cdot\Phi = \nabla\cdot\Psi = 0.$$

The basic idea of recursive RG analysis is to divide the wavenumber space $(0, k_0)$, where k_0 is Kolmogorov's scale, to a supergrid region $(0, k_c)$ and a subgrid region (k_c, k_0). The subgrid modes are then removed shell by shell by taking the subgrid average over a spherical shell (k_{n+1}, k_n). At the present stage, the cutoff ratio, defined by $\Lambda = k_{n+1}/k_n$ is maintained a constant, and will be later set to tend to 1 as the differential version of the RG analysis leading to the invariant effective eddy viscosity is sought.

For the renormalization group analysis, first of all, in order to distinguish the supergrid and subgrid modes, we introduce the following notations:

(i) for Φ field,

$$\Phi(\mathbf{k},t) = \begin{cases} \Phi^{<}(\mathbf{k},t) & \text{for } |\mathbf{k}| < k_1; \\ \Phi^{>}(\mathbf{k},t) & \text{for } |\mathbf{k}| > k_1, \end{cases}$$

(ii) for Ψ field,

$$\Psi_\alpha(\mathbf{k},t) = \begin{cases} \Psi^{\lessgtr}_\alpha(\mathbf{k},t) & \text{for } |\mathbf{k}| < k_1; \\ \Psi^{\gtrless}_\alpha(\mathbf{k},t) & \text{for } |\mathbf{k}| > k_1. \end{cases}$$

The momentum equations for the supergrid modes can be written

$$\mathsf{L}_{<}(k,t)\begin{pmatrix} \Phi^{\lessgtr}_\alpha(\mathbf{k},t) \\ \Psi^{\lessgtr}_\alpha(\mathbf{k},t) \end{pmatrix} = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j$$

$$\begin{pmatrix} \left[\Psi^{<}_\beta(\mathbf{j},t)\Phi^{<}_\gamma(\mathbf{k}-\mathbf{j},t) + 2\Psi^{\lessgtr}_\beta(\mathbf{j},t)\Phi^{\gtrless}_\gamma(\mathbf{k}-\mathbf{j},t) + \Psi^{>}_\beta(\mathbf{j},t)\Phi^{\gtrless}_\gamma(\mathbf{k}-\mathbf{j},t)\right] \\ \left[\Phi^{\lessgtr}_\beta(\mathbf{j},t)\Psi^{<}_\gamma(\mathbf{k}-\mathbf{j},t) + 2\Phi^{\lessgtr}_\beta(\mathbf{j},t)\Psi^{\gtrless}_\gamma(\mathbf{k}-\mathbf{j},t) + \Phi^{\gtrless}_\beta(\mathbf{j},t)\Psi^{\gtrless}_\gamma(\mathbf{k}-\mathbf{j},t)\right] \end{pmatrix},$$

and the momentum equation for the subgrid modes equation could be written

$$\mathsf{L}_{>}(j)\begin{pmatrix} \Phi^{\gtrless}_\beta(\mathbf{j},t) \\ \Psi^{\gtrless}_\beta(\mathbf{j},t) \end{pmatrix} = M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'$$

$$\begin{pmatrix} \left[\Psi^{<}_{\beta'}(\mathbf{j}',t)\Phi^{<}_{\gamma'}(\mathbf{j}-\mathbf{j}',t) + 2\Psi^{<}_{\beta'}(\mathbf{j}',t)\Phi^{>}_{\gamma'}(\mathbf{j}-\mathbf{j}',t) + \Psi^{>}_{\beta'}(\mathbf{j}',t)\Phi^{>}_{\gamma'}(\mathbf{j}-\mathbf{j}',t)\right] \\ \left[\Phi^{<}_{\beta'}(\mathbf{j}',t)\Psi^{<}_{\gamma'}(\mathbf{j}-\mathbf{j}',t) + 2\Phi^{<}_{\beta'}(\mathbf{j}',t)\Psi^{>}_{\gamma'}(\mathbf{j}-\mathbf{j}',t) + \Phi^{>}_{\beta'}(\mathbf{j}',t)\Psi^{>}_{\gamma'}(\mathbf{j}-\mathbf{j}',t)\right] \end{pmatrix},$$

and we have assumed that Markovian approximation holds for the subgrid modes. The matrix $\mathsf{L}_{>}(j)$ is defined by

$$\mathsf{L}_{>}(j) = \begin{pmatrix} \alpha_0 & \beta_0 \\ \beta_0 & \alpha_0 \end{pmatrix} j^2.$$

Before substantial progress can be made with the RG analysis, we shall make the following statistical hypotheses.

(i) The MHD fields have ensemble- mean-zero fluctuation,

$$\langle \Phi^{\gtrless}_\alpha(\mathbf{k},t)\rangle = \langle \Psi^{\gtrless}_\alpha(\mathbf{k},t)\rangle = 0.$$

(ii) Supergrid components are considered to be statistically independent of subgrid averaging,

$$\begin{cases} \langle \Phi^{\lessgtr}_\alpha(\mathbf{k},t)\rangle = \Phi^{\lessgtr}_\alpha(\mathbf{k},t); \\ \langle \Psi^{\lessgtr}_\alpha(\mathbf{k},t)\rangle = \Psi^{\lessgtr}_\alpha(\mathbf{k},t). \end{cases}$$

This assumption is simple (but not void) though its validity may be restrictive. But RG theory based on this assumption has not been explored to its full strength. Indeed, the RG results based on this assumption were found to be in remarkably close agreement with computed/measured data.

Under two conditions: (i) the mean magnetic induction is relatively weak compared to the mean flow velocity, and (ii) the Alfvén effect holds. Those two conditions directly imply that

$$Q(\mathbf{j})u^{\lessgtr}_\alpha(\mathbf{k},t) \gg R(\mathbf{j})B^{\lessgtr}_\alpha(\mathbf{k},t)$$

and

$$S(\mathbf{j})u^{\lessgtr}_\alpha(\mathbf{k},t) \gg R(\mathbf{j})B^{\lessgtr}_\alpha(\mathbf{k},t)$$

, and then:

$$\mathsf{L}_1(k,t)\begin{pmatrix} u^{\lessgtr}_\alpha(\mathbf{k},t) \\ B^{\lessgtr}_\alpha(\mathbf{k},t) \end{pmatrix}$$

$$= M_{\alpha\beta\gamma}(\mathbf{k})\int_{\Omega_1} d^3j \begin{pmatrix} u^{\lessgtr}_\beta(\mathbf{j},t)u^{\lessgtr}_\gamma(\mathbf{k}-\mathbf{j},t) - B^{\lessgtr}_\beta(\mathbf{j},t)B^{\lessgtr}_\gamma(\mathbf{k}-\mathbf{j},t) \\ B^{\lessgtr}_\beta(\mathbf{j},t)u^{\lessgtr}_\gamma(\mathbf{k}-\mathbf{j},t) - u^{\lessgtr}_\beta(\mathbf{j},t)B^{\lessgtr}_\gamma(\mathbf{k}-\mathbf{j},t) \end{pmatrix},$$

where

$$\mathsf{L}_1(k,t) = \begin{pmatrix} \partial/\partial t + \nu_1(k)k^2 & 0 \\ 0 & \partial/\partial t + \tau_1(k)k^2 \end{pmatrix}.$$

The effective eddy viscosity and magnetic resistivity after the first-step renormalization is given by $n_1(k)$ and $t_1(k)$ as follows

$$\nu_1(k) = \nu_0 + \delta\nu_0(k),$$

and

$$\tau_1(k) = \tau_0 + \delta\tau_0(k),$$

where

$$\delta\nu_0(k) = 2\int_{\Omega_0} d^3j \frac{L(\mathbf{k},\mathbf{k}-\mathbf{j})}{k^2}\left(\frac{Q(\mathbf{j})}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2} + \frac{S(\mathbf{j})}{\tau_0 j^2 + \tau_0|\mathbf{k}-\mathbf{j}|^2}\right),$$

and

$$\delta\tau_0(k) = 2\int_{\Omega_0} d^3j \frac{L(\mathbf{k},\mathbf{k}-\mathbf{j})}{k^2}\left[\frac{Q(\mathbf{j})-S(\mathbf{j})}{\tau_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2} - \frac{Q(\mathbf{j})-S(\mathbf{j})}{\nu_0 j^2 + \tau_0|\mathbf{k}-\mathbf{j}|^2}\right].$$

For MHD turbulence, we shall consider two kinds of energy contribution with wavenumber vectors lying within the spherical shell between k and $k+dk$:

$$\begin{cases} E^v(k)dk = 4\pi k^2 Q(\mathbf{k})dk \\ E^M(k)dk = 4\pi k^2 S(\mathbf{k})dk \end{cases}.$$

With all the exponents determined, the kinetic energy spectrum and the magnetic energy spectrum take respectively the following expressions:

$$\begin{cases} E_n^v(j) = C_K \varepsilon_v^{2/3} j^{-5/3} \phi_n(j/k_p); \\ E_n^M(j) = C_M \varepsilon_M^{2/3} j^{-5/3} \varphi_n(j/k_p), \end{cases}$$

and the effective eddy viscosity and the magnetic resistivity take respectively the expressions:

$$\begin{cases} \nu_n(j) = C_K^{1/2} \varepsilon_v^{1/3} j^{-4/3} \hat{\nu}_n(j/k_p); \\ \tau_n(j) = C_M^{1/2} \varepsilon_M^{1/3} j^{-4/3} \hat{\tau}_n(j/k_p), \end{cases}$$

Eq. (9) shows that the energy spectrum have the dependence of the power law of $j^{-5/3}$ which is exactly the Kolmogorov energy spectrum. Compared with laboratory experiments, the result is consistent with Alemany's measurement in passing a magnetized fluid to a grid mesh. Part of their experimental results provided a energy spectrum of the type:

$$E^v(k,t) \approx \frac{\varepsilon_v^{2/3} k^{-5/3}}{[1+N(t)]^{2/3}}.$$

Except the time dependence, our RG result is in good agreement with their experimental results. There are some other evidences from observations in astronomical physics that also support this Kolmogorov spectrum law. Matthaeus et al. discovered that the magnetic energy spectrum measured in the solar wind is often found to be close to $j^{-5/3}$. Velli et al. investigated a new phenomenology which involves the solar wind fluctuations near the sun and leads to a kinetic power spectrum scaling as k^{-a} where $a \approx 1$ for the largest scales, and $a \approx 1.5-1.7$ for the small scales. Moreover, the recent observations by Leamon et al. (the January 1997 event which involves the solar coronal mass ejections), also showed a power law, scaled as $k^{-1.67}$.

3. Concluding remarks

In this study, we have extended our previous RG analysis of incompressible to MHD turbulence. The Elsasser variables are introduced to write the MHD equations for the velocity and magnetic induction fields in a symmetric form. RG analysis is then performed in the wavenumber domain. Taking subgrid averaging of the equation governing the supergrid modes yields a renormalizable form of the MHD equations. To proceed further with the RG transformation, we have to the following two assumptions. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. That these conditions still warrant sufficient interest are illustrated by some available data from observations in astronomical physics. Under these conditions, renormalization does not incur an increment of the magnetic resistivity t, while the coupling effect tends to reduce the invariant effective eddy viscosity $n(k)$. Both the velocity and magnetic energy spectra are shown to follow the Kolmogorov $k^{-5/3}$ in the inertial subrange; this is consistent with some available laboratory measurements and observations in astronomical physics.

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