Intro. to Stochastic Processes

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Outline

- ❑ Stochastic Process
- ❑ Counting Process
- ❑ Poisson Process
- ❑ Markov Process

Stochastic Process

- ❑ A stochastic process $\underline{N}= \{N(t), t \in T\}$ is a collection of r.v., i.e., for each t in the index set T, N(t) is a random variable
  - t: time
  - N(t): state at time t
  - If T is a countable set, $\underline{N}$ is a discrete-time stochastic process
  - If T is continuous, $\underline{N}$ is a continuous-time stoc. proc.

Counting Process

- ❑ A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if N(t) is the total number of events that occurred up to time t. Hence, some properties of a counting process is
  - $N(t) \geq 0$
  - N(t) is integer valued
  - If $s < t$, $N(t) \geq N(s)$
  - For $s < t$, $N(t) - N(s)$ equals number of events occurring in the interval (s, t]

Counting Process

- Independent increments
  - If the number of events that occur in disjoint time intervals are independent
- Stationary increments
  - If the dist. of number of events that occur in any interval of time depends only on the length of time interval

Poisson Process

- Def. A: the counting process {N(t), t≥0} is said to be Poisson process having rate λ, λ>0 if
  - N(0) = 0;
  - The process has independent-increments
  - Number of events in any interval of length t is Poisson dist. with mean λt, that is for all s, t ≥0.

$$P[N(t+s)-N(s)=n]=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$n = 0,1,2,\ldots$$

Poisson Process

- Def. B: The counting process {N(t), t ≥0} is said to be a Poisson process with rate λ, λ>0, if
  - N(0) = 0
  - The process has stationary and independent increments
  - P[N(h) = 1] = λh +o(h)
  - P[N(h) ≥ 2] = o(h)
  - The func. f is said to be o(h) if $\lim_{h\to 0}\frac{f(h)}{h}=0$
  - Def A ⇔ Def B, i.e,. they are equivalent.
  - We show Def B ⇒ Def A
  - Def A ⇒ Def B is HW

Important Properties

- Property 1: mean number of event for any t ≥0, E[N(t)]=λt.
- Property 2: the inter-arrival time dist. of a Poisson process with rate λ is an exponential dist. with parameter λ.
- Property 3: the superposition of two independent Poisson process with rate λ1 and λ2 is a Poisson process with rate λ1+λ2

Properties (cont.)

- Property 4: if we perform Bernoulli trials to make independent random erasures from a Poisson process, the remaining arrivals also form a Poisson process
- Property 5: the time until rth arrival , i.e., τ_r is known as the rth order waiting time, is the sum of r independent experimental values of τ and is described by Erlan pdf.

Markov Process

- $P[X(t_{n+1}) \le X_{n+1} | X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots X(t_1) = x_1] = P[X(t_{n+1}) \le X_{n+1} | X(t_n) = x_n]$
  - Probabilistic future of the process depends only on the current state, not on the history
  - We are mostly concerned with discrete-space Markov process, commonly referred to as Markov chains
  - Discrete-time Markov chains
  - Continuous-time Markov chains

DTMC

- Discrete Time Markov Chain:
  - $P[X_{n+1} = j | X_n = k_n, X_{n-1} = k_{n-1}, \ldots X_0 = k_0] = P[X_{n+1} = j | X_n = k_n]$
- discrete time, discrete space
- A finite-state DTMC if its state space is finite
- A homogeneous DTMC if $P[X_{n+1} = j | X_n = i]$ does not depend on n for all i, j, i.e., $P_{ij} = P[X_{n+1} = j | X_n = i]$, where Pij is one step transition prob.

Definition

- P = [Pij] is the transition matrix

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0j} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1j} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i0} & \cdots & \cdots & p_{ij} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $p_{ij} \ge 0$ and $\sum_j p_{ij} = 1$

- A matrix that satisfies those conditions is called a stochastic matrix
- n-step transition prob.

$p_{ij}^n = P[x_n = j | x_0 = i]$

$i, j, n \ge 0$, p_{ij}^n is the prob. of going from state i to j in n step

Chapman-Kolmogorov Eq.

- Def.

  For all n ≥ 0, m ≥ 0, i , j ∈ I

  $$p_{ij}^{(n+m)} = \sum_{k \in I} p_{ik}^{n} p_{kj}^{m}$$

  in matrix form $P^{n+m} = P^n P^m$ where $P^n = [p_{ij}^n]$

- Proof:

Question

- We have only been dealing with conditional prob. but what we want is to compute the unconditional prob. that the system is in state j at time n, i.e.

  $$\pi_n(j) = p(x_n = j)$$

  So, given the initial dist. of x_0, i.e.,

  $$\pi_0(i) = p(x_0 = i) \text{ and } \sum_{i \in I} \pi_0 = 1$$

  we can get

  $$p[x_n = j] = \sum_{i \in I} p(x_n = j \mid x_0 = i) \pi_0(i)$$

  $$= \sum_{i \in I} p_{ij}^n \pi_0(i)$$

Result 1

- For all n ≥ 1, $\pi_n = \pi_0 P^n$, where $\pi_m = (\pi_m(0), \pi_m(1), \ldots)$ for all m ≥ 0. From the above equ., we deduce that $\pi_{n+1} = \pi_n P$. Assume that $\lim_{n \to \infty} \pi_n(i)$ exists for all i, and refer it as $\pi(i)$. The remaining question is how to compute π

  – Reachable: a state j is reachable from i. If

  $$p_{ij}^n > 0 \text{ for some } n \geq 1$$

  – Communicate: if j is reachable from i and if i is reachable form j, then we say that i and j communicate (i ↔ j)

Result 1 (cont.)

- Irreducible:
  – A M.C. is irreducible if i ↔ j for all i,j ∈I

- Aperiodic:
  – For every state i∈I, define d(i) to be largest common divisor of all integer n, s.t.,

  $$p_{ij}^n > 0 \text{ if } d(i) = 1 \text{ then the state is aperiodic}$$

Result 2

- Invariant measure of a M.C., if a M.C. with transition matrix P is irreducible and aperiodic and if the system of equation $\pi=\pi P$ and $\pi 1=1$ has a strict positive solution then $\pi(i) = \lim_{n\to\infty} \pi_n(i)$ independently of initial dist.
  - Invariant equ. : $\pi=\pi P$
  - Invariant measure π

CTMC

- Continuous-time Markov Chain
  - Continuous time, discrete state
  - $P[X(t)= j \mid X(s)=i, X(s_{n-1})= i_{n-1},\ldots X(s_0)= i_0]$
    $= P[X(t)= j \mid X(s)=i]$
  - A continuous M.C. is homogeneous if
    - $P[X(t+u)= j \mid X(s+u)=i] = P[X(t)= j \mid X(s)=i] = P_{ij}[t-s]$, where $t > s$
  - Chapman-Kolmogorov equ.

For all $t > 0$, $s > 0$, $i, j \in I$

$$p_{ij}(t+s) = \sum_{k\in I} p_{ik}(t)\, p_{kj}(s)$$

CTMC (cont.)

- $\pi(t)=\pi(0)e^{Qt}$
  - Q is called the infinitesimal generator
  - Proof:

Result 3

- If a continuous M.C. with infinitesimal generator Q is irreducible and if the system of equations $\pi Q = 0$, and $\pi 1=1$, has a strictly positive solution then $\pi(i)= \lim_{t\to\infty} p(x(t)=i)$ for all $i\in I$, independently of the initial dist.