Slides

he Me hod of Ins rumen al ariables for Generalized Linear Models

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Abstract

Th m thod of *in rumen al variable* (IVs) has b n d v lop d historically and us d xt nsiv ly in conom trics for stimating th structural param t rs in statistical mod ls wh r som r gr ssors ar corr lat d with th mod l's rror. Thus, as discuss d by Bowd n and Turkington (1984, pp. 3-10), th IV stimation m thod has four important applications: (1) th rror-in-variabl probl m, (2) th s lf-s l ction probl m, (3) th simultan ous quations mod l, and (4) th tim s ri s probl m. In this study, w g n raliz th curr nt IV m thod for lin ar and nonlin ar quations to allow th structural quations to b in a *generalized linear model* (GL) form so that th r spons variabl s of thos quations can b diff r nt kinds of m asur m nts such as continuous, binary, and count. How v r, wh n th structural quation is of a GL form, som additional difficulti s aris for th IV m thod b caus th *link func ion* of a GL mak s th m an of th r spons variabl nonlin ar in r gr ssion co ffici nts and th *variance func ion* of th r spons variabl in a GL usually d p nds on its m an. Thus, in g n ral, th IV stimator for a GL quation d riv d by minimizing a quadratic form lik th on d fin d for a lin ar or nonlin ar quation (s , .g., Am miya 198 , p. 246, Eq. (8.1.2)) would b inconsist nt. How v r, by applying th th ory of stimating functions (Godamb and Kal 1991, c. 1. , pp. 9-10, D smond 1991, p. 140, and cCullagh 1991, c. 11.3, sp., Eq. (3.2), pp. 269-271), w d v lop a g n ral IV m thodology for GL s, which has an applicability sp cially in th m asur m nt rror probl m and th simultan ous- quations bias of GL s. Th statistical prop rti s of our IV stimator ar xamin d. Th simulation studi s show promising r sults wh n it is appli d to stimat th structural param t rs in our g n raliz d factor analysis and g n raliz d path analysis (or g n raliz d simultan ous quations mod l).

Keywords:

Error-in-variabl probl m, asur m nt rror probl m, Factor analysis, imultan ous quations mod l, tructural quation mod l, Nonlin ar mod l, Discr t r spons .

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1 Review

1.1 Generali ed inear Models (G Ms)

. . Specification

L t i ind x obs rvations, $i = 1, 2,$

2. Logistic Reg ession Model:

. .3 Estimation

As shown by Dobson (1990, Chap. 4 and App ndix B, pp. 36-46 and 14 -146) and cCullagh and N ld r (1989, c. 2. , pp. 40-43), th *maximum likeli ood e ima or* (LEs) of th unknown r gr ssion co ffici nts β in a GL can b obtain d by a unifi d *i era ive reweig ed lea quare* (IRL) algorithm.

. .4 Infe ence

As discuss d in Dobson (1990, Chap. , pp. 49-66), th k y to th statistical inf r nc for GL s is th sampling distribution of th scor v ctor $U(\beta)$, which is asymptotically multivariat normal bas d on th *cen ral limi eorem* (CLT). Th n, th inf r nc on th r gr ssion co ffici nts β is bas d on th first-ord r Taylor xpansion of $U(\beta)$ for $\widehat{\beta}$.

1.2 Instrumental Variables (IVs)

.2. Definition

Th m thod of *in rumen al variable* (IVs) has b n d v lop d historically and us d xt nsiv ly in conom trics for stimating th structural param t rs in statistical mod- ls wh r th r gr ssor is corr lat d with th rror t rm. For asy computation, l t th "conv rg nc in probability" (i. ., $\xrightarrow{p}$) b d not d by "plim" h r inaft r.

1. he IV Estimators for a inear Equation:

A *linear* quation is sp cifi d as

$$Y = X\beta +$$

wh r $\mathbf{Y}$ is an $n \times 1$ v ctor of r spons variabl s, is an $n \times p$ matrix of covariat s, β is a $p \times 1$ v ctor of param t rs, and is an $n \times 1$ v ctor of rrors. uppos that

$$\text{plim}\left(\frac{1}{n}X^{\mathrm{T}}\right) = 0$$

so that th

2. he IV Estimators for a Nonlinear Equation:

N xt, a *nonlinear* quation can b sp cifi d as

$$Y = f(X;\beta) + \ .$$

On way to g n raliz th IV stimators from lin ar quations to nonlin ar on s is to r plac $(Y - X\beta)$ by $\left(Y - f(X;\beta)\right)$ in th minimization of th corr sponding quadratic form. Thus, giv n an instrum nt $\mathbf{Z}$ for , a consist nt IV stimator $\widetilde{\beta}_{IV}$ is th valu of β that minimiz s th following quadratic form

$$S_2(\beta \mid W) \ =$$

.2.2 Applications

As discuss d by Bowd n and Turkington (1984, pp. 3-10), th IV stimation m thod has four important applications:

. **The e o -in-va iable p oblem.** √

2. **The self-selection p oblem.**

3. **The simultaneous equations model.** √

4. **The time se ies p oblem.**

W ar particularly int r st d in th first and third on s.

2 roblem: IV for GLMs

In this study, w g n raliz th curr nt IV m thod for lin ar and nonlin ar quations to allow th structural quations to b in a *generalized linear model* (GL) form so that th r spons variabl s of such quations can b various kinds of m asur m nts such as continuous, binary, or counts.

2.1 Problem

How v r, wh n th structural quation is of a GL form, som additional difficulti s aris for th IV m thod b caus

1. th *link func ion* of a GL mak s it ”nonlin ar” and
2. th *variance func ion* of th r spons variabl in a GL usually d p nds on its m an.

Thus, in g n ral, th IV stimator for a GL quation d riv d by minimizing a quadratic form lik th on d fin d for a lin ar or nonlin ar quation (s , .g., Am miya 198 , p. 246, Eq. (8.1.2)) would b inconsist nt.

2.2 Solution

Y t, by applying

1. th *minimiza ion approac* of th IV m thod and
2. th th ory of *e ima ing func ion* for GL s (Godamb and Kal 1991, c. 1. , pp. 9-10 and cCullagh 1991, p. 269, Eq. (3.2)),

w d v lop a g n ral IV for GL s toward a g n ral IV m thodology, which has an applicability sp cially in th rror-in-variabl probl m and th simultan ous quations of GL s.

3 stimation: The IRIV Algorithm

By modifying th it rativ r w ight d l ast squar s (IRL) algorithm, w d v lop th *i era ive reweig ed in rumen al variable* (IRIV) algorithm for GL s.

3.1

Th n, th *core func ion* is

$$U(\beta) \equiv X^{\mathrm{T}} g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} \left[A^{\mathrm{T}} (y - \mu(\beta)) \right].$$

Notic that th "-1" can not b dir ctly appli d to th matrix s A^{T}, V, and A insid th "( )" sinc A is not a squar matrix.

It can b shown that th m an of U is

$$E(U) = 0$$

and th varianc of U, call d th *informa ion*, is

$$Var(U) \equiv E(U^2) = -E(U')$$

wh r

$$U' \equiv \frac{\partial U}{\partial \beta}$$

and th l m nts of th *informa ion ma rix I* ar

$$I \equiv Var(U(\beta)) = X^{\mathrm{T}} g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} A^{\mathrm{T}} g'(\mu)^{-1} X$$

On th oth r hand,

$$\begin{aligned} -E(U') &= -E\left(X^{\mathrm{T}} g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} A^{\mathrm{T}} \frac{\partial}{\partial \beta} \left[A^{\mathrm{T}} (y - \mu(\beta)) \right] \right) \\ &\quad - E\left(\frac{\partial}{\partial \beta} \left[X^{\mathrm{T}} g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} \right] \left[A^{\mathrm{T}} (y - \mu(\beta)) \right] \right) \\ &= X^{\mathrm{T}} g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} A^{\mathrm{T}} \frac{\partial \mu}{\partial \beta} \\ &= X^{\mathrm{T}} g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} A^{\mathrm{T}} g'(\mu)^{-1} X \\ &= X^{\mathrm{T}} W^{\star} X \end{aligned}$$

wh r

$$W^{\star} = g'(\mu)^{-1} A (A^{\mathrm{T}} V A)^{-1} A^{\mathrm{T}} g'(\mu)^{-1}.$$

Thus, w hav shown

$$I \equiv Var(U(\beta)) = -E(U')$$

If th *New on-Rap on me od* is us d to obtain th LEs $\widehat{\beta}$ of β, th n at th *m*th it ration,

$$\widehat{\beta}^{(m)} = \widehat{\beta}^{(m-1)} - \Big[\partial^2$$

Th n,

$$I^{(m-1)}\widehat{\beta}^{(m-1)} + U^{(m-1)} = X^{\mathrm{T}} W^{\star(m-1)} Z^{(m-1)}.$$

Finally, assuming that β and μ

3.2 Remarks

W find that "th IRL algorithm can b us d to lin ariz GL s" and sugg st to us it as a tool in d v loping th stimation m thod for mor complicat d statistical mod ls with r spons s of a mix d typ . Thr r marks ar mad in this r gard.

1. At ach it ration ind x d by (m), th IRL algorithm tak s a sp cial transformation on th original r spons variabl Y_i to obtain a *p eudo*-r spons variabl Z_i, no matt r what typ of Y_i is, to modify th prop rty of th original r spons variabl such that th original GL

   $$g(\mu_i) = \mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta}$$

   b com s a lin ar r gr ssion mod l

   $$E\left[Z_i^{(m)}\right] = \mathbf{x}_i^{\mathrm{T}}\boldsymbol{\beta}^{(m)}$$

   until th conv rg nc of $\widehat{\boldsymbol{\beta}}$. In th lin ar r gr ssion cas , th IRL algorithm r duc s to th usual *weig ed lea quare* (WL) proc dur .

2. Th n, inst ad of writing sp cific computing programs for a vari ty of GL s according to th ir lik lihood functions to obtain LEs, on can just apply on singl *unified* IRL algorithm for all GL s. , for xampl , th `PROC ENMOD` in th *SAS for indow* and th `gl` function in th *S-PLUS for indow* .

3. ost importantly, th d riv d lin ar r gr ssion mod l for th ps udo-r spons variabl Z_i at th *la* it ration of th IRL algorithm provid s an " quival nt" lin ar r gr ssion mod l for th original GL in th s ns that th y hav th sam valu s of th r gr ssion co ffici nts.

Our IRIV algorithm shar s th sam prop rti s as th IRL algorithm for GL s.

4 xample: The GFA Model

Th statistical prop rti s of our IV stimator ar xamin d in th following simulation

4.2 Specification

A GFA mod l is of th following form

$$
\begin{aligned}
g_{X_1}(\mu_{X_1}) &= \lambda_{11}F_1 + \lambda_{12}F_2 + \cdots + \lambda_{1m}F_m \\
g_{X_2}(\mu_{X_2}) &= \lambda_{21}F_1 + \lambda_{22}F
\end{aligned}
$$

4.3 Estimation

W introduc th E -CFA, IV1, and IV2 m thods for stimating th factor loadings of a GFA mod l in th following simulation, among which th IV2 stimator is obtain d through th IRIV algorithm.

. **The EM-CFA Estimato**

2. **The Inst umental a iable # (I) Estimato :**

   Notic that this m thod is th sam as that of Carroll and t fanski (1994). Carroll and t fanski (1994) and Carroll, Rupp rt, and t fanski (199) for th t chnical d tails.

3. The Inst umental a iable #2 (I 2) Estimato :

- **Step 0:** Fit th GL for th r spons variabl Y_2 on th covariat Y_1 using th IRL algorithm to obtain th initial stimat of th factor loadings λ_2 which is th GL 's slop co ffici nt, and fit th GL for th r spons variabl Y_3 on th covariat Y_1 using th IRL algorithm to obtain th initial stimat of th factor loadings λ_3 which is th GL 's slop co ffici nt, which ar th naiv stimat s of $\Lambda_{\boldsymbol{Y}} = [1, \lambda_2, \lambda_3]^{\mathrm{T}}$.
- **Step :** Obtain th pr dict d valu s of $Y_{1|2}$ and $Y_{1|3}$, $\widehat{Y}_{1|2}$ and $\widehat{Y}_{1|3}$, as th IV using th ordinary lin ar mod l (L).
- **Step 2:** Tr at th ps udo-r spons variabl s Z_i's d riv d from th indicator variabl s Y_i's in th IRL algorithm of GL as obs rv d continuous indicator variabl s to obtain th updat d stimat s of th factor loadings $\Lambda_{\boldsymbol{Y}} = [1, \lambda_2, \lambda_3]^{\mathrm{T}}$ using th IRL algorithm through th following formula, and it rat Z_i and $\Lambda_{\boldsymbol{Y}}$ until $\Lambda_{\boldsymbol{Y}}$ conv rg :

$$\begin{aligned}
\Lambda_{\boldsymbol{Y}_2} = [\lambda_{20}, \lambda_2]^{\mathrm{T}} &= \left[Y_1^{\mathrm{T}} g'(\mu)^{-1} \widehat{Y}_{1|3} (\widehat{Y}_{1|3}^{\mathrm{T}} V \widehat{Y}_{1|3})^{-1} \widehat{Y}_{1|3}^{\mathrm{T}} g'(\mu)^{-1} Y_1\right]^{-1} \times \\
&\quad Y_1^{\mathrm{T}} g'(\mu)^{-1} \widehat{Y}_{1|3} (\widehat{Y}_{1|3}^{\mathrm{T}} V \widehat{Y}_{1|3})^{-1} \widehat{Y}_{1|3}^{\mathrm{T}} g'(\mu)^{-1} Z_2, \text{ wh r } V = Var(Y_2) \\
\Lambda_{\boldsymbol{Y}_3} = [\lambda_{30}, \lambda_3]^{\mathrm{T}} &= \left[Y_1^{\mathrm{T}} g'(\mu)^{-1} \widehat{Y}_{1|2} (\widehat{Y}_{1|2}^{\mathrm{T}} V \widehat{Y}_{1|2})^{-1} \widehat{Y}_{1|2}^{\mathrm{T}} g'(\mu)^{-1} Y_1\right]^{-1} \times \\
&\quad Y_1^{\mathrm{T}} g'(\mu)^{-1} \widehat{Y}_{1|2} (\widehat{Y}_{1|2}^{\mathrm{T}} V \widehat{Y}_{1|2})^{-1} \widehat{Y}_{1|2}^{\mathrm{T}} g'(\mu)^{-1} Z_3, \text{ wh r } V = Var(Y_3).
\end{aligned}$$

4. The Naive Estimato :

For th purpos of comparison, w also comput th *naive* stimat s of $\widetilde{\lambda}_2$ and $\widetilde{\lambda}_3$ by dir ctly fitting th GL s with th proxy of th lat nt variabl F to th data, which ar also tak n as th initial valu s of th IV2 m thod.

4.4 Simulation

4.4. Setup

R call that th pr viously sp cifi d on -factor thr -indicator GFA mod l is

$$\begin{aligned} Y_1 &= \alpha_1 + 1 \cdot F + \delta_{Y_1} \\ \text{logit}(\mu_2) &= \alpha_2 + \lambda_2 F \\ \log(\mu_3) &= \alpha_3 + \lambda_3 F \end{aligned}$$

wh r μ_2 and μ_3 ar th *mean* of th indicator variabl s Y_2 and Y_3, th commonly us d ”logit” and ”log” *link* functions ar chos n for μ_{2i} and μ_{3i}, δ_{Y_1} is th m asur m nt rror of th indicator variabl Y_1, and th lat nt variabl F is a continuous variabl m asur d by thr indicator variabl s Y_1, Y_2, and Y_3 of a mix d typ — *con inuou* , *binary*, and *coun* — r sp ctiv ly.

In th following simulations, th sampl siz (n) is 1000 and th numb r of r p tition (m) is 00. W follow th following st ps to g n rat th simulat d data.

1. Firstly, w s t th tru param t r valu s.

   (a) For simplicity, w s t all th int rc pts t

4.4.2 Result

Th simulation r sults ar list d in **Table 4**. W find th following int r sting r sults.

1. Th naiv stimator always s riously und r stimat s th tru valu s of (λ_2, λ_3).

2. G n ral sp aking, th E -CFA, IV1, and IV2 stimators p rform w ll for an ”id ntity” or a ”log” link function.

3. For logistic link,

   (a) E -CFA stimator p rforms w ll xc pt wh n th tru valu s of $\lambda_2 = 1.25$:

   $$|\widehat{\lambda}_2 - \lambda_2| = |1.4022 - 1.25| = 0.1522.$$

   (b) IV1 stimator p rforms w ll xc pt wh n th tru valu s of $\lambda_2 = 1.0$ and 1.25:

   - $\lambda_2 = 1.0$: $|1.2453 - 1.0| = 0.2453$
   - $\lambda_2 = 1.25$: $|2.2607 - 1.25| = 1.0107$

   (c) IV2 stimator p rforms w ll xc pt wh n th tru valu s of $\lambda_2 = 0.25$:

   $$|\widehat{\lambda}_2 - \lambda_2| = |0.1882 - 0.25| = 0.0618.$$

   Among th m, E -CFA stimator always has a larg r varianc .

4. Wh n th valu s of λ_2 and λ_3 ar diff r nt such as $(0.5, 1.0)$ and $(1.0, 0.5)$, th p rformanc of IV1 has a larg r bias.

. In all th s valu s of (λ_2, λ_3), IV2 always has l ss bias and small r varianc xc pt wh n $(\lambda_2, \lambda_3) = (0.25, 0.25)$,

   - $\lambda_2 = 0.25$: $|0.1882 - 0.25| = 0.0618$
   - $\lambda_3 = 0.25$: $|0.1830 - 0.25| = 0.067$

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6 Tables

Table 4: The Sim lation Res lts for a One-Factor Three-Indicator GFA Model

$Var(\delta_{Y_1})$ = 0.25.   ampl siz (n) = 1000.   Numb r of r p titions (m) = 500.

Estimato s		λ_2	λ_3
T ue alue		0.2	0.2
EM-CFA	an	0.2 092649693187	0.2 077280393814
	D	(0.271889178046 1)	(0.2 77981 4 1420)
I	an	0.23917803 9980	0.2417 813023387
	D	(0.2 016339007073)	(0.2639 946298 6)
I 2	an	0.18820484324 42	0.182964732324 0
	D	(0.189622230030 8)	(0.18430 1690413)
Naive	an	0.2038 240382896	0.19896037286348
	D	(0.0 616941128867)	(0.028291328 4139)
T ue alue		0.	0.
EM-CFA	an	0. 062624 6399002	0. 27893368 0906
	D	(0.169496397921747)	(0.164492467426 08)
I	an	0.47994800902810	0.49864687709039
	D	(0.143898300631222)	(0.14016649068670)
I 2	an	0.491471 72991417	0. 2893813710281
	D	(0.1469 3098636 2)	(0.1402797363997)
Naive	an	0.394203707107843	0.39879624899906
	D	(0.06049646 011727)	(0.02688417 697182)
T ue alue		0.7	0.7
EM-CFA	an	0.77080 776916108	0.803 664 3 4 486
	D	(0.148 48 12 32787)	(0.112 0796094448)
I	an	0.7 992332797742	0.7 170694969821
	D	(0.12 16 60084339)	(0.1013 1033826447)
I 2	an	0.741 2313269 77	0.7 20347218 432
	D	(0.122236971276649)	(0.1014 3 948 326)
Naive	an	0. 840 3624312387	0. 98897441281704
	D	(0.061932729997401)	(0.02890 63024 148)
T ue alue		1.0	1.0
EM-CFA	an	1.06302978 40374	1.03714873348 6
	D	(0.18 30767078 744)	(0.090 2921306 184)
I	an	1.24 2880 940761	1.0088181997 644
	D	(0.23 194 141 4233)	(0.094960013209768)
I 2	an	0.977987102 82031	1.01 3413 67 28
	D	(0.139790973079 0)	(0.08842 6048 614)
Naive	an	0.76 614819012 9	0.799832361822084
	D	(0.071004841008492)	(0.033 06419 148907)

Estimato s		λ_2	λ_3
T ue alue		1.2	1.2
EM-CFA	an	1.40220449 09103	1.16619704621377
	D	(0.222178890668 4)	(0.14 73630432616)
I	an	2.260721 2841269	1.2491914 13 0 7
	D	(0. 9010017172323)	(0.0936 18799 617)
I 2	an	1.2399 9 06800 7	1.27412203789911
	D	(0.183196647808 1)	(0.13824740378273)
Naive	an	0.93681694772132	0.9898706 901224
	D	(0.070 0462839213)	(0.04 1823 18 069)
T ue alue		0.	1.0
EM-CFA	an	0. 1 016 018093	1.0183028 9130 2
	D	(0.12144 617446 2)	(0.1272447814200)
I	an	0. 7788827699316	1.00381246264769
	D	(0.160 6702818 23)	(0.16801078221817)
I 2	an	0.499 039773 1 2	1.06309367337369
	D	(0.111 8406617320)	(0.26484241789319)
Naive	an	0.39673 43 6 992	0.7999871 2 9196
	D	(0.06377698891446)	(0.0348113 326788)
T ue alue		1.0	0.
EM-CFA	an	1.02781 23803823	0. 2 066319497778
	D	(0.232 44 8669213)	(0.0894179464868 79)
I	an	0.881 8911810 2	0. 02297498691286
	D		