Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H.
- A knock-out option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

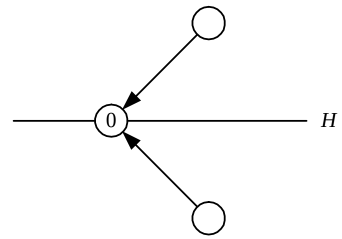
[a] A former student told me on March 26, 2004, she did not understand what I meant by barrier options until she started working in a bank.

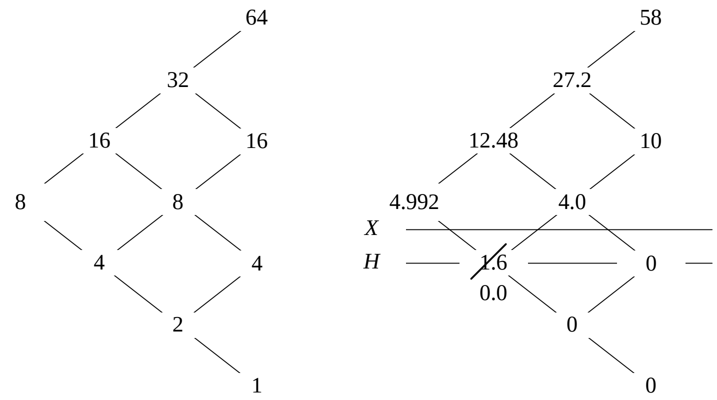
Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all kinds of barrier options.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.

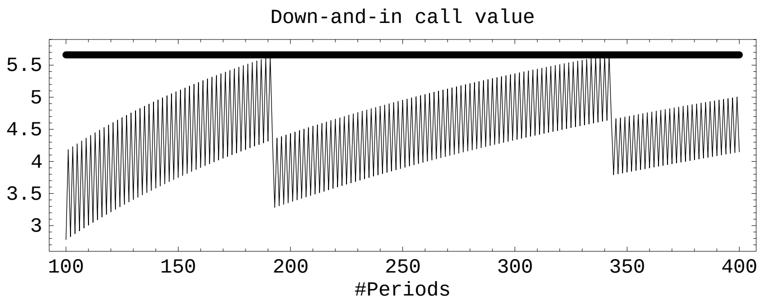




$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.
Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (concluded)

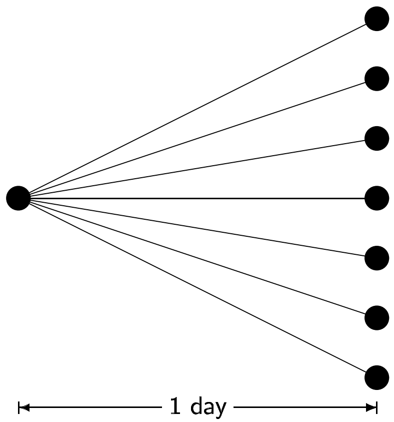
- But convergence is erratic because H is not at a price level on the tree (see plot on next page).
  - Typically, the barrier has to be adjusted to be at a price level.
- Solutions will be presented later.



Daily Monitoring

- Almost all barrier options monitor the barrier only for the daily closing prices.
- In that case, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d + 1$ nodes if each day is partitioned into d periods.
- This saves time and space.

A Heptanomial Tree (6 Periods Per Day)



Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.
- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

Foreign Exchange Options

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases $100{,}000{,}000/6{,}250{,}000 = 16$ puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March 2000.
- This gives the company the right to sell 100,000,000 Japanese yen for $100{,}000{,}000 \times .0088 = 880{,}000$ U.S. dollars.

Foreign Exchange Options (concluded)

- The formulas derived for stock index options in Eqs. (26) on p. 268 apply with the dividend yield equal to $\hat{r}$:

$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{29}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x), \tag{29'}$$

  - where

$$x \equiv \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

Bar the roads!
Bar the paths!
Wert thou to flee from here, wert thou
to find all the roads of the world,
the way thou seekst
the path to that thou'dst find not[.]
— Richard Wagner (1813–1883), *Parsifal*

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- In contrast, some derivatives are path-dependent in that their terminal payoff depends "critically" on the path.
- The (arithmetic) average-rate call has a terminal value given by
$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$
- The average-rate put's terminal value is given by
$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (continued)

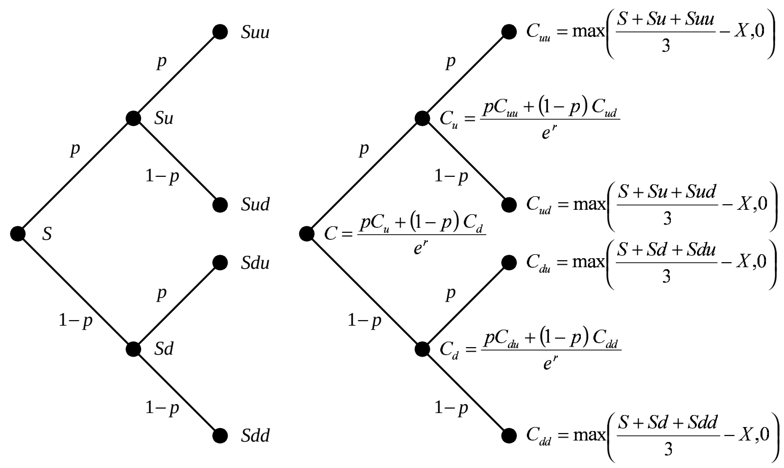
- Average-rate options are also called Asian options.
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.

Path-Dependent Derivatives (concluded)

- A lookback call option on the minimum has a terminal payoff of $S_n - \min_{0 \le i \le n} S_i$.
- A lookback put option on the maximum has a terminal payoff of $\max_{0 \le i \le n} S_i - S_n$.
- The fixed-strike lookback option provides a payoff of $\max(\max_{0 \le i \le n} S_i - X, 0)$ for the call and $\max(X - \min_{0 \le i \le n} S_i, 0)$ for the put.
- Lookback call and put options on the average are called average-strike options.

Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine.
- A straightforward algorithm is to enumerate the 2^n price paths for an n-period binomial tree and then average the payoffs.
- But the exponential complexity makes this naive algorithm impractical.
- As a result, the Monte Carlo method and approximation algorithms are some of the alternatives left.



Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
- Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),$$
$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas.
  - With the volatility set to $\sigma_{\mathrm{a}} \equiv \sigma/\sqrt{3}$.
  - With the dividend yield set to $q_{\mathrm{a}} \equiv (r + q + \sigma^2/6)/2$.
- The formula is therefore

$$C = S e^{-q_{\mathrm{a}} \tau} N(x) - X e^{-r\tau} N(x - \sigma_{\mathrm{a}} \sqrt{\tau}),$$
$$P = X e^{-r\tau} N(-x + \sigma_{\mathrm{a}} \sqrt{\tau}) - S e^{-q_{\mathrm{a}} \tau} N(-x),$$

  - where $x \equiv \frac{\ln(S/X) + \left(r - q_{\mathrm{a}} + \sigma_{\mathrm{a}}^2/2\right)\tau}{\sigma_{\mathrm{a}}\sqrt{\tau}}$.

Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.
- Name such a node $N(j, i)$.
- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

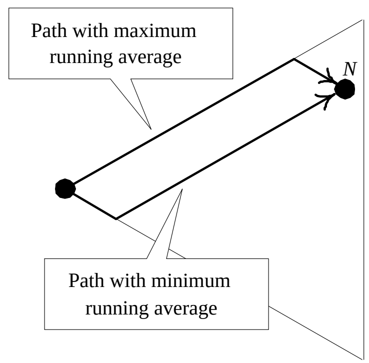
$$S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i} d + \cdots + u^{j-i} d^i}^{j})$$
$$= S_0 \frac{1 - u^{j-i+1}}{1 - u} + S_0 u^{j-i} d \, \frac{1 - d^i}{1 - d}.$$

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j + 1$ and call it $A_{\max}(j, i)$.
- Similarly, the running sum has a minimum value of

$$S_0(1 + \overbrace{d + d^2 + \cdots + d^i + d^i u + \cdots + d^i u^{j-i}}^{j})$$
$$= S_0 \frac{1 - d^{i+1}}{1 - d} + S_0 d^i u \, \frac{1 - u^{j-i}}{1 - u}.$$

- Divide this value by $j + 1$ and call it $A_{\min}(j, i)$.
- $A_{\min}$ and $A_{\max}$ are running averages.



Approximation Algorithm for Asian Options (continued)

- The possible running averages at $N(j,i)$ are far too many: $\binom{j}{i}$.
- But all lie between $A_{\min}(j,i)$ and $A_{\max}(j,i)$.
- Pick $k+1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j,i) \equiv \left(\frac{k-m}{k}\right) A_{\min}(j,i) + \left(\frac{m}{k}\right) A_{\max}(j,i)$$

for $m = 0, 1, \ldots, k$.

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" introduces errors, but it works reasonably well in practice.[a]
- A better alternative is to pick values whose logarithms are equally spaced.
- Still other alternatives are possible.
- Generally, k must scale with at least n to show convergence.[b]

[a] Hull and White (1993).
[b] Dai, Huang, and Lyuu (2002).

Approximation Algorithm for Asian Options (continued)

- Backward induction calculates the option values at each node for the $k+1$ running averages.
- Suppose the current node is $N(j,i)$ and the running average is a.
- Assume the next node is $N(j+1,i)$, after an up move.
- As the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the running average

$$A_{\mathrm{u}} \equiv \frac{(j+1)\,a + S_0 u^{j+1-i} d^i}{j+2}.$$

Approximation Algorithm for Asian Options (continued)

- But A_{u} is not likely to be one of the $k+1$ running averages at $N(j+1,i)$!
- Find the running averages that bracket it, that is,

$$A_\ell(j+1,i) \le A_{\mathrm{u}} \le A_{\ell+1}(j+1,i).$$

- Express A_{u} as a linearly interpolated value of the two running averages,

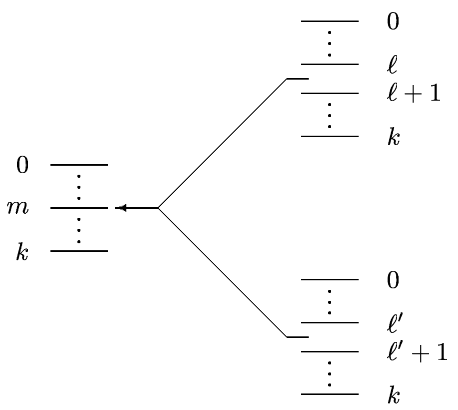
$$A_{\mathrm{u}} = x A_\ell(j+1,i) + (1-x)\,A_{\ell+1}(j+1,i), \quad 0 \le x \le 1.$$

Approximation Algorithm for Asian Options (continued)

- Obtain the approximate option value given the running average $A_{\rm u}$ via

$$C_{\rm u} \equiv x C_\ell(j+1, i) + (1-x)\, C_{\ell+1}(j+1, i).$$

  - $C_\ell(t, s)$ denotes the option value at node $N(t, s)$ with running average $A_\ell(t, s)$.
- This interpolation introduces the second source of error.



Approximation Algorithm for Asian Options (continued)

- The same steps are repeated for the down node $N(j+1, i+1)$ to obtain another approximate option value $C_{\rm d}$.
- Finally obtain the option value as

$$[\, pC_{\rm u} + (1-p)\, C_{\rm d}\,]\, e^{-r\Delta t}.$$

- The running time is $O(kn^2)$.
  - There are $O(n^2)$ nodes.
  - Each node has $O(k)$ buckets.

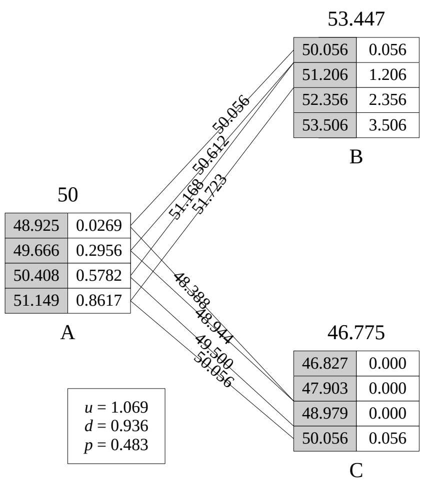
Approximation Algorithm for Asian Options (concluded)

- Arithmetic average-rate options were assumed to be newly issued: There was no historical average to deal with.
- This problem can be easily dealt with (see text).
- How about the Greeks?[a]

[a]Thanks to a lively class discussion on March 31, 2004.

A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.
- Assume zero interest rate in order to dispense with discounting.
- The minimum running average at node A in the figure on p. 350 is 48.925.
- The maximum running average at node A in the same figure is 51.149.



A Numerical Example (continued)

- Each node picks $k = 3$ for 4 equally spaced running averages.
- The same calculations are done for node A's successor nodes B and C.
- Suppose node A is 2 periods from the root node.
- Consider the up move from node A with running average 49.666.

A Numerical Example (continued)

- Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

- With 50.612 lying between 50.056 and 51.206 at node B, we solve

$$50.612 = x \times 50.056 + (1 - x) \times 51.206$$

to obtain $x \approx 0.517$.

A Numerical Example (continued)

- The option values corresponding to running averages 50.056 and 51.206 at node B are 0.056 and 1.206, respectively.
- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

$$x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$$

A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.
- Because the stock price at node C is 46.775, the new running average will be

$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$

- With 48.944 lying between 47.903 and 48.979 at node C, we solve

$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$

to obtain $x \approx 0.033$.

A Numerical Example (concluded)

- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.
- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.
- Finally, the option value corresponding to running average 49.666 at node A equals

$$p \times 0.611 + (1 - p) \times 0.0 \approx 0.2956,$$

where $p = 0.483$.

- The remaining three option values at node A can be computed similarly.

Remarks on Asian Option Pricing

- Asian option pricing is an active research area.
- The above algorithm overestimates the "true" value.[a]
- To guarantee convergence, k needs to grow with n.
- Analytical approximations for European Asian options exist.
- There is a convergent approximation algorithm that does away with interpolation with a provable running time of $2^{O(\sqrt{n})}$.[b]

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[a] Dai, Huang, and Lyuu (2002).
[b] Dai and Lyuu (2002, 2004).

Remarks on Asian Option Pricing (continued)

- There is an $O(kn^2)$-time algorithm with an error bound of $O(Xn/k)$ from the naive $O(2^n)$-time binomial tree algorithm in the case of European Asian options.[a]
  - k can be varied for trade-off between time and accuracy.
  - So if we pick $k = O(n^2)$, then the error is $O(1/n)$, and the running time is $O(n^4)$.
- In practice, log-linear interpolation works better.

[a] Aingworth, Motwani, and Oldham (2000).

Remarks on Asian Option Pricing (concluded)

- Another approximation algorithm reduces the error to $O(X\sqrt{n}/k)$.[a]
  - It varies the number of buckets per node.
  - If we pick k proportional to n, the error is $O(n^{-0.5})$.
  - So if we pick $k = O(n^{1.5})$, then the error is $O(1/n)$, and the running time is $O(n^{3.5})$.
- Under some "reasonable assumptions," Hsu and Lyuu (2004) produce an $O(n^2)$-time algorithm with an error bound of $O(1/n)$.

[a] Dai, Huang, and Lyuu (2002).