Martingale Pricing

- Recall that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [\,pC_u + (1-p)\,C_d\,]/R.$$

  - p is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time i.
- Consider the discount process

$$\{\,C(i)/R^i, i = 0, 1, \dots, n\,\}.$$

- Then,

$$E\left[\left.\frac{C(i+1)}{R^{i+1}}\,\right|\,C(i) = C\right] = \frac{pC_u + (1-p)\,C_d}{R^{i+1}} = \frac{C}{R^i}.$$

Martingale Pricing (continued)

- In general,

$$E\left[\left.\frac{C(k)}{R^k}\,\right|\,C(i) = C\right] = \frac{C}{R^i}, \quad i \le k. \tag{42}$$

- The discount process is a martingale:

$$\frac{C(i)}{R^i} = E_i^{\pi}\left[\frac{C(k)}{R^k}\right], \quad i \le k. \tag{43}$$

  - E_i^{π} is taken under the risk-neutral probability conditional on the price information up to time i.
- This risk-neutral probability is also called the EMM, or the equivalent martingale (probability) measure.

Martingale Pricing (continued)

- In general, Eq. (43) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right], \quad i \le k. \tag{44}$$

  - $M(j)$ is the balance in the money market account at time j using the rollover strategy with an initial investment of \$1.
  - So it is called the bank account process.
- It says the discount process is a martingale under π.

Martingale Pricing (concluded)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0) = 1$.
  - $M(j)$ is known at time $j - 1$.
- Identity (44) on p. 427 is the general formulation of risk-neutral valuation.

Theorem 18 *A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale. This probability measure is called the risk-neutral probability measure.*

Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is

$$p_{\mathrm{f}} F u + (1 - p_{\mathrm{f}})\, F d = F \left(\frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F$$

  (see p. 394).
- Can be generalized to

$$F_i = E_i^{\pi}[\, F_k \,], \quad i \leq k,$$

  where F_i is the futures price at time i.
- It holds under stochastic interest rates.

Martingale Pricing and Numeraire

- The martingale pricing formula (44) on p. 427 uses the money market account as numeraire.[a]
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset S's value is positive at all times.

[a] Walras (1834–1910).

Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi} \left[\frac{C(k)}{S(k)} \right], \quad i \leq k.$$

  - $S(j)$ denotes the price of S at time j.
- So the discount process remains a martingale.

Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to S_1 or S_2.
- In a period, asset two's price can go from P to P_1 or P_2.
- Assume
$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}$$
to rule out arbitrage opportunities.

Example (continued)

- For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1.
- Let C_2 be its price at time one if asset one's price moves to S_2.
- Replicate the derivative by solving
$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$
using α units of asset one and β units of asset two.

Example (continued)

- This yields
$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$
- The derivative costs
$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

Example (concluded)

- It is easy to verify that
$$\frac{C}{P} = p \, \frac{C_1}{P_1} + (1 - p) \, \frac{C_2}{P_2}.$$
  - Above,
$$p \equiv \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$
- The derivative's price using asset two as numeraire is thus a martingale under the risk-neutral probability p.
- The expected returns of the two assets are irrelevant.

Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.
  1. $X(0) = 0$, unless stated otherwise.
  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables
  $$X(t_k) - X(t_{k-1})$$
  for $1 \leq k \leq n$ are independent.[b]
  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

[a]Robert Brown (1773–1858).
[b]So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- Such a process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is also called the Wiener process.

[a]Norbert Wiener (1894–1964).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t-s)$.
- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:
  $$Y(t) = \mu t + \sigma X(t).$$
- As $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$, uncertainty about the future value of Y grows as the square root of how far we look into the future.

Brownian Motion as Limit of Random Walk

Claim 1 *A (μ, σ) Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1 - p$.
- It moves to the right with probability p after Δt time.
- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time t is
  $$Y(t) \equiv \Delta x \left(X_1 + X_2 + \cdots + X_n \right).$$

Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here

$$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  - X_i are independent with $\text{Prob}[\,X_i = 1\,] = p = 1 - \text{Prob}[\,X_i = -1\,]$.
- Recall $E[\,X_i\,] = 2p - 1$ and $\text{Var}[\,X_i\,] = 1 - (2p-1)^2$.

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\text{Var}[\,Y(t)\,] = n(\Delta x)^2 \left[\,1 - (2p-1)^2\,\right].$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv [\,1 + (\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \text{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1 - (\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

  as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{\,Y(t), t \geq 0\,\}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned} &\text{Var}[\,Y(t+\Delta t) - Y(t)\,] \\ =&\text{Var}[\,\Delta x\, X_{n+1}\,] = (\Delta x)^2 \times \text{Var}[\,X_{n+1}\,] \to \sigma^2 \Delta t. \end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (25) on p. 249 and $\Delta x = \ln u$.

Geometric Brownian Motion

- Let $X \equiv \{\,X(t), t \geq 0\,\}$ be a Brownian motion process.
- The process

$$\{\,Y(t) \equiv e^{X(t)}, t \geq 0\,\},$$

  is called geometric Brownian motion.
- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

$$E\left[\,e^{sX(t)}\,\right] = E\left[\,Y(t)^s\,\right] = e^{\mu t s + (\sigma^2 t s^2/2)}$$

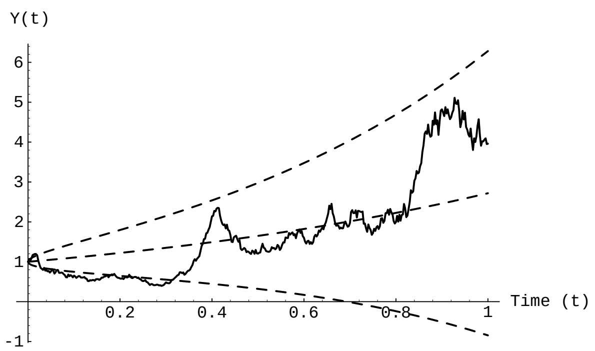
  from Eq. (17) on p 146.

Geometric Brownian Motion (continued)

- In particular,

$$E[\,Y(t)\,] = e^{\mu t + (\sigma^2 t/2)},$$

$$\begin{aligned}\mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2\\ &= e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right).\end{aligned}$$



Geometric Brownian Motion (concluded)

Useful for situations in which percentage changes are independent and identically distributed.

- Let Y_n denote the stock price at time n and $Y_0 = 1$.
- Assume relative returns $X_i \equiv Y_i / Y_{i-1}$ are independent and identically distributed.
- Then

$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$

  is a sum of independent, identically distributed random variables.
- Thus $\{\,\ln Y_n, n \geq 0\,\}$ is approximately Brownian motion.

Continuous-Time Financial Mathematics

Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X\, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X\, dW$.

[a] Ito (1915–).

Stochastic Integrals (concluded)

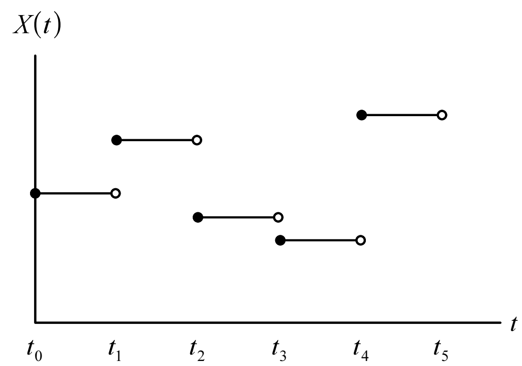
- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{ X(s), 0 \leq s \leq t \}$ is independent of $\{ W(t+u) - W(t), u > 0 \}$.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [t_{k-1}, t_k), \; k = 1, 2, \ldots$$

for any realization (see figure next page).



Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{45}$$

where $t_n = t$.

  - The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{\, X(t), t \geq 0 \,\}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
- A corollary is the mean value formula

$$E\left[\int_a^b X\, dW\right] = 0.$$

Theorem 19 *The Ito integral $\int X\, dW$ is a martingale.*

Discrete Approximation

- Recall Eq. (45) on p. 452.
- The following simple stochastic process $\{\, \widehat{X}(t) \,\}$ can be used in place of X to approximate the stochastic integral $\int_0^t X\, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \quad \text{for } s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{\, \widehat{X}(t), W(t), 0 \leq t \leq s \,\},$$

  cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

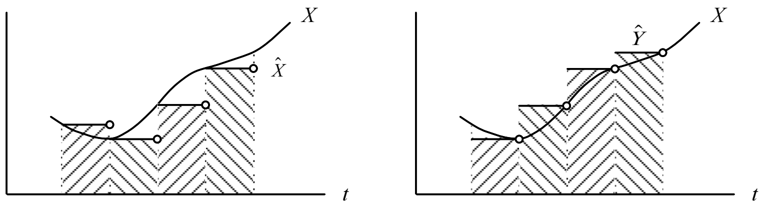
- Suppose we defined the stochastic integral as

$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \quad \text{for} \quad s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.



(a) (b)

Ito Process

- The stochastic process $X = \{\, X_t, t \geq 0 \,\}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{\, a(X_t, t) : t \geq 0 \,\}$ and $\{\, b(X_t, t) : t \geq 0 \,\}$ are stochastic processes satisfying certain regularity conditions.

- The terms $a(X_t, t)$ and $b(X_t, t)$ are the drift and the diffusion, respectively.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{46}$$

  - Or simply $dX_t = a_t\, dt + b_t\, dW_t$.

- This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.
- X is a martingale if the drift a_t is zero by Theorem 19 (p. 454).

[a]Langevin (1904).

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form to Eq. (46) is
$$dX_t = a_t\,dt + b_t\sqrt{dt}\;\xi, \tag{47}$$
where $\xi \sim N(0,1)$.
- This formulation makes it easy to derive Monte Carlo simulation algorithms.

Euler Approximation

- The following approximation follows from Eq. (47),
$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n),\end{aligned} \tag{48}$$
where $t_n \equiv n\Delta t$.
- It is called the Euler or Euler-Maruyama method.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.

More Discrete Approximations

- Under fairly loose regularity conditions, approximation (48) on p. 461 can be replaced by
$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).\end{aligned}$$
  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- A simpler discrete approximation scheme:
$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,\xi.\end{aligned} \tag{49}$$
  - $\text{Prob}[\,\xi = 1\,] = \text{Prob}[\,\xi = -1\,] = 1/2$.
  - Note that $E[\,\xi\,] = 0$ and $\text{Var}[\,\xi\,] = 1$.
- This clearly defines a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.