Interest Rate Swaps

- Consider an interest rate swap made at time t with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the floating-rate payer.
  - Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} (P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i)) \\
= & \; P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t,t_0) - P(t,t_n)}{\sum_{i=1}^{n} P(t,t_i)\,\Delta t}. \tag{96}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t,t_0) = 1$.

The Term Structure Equation

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r,t,T)$ follow

$$\frac{dP}{P} = \mu_p\,dt + \sigma_p\,dW.$$

- Suppose an investor at time t shorts one unit of a bond maturing at time s_1 and at the same time buys α units of a bond maturing at time s_2.

The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r,t,s_1) + \alpha\,dP(r,t,s_2) \\ = \; & (-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2))\,dt \\ & + (-P(r,t,s_1)\,\sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\,\sigma_p(r,t,s_2))\,dW. \end{aligned}$$

- Pick

$$\alpha \equiv \frac{P(r,t,s_1)\,\sigma_p(r,t,s_1)}{P(r,t,s_2)\,\sigma_p(r,t,s_2)}.$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1) + \alpha P(r,t,s_2)} = r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2) - \sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1) - \sigma_p(r,t,s_2)} = r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2) - r}{\sigma_p(r,t,s_2)} = \frac{\mu_p(r,t,s_1) - r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \equiv \lambda(r,t) \tag{97}$$

for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.
- The term $\lambda(r,t)$ is called the market price of risk.
- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model, $dr = \mu(r,t)\,dt + \sigma(r,t)\,dW$.
- Then the bond price process is also Markovian.
- By Eq. (56) on p. 478,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2}\right)/P, \tag{98}$$

$$\sigma_p = \left(\sigma(r,t)\,\frac{\partial P}{\partial r}\right)/P, \tag{98$'$}$$

subject to $P(\,\cdot\,, T, T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (97) on p. 790 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{99}$$

- This is called the term structure equation.
- Once P is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

- Equation (99) applies to all interest rate derivatives, the difference being the terminal and the boundary conditions.

Risk-Neutral Process

- The local expectations theory is usually imposed for convenience.
- In fact, a probability measure exists such that bonds can be priced as if the theory were true to preclude arbitrage opportunities.
- In the world where the local expectations theory holds, $\mu_p(r,t,s) = r$ and the market price of risk is zero.
  - No risk adjustment is needed.

Risk-Neutral Process (continued)

- The term structure equation becomes

$$-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{100}$$

- The bond price dynamics (98) on p. 791 is simplified to

$$dP = rP\,dt + \sigma(r,t)\,\frac{\partial P}{\partial r}\,dW.$$

Risk-Neutral Process (concluded)

- Alternatively, derivatives can be priced by assuming that the short rate follows the risk-neutral process:

$$dr = (\mu(r,t) - \lambda(r,t)\,\sigma(r,t))\,dt + \sigma(r,t)\,dW.$$

- The market price of risk will be assumed to be zero in pricing unless stated otherwise.

The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1-q$ to Pd, where $u > d$:

$$\begin{tikzcd}[row sep=small] & Pd \\ P \arrow[ur, "1-q"] \arrow[dr, "q"'] & \\ & Pu \end{tikzcd}$$

The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{101}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{102}$$

- The bond whose maturity is only one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate.

The Binomial Model (continued)

- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$, analogous to Eq. (97) on p. 790.
- The same arbitrage argument as in the continuous-time case can be employed to show that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \tag{103}$$

which is independent of bond maturity and q.
  - Recall the BOPM.
- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

$$\begin{tikzcd}[row sep=small] & 8\% \\ 4\% \arrow[ur] \arrow[dr] & \\ & 2\% \end{tikzcd}$$

Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$100/1.04 = 96.154, 100/(1.05)^2 = 90.703.$$

- They follow the binomial processes on p. 802.

Numerical Examples (continued)

$$
\begin{tikzcd}[row sep=small]
 & 92.593\ (= 100/1.08) \\
90.703 \arrow[ur] \arrow[dr] & \\
 & 98.039\ (= 100/1.02)
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
 & 100 \\
96.154 \arrow[ur] \arrow[dr] & \\
 & 100
\end{tikzcd}
$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a $95 strike price has the payoffs,

$$
\begin{tikzcd}[row sep=small]
 & 0.000 \\
C \arrow[ur] \arrow[dr] & \\
 & 3.039
\end{tikzcd}
$$

- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity, as shown below.

$$\begin{tikzcd} & 92\ (= 100 - 8) \\ F \arrow[ur] \arrow[dr] & \\ & 98\ (= 100 - 2) \end{tikzcd}$$

- As the futures price F is the expected future payoff (see text), $F = (1-p) \times 92 + p \times 98 = 93.914$.
- On the other hand, the forward price for a one-year forward contract on a one-year zero-coupon bond equals $90.703/96.154 = 94.331\%$.
- The forward price exceeds the futures price.

Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 811.
- Its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

- Identical results could have been obtained via arbitrage considerations.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 105 & & \\
 & 5 \arrow[ur] \arrow[dr] & & & 102.222\ (= 5 + (105/1.08)) \\
 & & 105 & & \\
0 \arrow[uur] \arrow[ddr] & & & M \arrow[uur] \arrow[ddr] & \\
 & & 105 & & \\
 & 5 \arrow[ur] \arrow[dr] & & & 107.941\ (= 5 + (105/1.02)) \\
 & & 105 & &
\end{tikzcd}$$

The left diagram depicts the cash flow; the right diagram illustrates the price process.

Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

$$\begin{tikzcd}[row sep=small]
 & 102.222 \\
M \arrow[ur] \arrow[dr] & \\
 & 105
\end{tikzcd}$$

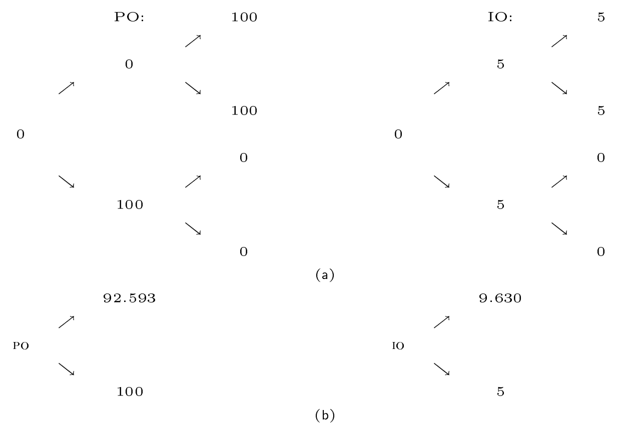
- The security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage's principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (see p. 814(a)).
- Their prices hence follow the processes on p. 814(b).
- The fair prices are

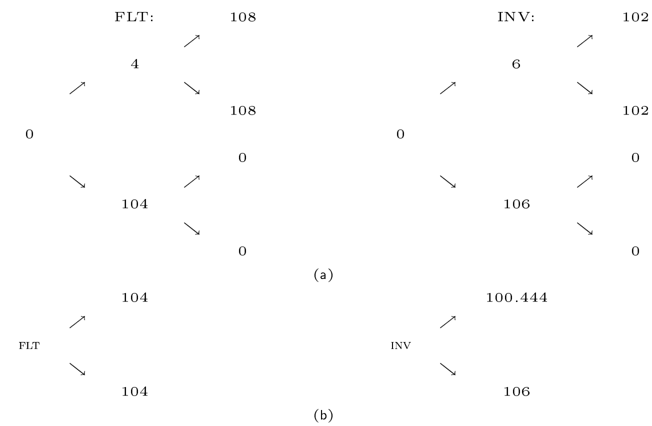
$$\begin{aligned}
\text{PO} &= \frac{(1-p) \times 92.593 + p \times 100}{1.04} = 91.304, \\
\text{IO} &= \frac{(1-p) \times 9.630 + p \times 5}{1.04} = 7.839.
\end{aligned}$$



The price 9.630 is derived from $5 + (5/1.08)$.

Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of
  $$(10\% - \textsf{one-year rate})$$
  to make the overall coupon rate 5%.
- Their cash flows as percentages of par and values are shown on p. 816.



Numerical Examples: MBSs (concluded)

- On p. 816, the floater's price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater's price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are

$$\begin{aligned} \text{FLT} &= \frac{1}{2} \times \frac{104}{1.04} = 50, \\ \text{INV} &= \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142. \end{aligned}$$

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \,|\, r(t) = r\,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (55) on p. 474.

[a] Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \qquad (104)$$

where

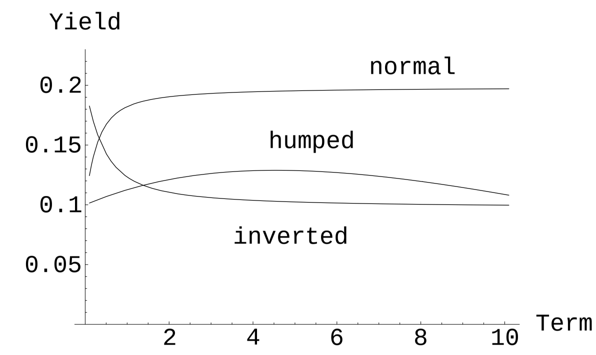
$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve $(\partial r(t,T)/\partial r)\, \sigma = \sigma B(t,T)/(T-t)$.
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve; indeed, higher β leads to greater attenuation of volatility with maturity.



The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t,s)\, N(x) - X P(t,T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$\begin{aligned} x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X} \right) + \frac{\sigma_v}{2}, \\ \sigma_v &\equiv v(t,T)\,B(T,s), \\ v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}. \end{aligned}$$

- By the put-call parity, the price of a European put is

$$X P(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\;\xi(k), \quad 0 \leq k < n.$$

[a] Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.
- For a general process Y with nonconstant volatility, the resulting binomial tree may not combine.
- A way out is to transform Y into one with constant volatility (see later).

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\,dt + \sigma\sqrt{r}\,dW. \tag{105}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a]Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\,dt + dW,$$

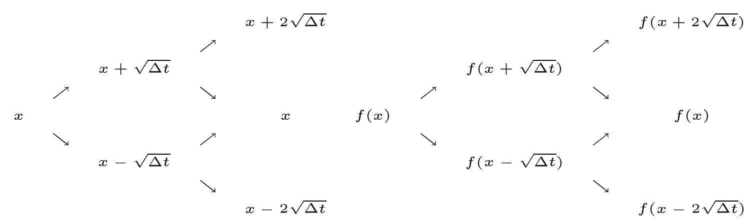
where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation $r = f(x) \equiv x^2\sigma^2/4$ (see p. 834).



Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \qquad (106)$$

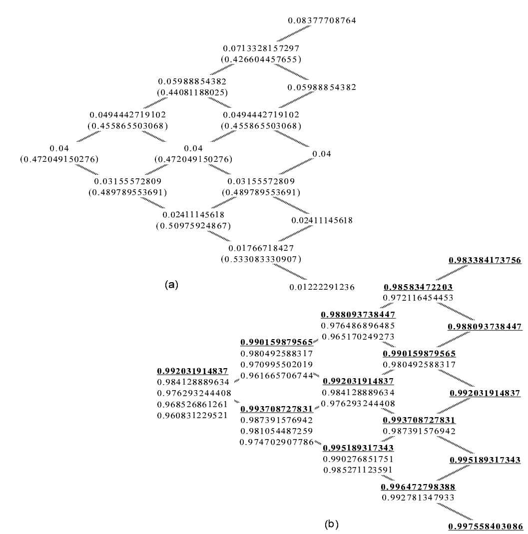
  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.
- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 837(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.



Numerical Examples (continued)

- To give an idea how these numbers come into being, consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process $dy = \alpha(y,t)\, dt + \sigma(y,t)\, dW$.
- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\mathrm{u}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.

[a] Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine:

$$\sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}},t)\sqrt{\Delta t} \neq -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}},t)\sqrt{\Delta t}$$

  in general.
- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.
- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows $dx = m(y,t)\, dt + dW$ for some $m(y,t)$ (see text).

A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.
- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

A General Method (concluded)

- The transformation is

$$\int^{r} (\sigma\sqrt{z})^{-1}\,dz = 2\sqrt{r}/\sigma$$

  for the CIR model.
- The transformation is

$$\int^{S} (\sigma z)^{-1}\,dz = (1/\sigma)\ln S$$

  for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

Model Calibration

- In the time-series approach, the time series of short rates is used to estimate the parameters of the process.
- This approach may help in validating the proposed interest rate process.
- But it alone cannot be used to estimate the risk premium parameter λ.
- The model prices based on the estimated parameters may also deviate a lot from those in the market.

Model Calibration (concluded)

- The cross-sectional approach uses a cross section of bond prices observed at the same time.
- The parameters are to be such that the model prices closely match those in the market.
- After this procedure, the calibrated model can be used to price interest rate derivatives.
- Unlike the time-series approach, the cross-sectional approach is unable to separate out the interest rate risk premium from the model parameters.

An Example

- Suppose the short rate r follows $dr = \mu(r)\, dt + \sigma(r)\, dW$ in the real world.
- It follows $dr = \mu^*(r)\, dt + \sigma\, dW^*$ in the risk-neutral world.
- As prices are derived in the risk-neutral world, cross-sectional data are used to estimate $\mu^*(r)$.
- Because short rates are generated under the real-world probability measure, historical short-rate time series is used to estimate $\mu(r)$ and $\sigma(r)$.
- Set the market price of risk to $[\,\mu(r) - \mu^*(r)\,]/\sigma(r)$.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds[a]

- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.
- The payoff for the option is

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0\right).$$

[a] Jamshidian (1989).

Options on Coupon Bonds (continued)

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.
- This r^* can be obtained by solving $X = \sum_i c_i P(r, T, t_i)$ numerically for r.
- The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let $X_i \equiv P(r^*, T, t_i)$, the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

Options on Coupon Bonds (concluded)

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.
- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of n options on the underlying zero-coupon bond.

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