Bond Price Volatility

Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest-rate-sensitive securities.
- Assume level-coupon bonds throughout.

Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-\frac{\frac{\partial P}{\partial y}}{P}.$$

Price Volatility of Bonds

- The price volatility of a coupon bond is

$$-\frac{(C/y)\,n-\left(C/y^2\right)\left((1+y)^{n+1}-(1+y)\right)-nF}{(C/y)\left((1+y)^{n+1}-(1+y)\right)+F(1+y)},$$

  where F is the par value, and C is the coupon payment per period.
- For bonds without embedded options,

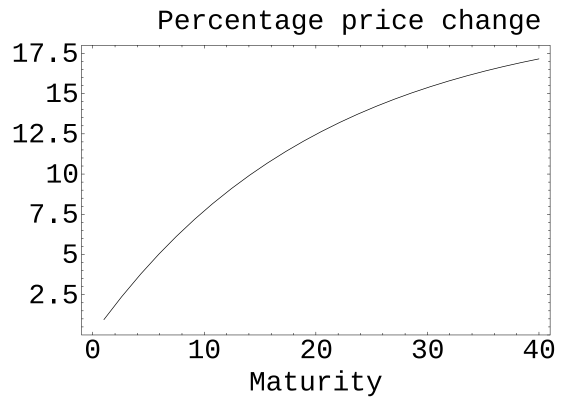
$$-\frac{\frac{\partial P}{\partial y}}{P}>0.$$

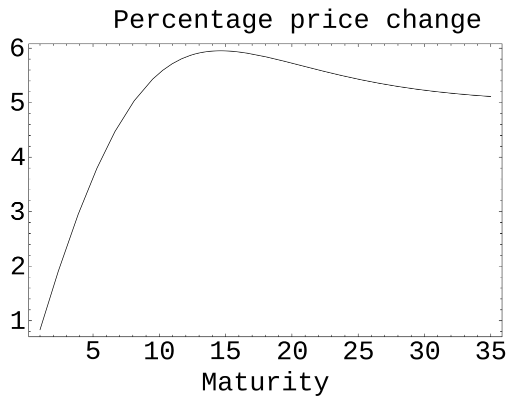
Behavior of Price Volatility (1)

- Price volatility increases as the coupon rate decreases.
  - Zero-coupon bonds are the most volatile.
  - Bonds selling at a deep discount are more volatile than those selling near or above par.
- Price volatility increases as the required yield decreases.
  - So bonds traded with higher yields are less volatile.

Behavior of Price Volatility (2)

- For bonds selling above par or at par, price volatility increases as the term to maturity lengthens (see figure on next page).
  - Bonds with a longer maturity are more volatile.
- For bonds selling below par, price volatility first increases then decreases (see the figure on p. 74).
  - Longer maturity here cannot be equated with higher price volatility.





Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,

$$\mathrm{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i}.$$

- The Macaulay duration, in periods, is equal to

$$\mathrm{MD} = -(1+y)\,\frac{\partial P}{\partial y}\,\frac{1}{P}. \tag{7}$$

MD of Bonds

- The MD of a coupon bond is

$$\mathrm{MD} = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \tag{8}$$

- It can be simplified to

$$\mathrm{MD} = \frac{c(1+y)\,[\,(1+y)^n - 1\,] + ny(y-c)}{cy\,[\,(1+y)^n - 1\,] + y^2},$$

  where c is the period coupon rate.
- The MD of a zero-coupon bond equals its term to maturity n.
- The MD of a coupon bond is less than its maturity.

Finesse

- Equations (7) on p. 75 and (8) on p. 76 hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y.
- That is, if the cash flow is independent of yields.

How Not To Think of MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But you use it that way at your peril.
- The MD should be seen mainly as measuring price volatility.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.

Modified Duration

- Modified duration is defined as

$$\text{modified duration} \equiv -\frac{\partial P}{\partial y}\frac{1}{P} = \frac{\text{MD}}{(1+y)}. \qquad (9)$$

- By Taylor expansion,

$$\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.$$

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Modified Duration of a Portfolio

- The modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

  - D_i is the modified duration of the ith asset.
  - ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}.$$

  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.

Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0 \, \Delta y}.$$

  - More economical but less accurate.

The Practices

- Duration is usually expressed in percentage terms—call it $D_{\%}$—for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by

$$-D_{\%} \times \Delta r$$

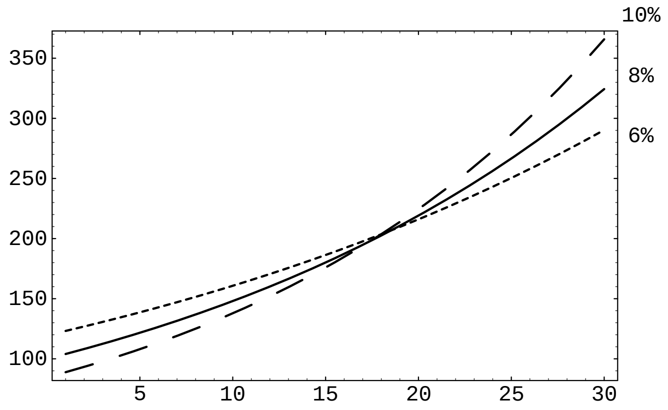
  when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 20% if $D_{\%} = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.
- In fact, $D_{\%}$ equals modified duration as originally defined (prove it!).

Meeting Liabilities

- Buy coupon bonds to meet a future liability.
- What happens at the horizon date when the liability is due?
- Say interest rates rise subsequent to the purchase:
  - The interest on interest from the reinvestment of the coupon payments will increase.
  - But a capital loss will occur for the sale of the bonds.
- The reverse is true if interest rates fall.
- Uncertainties in meeting the liability.

Immunization

- A portfolio immunizes a liability if its value at horizon covers the liability for small rate changes now.
- A bond portfolio whose MD equals the horizon and whose PV equals the PV of the single future liability.
  - At horizon, losses from the interest on interest will be compensated by gains in the sale price when interest rates fall.
  - Losses from the sale price will be compensated by the gains in the interest on interest when interest rates rise (see figure on p. 87).



Example

- Consider a \$100,000 liability 12 years from now.
- It should be matched by a portfolio with an MD of 12 years and a FV of \$100,000.

The Proof

- Assume the liability is L at time m and the current interest rate is y.
- Want a portfolio such that

  (1) Its FV is L at the horizon m;

  (2) $\partial \text{FV}/\partial y = 0$;

  (3) FV is convex around y.
- Condition (1) says the obligation is met.
- Conditions (2) and (3) mean L is the portfolio's minimum FV at horizon for small rate changes.

The Proof (continued)

- Let $\text{FV} \equiv (1+y)^m P$, where P is the PV of the portfolio.
- Now,

$$\frac{\partial \text{FV}}{\partial y} = m(1+y)^{m-1} P + (1+y)^m \frac{\partial P}{\partial y}.$$

- Imposing Condition (2) leads to

$$m = -(1+y)\,\frac{\partial P/P}{\partial y}.$$

- The MD is equal to the horizon m.

The Proof (concluded)

- Employ a coupon bond for immunization.
- Since

$$\text{FV} = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}},$$

it follows that

$$\frac{\partial^2 \text{FV}}{\partial y^2} > 0 \tag{10}$$

for $y > -1$.

- Since FV is convex for $y > -1$, the minimum value of FV is indeed L.

Rebalancing

- Immunization has to be rebalanced constantly to ensure that the MD remains matched to the horizon.
- The MD decreases as time passes.
- But, except for zero-coupon bonds, the decrement is not identical to that in the time to maturity.
  - Consider a coupon bond whose MD matches horizon.
  - Since the bond's maturity date lies beyond the horizon date, its MD will remain positive at horizon.
- So immunization needs to be reestablished even if interest rates never change.

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price } (\% \text{ of par}) = -\frac{\partial P}{\partial y}.$$

- The approximate dollar price change per \$100 of par value is

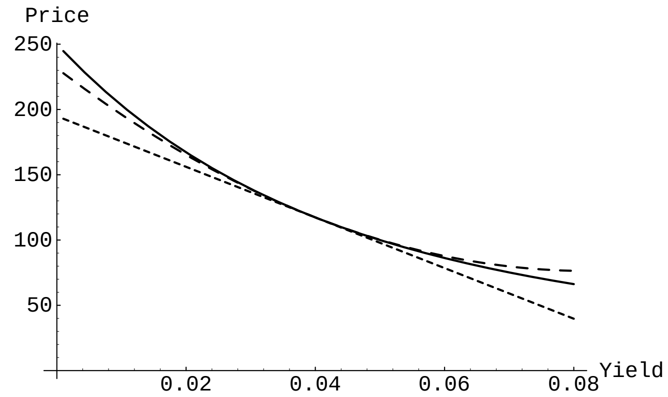
$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

Convexity

- Convexity is defined as

$$\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.$$

- The convexity of a coupon bond is positive (see Eq. (10) on p. 91).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.
- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.



Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by
$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$
when there are k periods per annum.
- The convexity of a coupon bond increases as its coupon rate decreases.
- For a given yield and duration, the convexity decreases as the coupon decreases.

Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use
$$\begin{aligned} \frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2 \\ &= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2. \end{aligned}$$
- Recall the figure on p. 95.

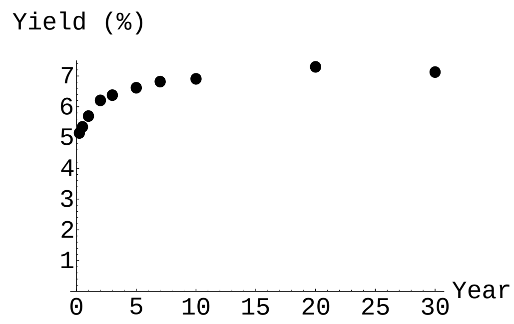
The Practices

- Convexity is usually expressed in percentage terms—call it $C_{\%}$—for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by $-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because
$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$
- In fact, $C_{\%}$ equals convexity divided by 100 (prove it!).

Term Structure of Interest Rates

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.



Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.

Four Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is

$$[\,1 + S(i)\,]^{-i}.$$

- The one-period spot rate is called the short rate.
- A spot rate curve is a plot of spot rates against maturity.

Problems with the PV Formula

- In the bond price formula,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield y.

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams.
- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn't they be discounted at the same rate?
- Enter the spot rate methodology.

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.
- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1+S(i)\,]^i} + \frac{F}{[\,1+S(n)\,]^n}. \tag{11}$$

- This pricing method incorporates information from the term structure.
- Discount each cash flow at the corresponding spot rate.

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  - Above, $d(i) \equiv [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.
  - $d(i)$ is the PV of one dollar i periods from now.
- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[\,1 + S(2)\,]^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price P of the n-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.
- Then $S(n)$ can be computed from Eq. (11), repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}.$$

- The running time is $O(n)$.
- The procedure is called bootstrapping.

Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  - Lack economic justifications.

Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Coupon Effect on the Yield to Maturity

- Under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity.
- Picking a zero-coupon bond over a coupon bond based purely on the zero's higher yield to maturity is flawed.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But only a trend not a mathematical truth.

Shapes (concluded)

- When the final principal payment is relatively insignificant, the spot rate curve and the yield curve do share the same shape.
  - Bonds of high coupon rates and long maturities.
- By the agreement in shape, remember the above proviso.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1+S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j-i$ periods where $j>i$.
- Will have $[\,1+S(i)\,]^i[\,1+S(i,j)\,]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j-i)$-period spot rate i periods from now.
  - The rollover strategy.

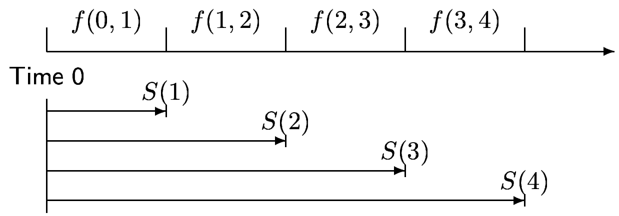
Forward Rates (concluded)

- When $S(i,j)$ equals

$$f(i,j) \equiv \left[\frac{(1+S(j))^j}{(1+S(i))^i}\right]^{1/(j-i)} - 1, \tag{12}$$

  we will end up with $[\,1+S(j)\,]^j$ dollars again.
- By definition, $f(0,j)=S(j)$.
- $f(i,j)$ is called the (implied) forward rates.
  - More precisely, the $(j-i)$-period forward rate i periods from now.

Time Line



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i,j)$ and future spot rate $S(i,j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate two investment strategies.
- $f(i,i+1)$ are instantaneous forward rates or one-period forward rates.

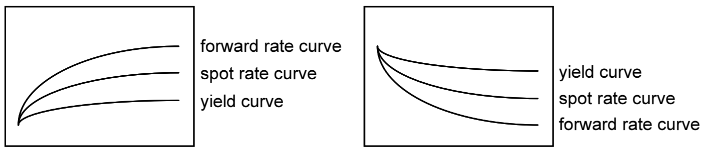
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i,j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i,j) < S(j) < \cdots < S(i).$$



(a) (b)

Forward Rates≡Spot Rates≡Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive $[\,1 + S(n)\,]^n$.
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is $[\,1 + S(1)\,][\,1 + f(1,2)\,] \cdots [\,1 + f(n-1,n)\,]$.

Forward Rates≡Spot Rates≡Yield Curve (concluded)

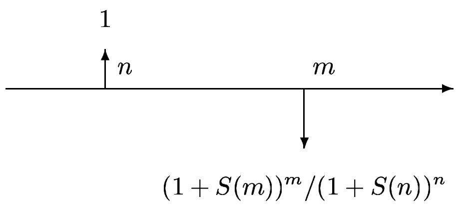
- Since they are identical,

$$\begin{aligned} S(n) = {} & ((1 + S(1))(1 + f(1,2)) \\ & \cdots (1 + f(n-1,n)))^{1/n} - 1. \end{aligned} \tag{13}$$

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalency can be derived similarly.
- Show that $f(T, T+1) = d(T)/d(T+1) - 1$.

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-year zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
- This implies the rate $f(n, m)$ between times n and m.



Forward Contracts

- We generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to borrow money at time n in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.
- Can the spot rate curve be an arbitrary curve?

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

Spot and Forward Rates under Continuous Compounding (concluded)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j-i}.$$

- The one-period forward rate:

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

-

$$f(T) \equiv \lim_{\Delta T \to 0} f(T, T+\Delta T) = S(T) + T\,\frac{\partial S}{\partial T}.$$

- $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\,S(a,b)\,]. \tag{14}$$

- Does not imply that the forward rate is an accurate predictor for the future spot rate.
- Implies that the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

Unbiased Expectations Theory and Spot Rate Curve

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \ \text{ for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.
- Holding premium:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} - (1+S(1)).$$

  - Zero under the local expectations theory.

Duration Revisited

- Let $P(y) \equiv \sum_i C_i/(1+S(i)+y)^i$ be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0} = \frac{\sum_i \frac{iC_i}{(1+S(i))^{i+1}}}{\sum_i \frac{C_i}{(1+S(i))^i}}.$$

  - The curve is shifted in parallel to $S(1)+\Delta y, S(2)+\Delta y, \ldots$ before letting Δy go to zero.
- The percentage price change roughly equals duration times the size of the parallel shift in the spot rate curve.

Duration Revisited (continued)

- The simple linear relation between duration and MD in Eq. (9) on p. 79 breaks down.
- One way to regain it is to resort to a different kind of shift, the proportional shift:

$$\frac{\Delta(1+S(i))}{1+S(i)} = \frac{\Delta(1+S(1))}{1+S(1)}$$

  for all i.
  - $\Delta(x)$ denotes the change in x when the short-term rate is shifted by Δy.

Duration Revisited (concluded)

- Duration now becomes

$$\frac{1}{1+S(1)}\left[\frac{\sum_i \frac{iC_i}{(1+S(i))^i}}{\sum_i \frac{C_i}{(1+S(i))^i}}\right]. \tag{15}$$

- Define Macaulay's second duration to be the number within the brackets in Eq. (15).
- Then

$$\text{duration} = \frac{\text{Macaulay's second duration}}{(1+S(1))}.$$

Immunization Revisited

- Recall that a future liability can be immunized by matching PV and MD under flat spot rate curves.
- If only parallel shifts are allowed, this conclusion continues to hold under general spot rate curves.
- Assume liability L is T periods from now.
- Assume $L = 1$ for simplicity.
- Assume the matching portfolio consists only of zero-coupon bonds maturing at t_1 and t_2 with $t_1 < T < t_2$.

Immunization Revisited (continued)

- Let there be n_i bonds maturing at time t_i, $i = 1, 2$.
- The portfolio's PV is

$$V \equiv n_1 e^{-S(t_1)\, t_1} + n_2 e^{-S(t_2)\, t_2} = e^{-S(T)\, T}.$$

- Its MD is

$$\frac{n_1 t_1 e^{-S(t_1)\, t_1} + n_2 t_2 e^{-S(t_2)\, t_2}}{V} = T.$$

- These two equations imply

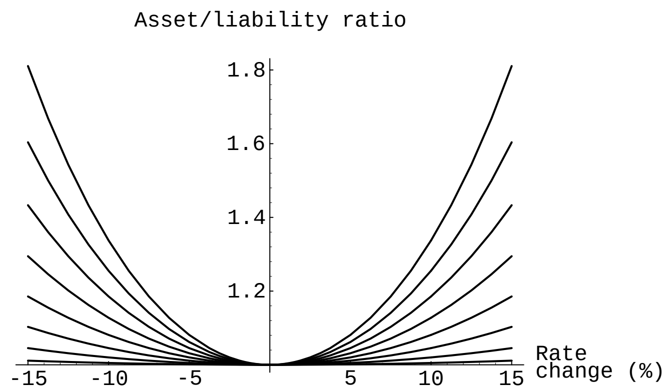
$$n_1 e^{-S(t_1)\, t_1} = \frac{V(t_2 - T)}{t_2 - t_1} \text{ and } n_2 e^{-S(t_2)\, t_2} = \frac{V(t_1 - T)}{t_1 - t_2}.$$

Immunization Revisited (concluded)

- Now shift the spot rate curve uniformly by $\delta \neq 0$.
- The portfolio's PV becomes

$$\begin{aligned} & n_1 e^{-(S(t_1)+\delta)\, t_1} + n_2 e^{-(S(t_2)+\delta)\, t_2} \\ = \;& e^{-\delta t_1} \frac{V(t_2 - T)}{t_2 - t_1} + e^{-\delta t_2} \frac{V(t_1 - T)}{t_1 - t_2} \\ = \;& \frac{V}{t_2 - t_1} \left(e^{-\delta t_1}(t_2 - T) + e^{-\delta t_2}(T - t_1) \right). \end{aligned}$$

- The liability's PV after shift is $e^{-(S(T)+\delta)\, T} = e^{-\delta T} V$.
- And $\frac{V}{t_2 - t_1} \left(e^{-\delta t_1}(t_2 - T) + e^{-\delta t_2}(T - t_1) \right) > e^{-\delta T} V$.



Two Intriguing Implications

- A duration-matched position under parallel shifts implies free lunch as any interest rate change generates profits.
- No investors would hold the T-period bond because a portfolio of t_1- and t_2-period bonds has a higher return for any interest rate shock.
  - They would own only bonds of the shortest and longest maturities.
- The logic seems impeccable.
- What gives?