Midterm Exam. (Special topics on graph algorithms)

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Unless specified explicitly, a graph G is assumed to be simple and undirected, and the edge weights are nonnegative.

1. (10%) Assume the vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$. Decode the following Prüfer sequences: (a) $P = (1, 2, 3, 4, 5)$, and (b) $P = (1, 1, 3, 5, 7)$.

2. (10%) Let $F_1, F_2, \ldots, F_k$ be a spanning forest of G, and let (u, v) be the smallest of all edges with only one endpoint $u \in V(F_1)$. Prove that there is a minimum spanning tree containing (u, v) among all spanning trees containing all edges in $\cup_{i=1}^{k} E(F_i)$.

3. (10%) Apply the Bellman-Ford algorithm to Figure 1, and show how it detects the negative cycle in the graph.

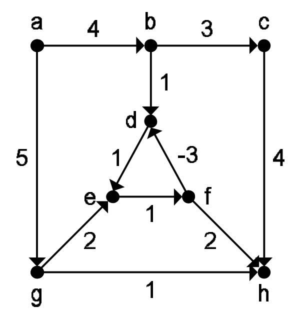

Figure 1: A directed graph with a negative-weight cycle.

4. (15%) (a) What is a minimum routing cost spanning tree of a complete graph with unit length on each edge? Prove your answer. (b) What is a *maximum* routing cost spanning tree of a complete graph with unit length on each edge? Prove your answer.

5. (10%) (a) Give a tree with two centroids. (b) Show that any tree can have at most two centroids.

6. (10%) Construct an example where its minimum spanning tree has a routing cost $\Theta(n)$ times that of a minimum routing cost spanning tree.

7. (15%) Prove that a shortest-paths tree rooted at the median of a graph is a 2-approximation of a minimum routing cost spanning tree of the graph.

8. (10%) Let $P = (p_1, p_2, ..., p_k)$ be a path separator of $\widehat{T}$. It is easy to see that a centroid must be in $V(P)$. Let p_q be a centroid of $\widehat{T}$. Construct $R = SP_G(p_1, p_q) \cup SP_G(p_q, p_k)$. In class, we show that
$$\sum_{v \in V} d_G(v, R) \leq \sum_{v \in V} d_{\widehat{T}}(v, P) + (n/12)w(P).$$
Explain why we could have the coefficient $n/12$ instead of $n/6$ as in the case using only two end vertices p_1 and p_k.

9. (10%) We are given a tree T with positive edge weights. Suppose that $P = SP_T(v_1, v_2)$ is a diameter. Starting at v_1 and traveling along the path P, we compute the distance $d_T(u, v_1)$ for each vertex u on the path. Let u_1 be the last encountered vertex such that $d_T(v_1, u_1) \leq \frac{1}{2}w(P)$ and u_2 be the next vertex to u_1. Prove that u_1 or u_2 is a center of the tree.