Remarks on Asian Option Pricing

- Asian option pricing is an active research area.
- The above algorithm overestimates the "true" value.[a]
- To guarantee convergence, k needs to grow with n.
- There is a convergent approximation algorithm that does away with interpolation with a provable running time of $2^{O(\sqrt{n})}$.[b]

[a]Dai, Huang, and Lyuu (2002).
[b]Dai and Lyuu (2002, 2004).

Remarks on Asian Option Pricing (continued)

- There is an $O(kn^2)$-time algorithm with an error bound of $O(Xn/k)$ from the naive $O(2^n)$-time binomial tree algorithm in the case of European Asian options.[a]
  - k can be varied for trade-off between time and accuracy.
  - If we pick $k = O(n^2)$, then the error is $O(1/n)$, and the running time is $O(n^4)$.
- In practice, log-linear interpolation works better.

[a]Aingworth, Motwani, and Oldham (2000).

Remarks on Asian Option Pricing (concluded)

- Another approximation algorithm reduces the error to $O(X\sqrt{n}/k)$.[a]
  - It varies the number of buckets per node.
  - If we pick k proportional to n, the error is $O(n^{-0.5})$.
  - If we pick $k = O(n^{1.5})$, then the error is $O(1/n)$, and the running time is $O(n^{3.5})$.
- Under "reasonable assumptions," an $O(n^2)$-time algorithm with an error bound of $O(1/n)$ exists.[b]

[a]Dai, Huang, and Lyuu (2002).
[b]Hsu and Lyuu (2004).

A Grand Comparison[a]

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
95	0.05	0.05	7.1777275	7.1777244	7.1777279	7.178812	7.177726
100			2.7161745	2.7161755	2.7161744	2.715613	2.716168
105			0.3372614	0.3372601	0.3372614	0.338863	0.337231
95		0.09	8.8088392	8.8088441	8.8088397	8.808717	8.808839
100			4.3082350	4.3082253	4.3082331	4.309247	4.308231
105			0.9583841	0.9583838	0.9583841	0.960068	0.958331
95		0.15	11.0940944	11.0940964	11.0940943	11.093903	11.094094
100			6.7943550	6.7943510	6.7943553	6.795678	6.794354
105			2.7444531	2.7444538	2.7444531	2.743798	2.744406
90	0.10	0.05	11.9510927	11.9509331	11.9510871	11.951610	11.951076
100			3.6413864	3.6414032	3.6413875	3.642325	3.641344
110			0.3312030	0.3312563	0.3311968	0.331348	0.331074
90		0.09	13.3851974	13.3851165	13.3852048	13.385563	13.385190
100			4.9151167	4.9151388	4.9151177	4.914254	4.915075
110			0.6302713	0.6302538	0.6302717	0.629843	0.630064
90		0.15	15.3987687	15.3988062	15.3987860	15.398885	15.398767
100			7.0277081	7.0276544	7.0277022	7.027385	7.027678
110			1.4136149	1.4136013	1.4136161	1.414953	1.413286

[a]Hsu and Lyuu (2004); Zhang (2001,2003); Chen and Lyuu (2006).

A Grand Comparison (concluded)

X	σ	r	Exact	AA2	AA3	Hsu-Lyuu	Chen-Lyuu
90	0.20	0.05	12.5959916	12.5957894	12.5959304	12.596052	12.595602
100			5.7630881	5.7631987	5.7631187	5.763664	5.762708
110			1.9898945	1.9894855	1.9899382	1.989962	1.989242
90		0.09	13.8314996	13.8307782	13.8313482	13.831604	13.831220
100			6.7773481	6.7775756	6.7773833	6.777748	6.776999
110			2.5462209	2.5459150	2.5462598	2.546397	2.545459
90		0.15	15.6417575	15.6401370	15.6414533	15.641911	15.641598
100			8.4088330	8.4091957	8.4088744	8.408966	8.408519
110			3.5556100	3.5554997	3.5556415	3.556094	3.554687
90	0.30	0.05	13.9538233	13.9555691	13.9540973	13.953937	13.952421
100			7.9456288	7.9459286	7.9458549	7.945918	7.944357
110			4.0717942	4.0702869	4.0720881	4.071945	4.070115
90		0.09	14.9839595	14.9854235	14.9841522	14.984037	14.982782
100			8.8287588	8.8294164	8.8289978	8.829033	8.827548
110			4.6967089	4.6956764	4.6969698	4.696895	4.694902
90		0.15	16.5129113	16.5133090	16.5128376	16.512963	16.512024
100			10.2098305	10.2110681	10.2101058	10.210039	10.208724
110			5.7301225	5.7296982	5.7303567	5.730357	5.728161

Forwards, Futures, Futures Options, Swaps

Summon the nations to come to the trial.
Which of their gods can predict the future?
— Isaiah 43:9

Terms

- r will denote the riskless interest rate.
- The current time is t.
- The maturity date is T.
- The remaining time to maturity is $\tau \equiv T - t$ (all measured in years).
- The spot price S, the spot price at maturity is S_T.
- The delivery price is X.

Terms (concluded)

- The forward or futures price is F for a newly written contract.
- The value of the contract is f.
- A price with a subscript t usually refers to the price at time t.
- Continuous compounding will be assumed.

Forward Contracts

- Forward contracts are for the delivery of the underlying asset for a certain delivery price on a specific time.
  - Foreign currencies, bonds, corn, etc.
- Ideal for hedging purposes.
- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for $2.5 per bushel on September 27, 1995.
- The farmer is assured of a buyer at an acceptable price.
- The processor knows the cost of corn in advance.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.
  - If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.
  - If the price declines, the processor will be paying more than it would.
- Either side has an incentive to default.
- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.
- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).
- r_{f} denotes the annual interest rates of the foreign currency.
- r_ℓ denotes the annual interest rates of the local currency.
- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity[a]

$$\frac{F}{S} = e^{r_\ell - r_{\mathrm{f}}}. \tag{30}$$

- A holder of the local currency can do either of:
  - Lend the money in the domestic market to receive e^{r_ℓ} one year from now.
  - Convert local currency for foreign currency, lend for 1 year in foreign market, and convert foreign currency into local currency at the fixed forward exchange rate, F, by selling forward foreign currency now.

[a]Keynes (1923). John Maynard Keynes (1883–1946) was one of the greatest economists in history.

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.
- One unit of local currency will hence become $Fe^{r_{\mathrm{f}}}/S$ one year from now in the 2nd case.
- If $Fe^{r_{\mathrm{f}}}/S > e^{r_\ell}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.
- If $Fe^{r_{\mathrm{f}}}/S < e^{r_\ell}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

- Forward contracts do not involve any initial cash flow.
- The forward price is the delivery price which makes the forward contract zero valued.
  - That is, $f = 0$ when $F = X$.

Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.
- But the forward price may change after the contract comes into existence.
  - The value of a forward contract, f, is 0 at the outset.
  - It will fluctuate with the spot price thereafter.
  - This value is enhanced when the spot price climbs and depressed when the spot price declines.
- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 9 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{31}$$

- If $F > Se^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, sell the asset for F and use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of $F - Se^{r\tau} > 0$.
- If $F < Se^{r\tau}$, do the opposite.

Example

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on a 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}.$$

- Consider a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and one short position in the underlying asset.
- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.

Forward Price: Underlying Pays Predictable Income

Lemma 10 *For a forward contract on an underlying asset providing a predictable income with a PV of I,*

$$F = (S - I)\, e^{r\tau}. \tag{32}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, the asset is sold for F, and $(S - I)\, e^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - (S - I)\, e^{r\tau} > 0$.
- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)}.$$

  - r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

The value of a forward contract at any time prior to T is

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{33}$$

- Consider a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and a short position in $e^{-q\tau}$ units of the underlying asset.
- All dividends are paid for by shorting additional units of the underlying asset.
- The cash will grow to X at maturity.
- The short position will grow to exactly one unit of the underlying asset.

Underlying Pays a Continuous Dividend Yield (concluded)

- There is sufficient fund to take delivery of the forward contract.
- This offsets the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.
- One consequence of Eq. (33) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{34}$$

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.
  - The settlement price is some kind of average traded price immediately before the end of trading.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the CBT.
  - One million U.S. dollars for the Eurodollar futures on the CME.
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell a food processor 100,000 bushels of corn at $2.00 per bushel in November.
  - Suppose the price of corn rises to $2.5 by November.

Daily Settlements (concluded)

- (continued)
  - The farmer has incentive to sell his harvest in the spot market at $2.5.
  - With marking to market, the farmer has transferred $0.5 per bushel from his futures account to that of the food processor by November.
  - When the farmer makes delivery, he is paid the spot price, $2.5 per bushel.
  - The farmer has little incentive to default.
  - The net price remains $2.00 per bushel, the original delivery price.

Delivery and Hedging

- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.
- Changes in futures prices usually track those in spot prices.
- This makes hedging possible.
- Before the delivery date, the futures price could be above or below the spot price.

Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$\begin{aligned}(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) \\ = F_n - F_0 = S_T - F_0.\end{aligned}$$

- A futures contract has the same accumulated payoff $S_T - F_0$ as a forward contract.
- The actual payoff may differ because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Forward and Futures Prices[a]

Futures price equals forward price if interest rates are nonstochastic![b]

- See text for proof.

[a]Cox, Ingersoll, and Ross (1981).

[b]This "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a long futures position.
  - A put holder acquires a short futures position.
- The futures contract is then marked to market, and the futures position of the two parties will be at the prevailing futures price.

Futures Options (concluded)

- It works as if the call writer delivered a futures contract to the option holder and paid the holder the prevailing futures price minus the strike price.
- It works as if the put writer took delivery a futures contract from the option holder and paid the holder the strike price minus the prevailing futures price.
- The amount of money that changes hands upon exercise is the difference between the strike price and the prevailing futures price.

Forward Options

- Similar to futures options except that what is delivered is a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a long position in a forward contract.
  - Exercising a put forward option results in a short position in a forward contract.
- Exercising a forward option incurs no immediate cash flows.

Example

- Consider a call with strike \$100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.
- If an offsetting position is then taken in the forward market, a \$10 profit in December will be assured.
- A call on the futures would realize the \$10 profit in July.

Some Pricing Relations

- Let delivery take place at time T, the current time be 0, and the option on the futures or forward contract have expiration date t $(t \leq T)$.
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does not have the same value as a futures option.
- The payoffs of calls at time t are

$$\begin{aligned} \text{futures option} &= \max(F_t - X, 0), & (35) \\ \text{forward option} &= \max(F_t - X, 0)\, e^{-r(T-t)}. & (36) \end{aligned}$$

Some Pricing Relations (concluded)

- A European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the options.
  - Futures price equals spot price at maturity.
  - This conclusion is independent of the model for the spot price.

Put-Call Parity

The put-call parity is slightly different from the one in Eq. (18) on p. 178.

Theorem 11 *(1) For European options on futures contracts,* $C = P - (X - F)\, e^{-rt}$*. (2) For European options on forward contracts,* $C = P - (X - F)\, e^{-rT}$*.*

- See text for proof.

Early Exercise and Forward Options

The early exercise feature is not valuable.

Theorem 12 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

- See text for proof.

Early exercise may be optimal for American futures options even if the underlying asset generates no payouts.

Theorem 13 *American futures options may be exercised optimally before expiration.*

Black Model[a]

- Formulas for European futures options:

$$\begin{aligned} C &= Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma\sqrt{t}), \qquad (37)\\ P &= Xe^{-rt}N(-x + \sigma\sqrt{t}) - Fe^{-rt}N(-x), \end{aligned}$$

  where $x \equiv \frac{\ln(F/X) + (\sigma^2/2)\, t}{\sigma\sqrt{t}}$.
- Formulas (37) are related to those for options on a stock paying a continuous dividend yield.
- In fact, they are exactly Eqs. (24) on p. 256 with the dividend yield q set to the interest rate r and the stock price S replaced by the futures price F.

[a] Black (1976).

Black Model (concluded)

- This observation incidentally proves Theorem 13 (p. 377).
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$ based on Eqs. (35)–(36).

Binomial Model for Forward and Futures Options

- Futures price behaves like a stock paying a continuous dividend yield of r.
- Under the BOPM, the risk-neutral probability for the futures price is

$$p_{\mathrm{f}} \equiv (1-d)/(u-d)$$

by Eq. (25) on p. 257.
  - The futures price moves from F to Fu with probability p_{f} and to Fd with probability $1-p_{\mathrm{f}}$.
- The binomial tree algorithm for forward options is identical except that Eq. (36) on p. 374 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via $F = Se^{rT}$ if the underlying asset pays no dividends.
- The stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

with probability p_{f} per period.
- The stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

with probability $1-p_{\mathrm{f}}$ per period.

Negative Probabilities Revisited

- As $0 < p_{\mathrm{f}} < 1$, we have $0 < 1-p_{\mathrm{f}} < 1$ as well.
- The problem of negative risk-neutral probabilities is now solved:
  - Suppose the stock pays a continuous dividend yield of q.
  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
  - Calculate S from F at each node via $S = Fe^{-(r-q)(T-t)}$.

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to a predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_{\mathrm{A}}\%$	$Y_{\mathrm{A}}\%$
B	$D_{\mathrm{B}}\%$	$Y_{\mathrm{B}}\%$

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_{\mathrm{A}}\%$ and B to borrow dollars at $D_{\mathrm{B}}\%$.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, and vice versa for B.
  - That is, $Y_{\mathrm{B}} - Y_{\mathrm{A}} < D_{\mathrm{B}} - D_{\mathrm{A}}$.
- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total gain is $((D_{\mathrm{B}} - D_{\mathrm{A}}) - (Y_{\mathrm{B}} - Y_{\mathrm{A}}))\%$:
  - The total interest rate is originally $(Y_{\mathrm{A}} + D_{\mathrm{B}})\%$.
  - The new arrangement has a smaller total rate of $(D_{\mathrm{A}} + Y_{\mathrm{B}})\%$.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

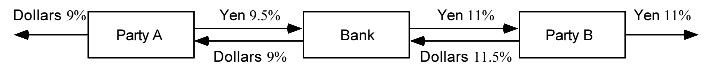
- A and B face the following borrowing rates:

	Dollars	Yen
A	9%	10%
B	12%	11%

- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

Example (concluded)

- As the rate differential in dollars (3%) is different from that in yen (1%), a currency swap with a total saving of $3 - 1 = 2\%$ is possible.
- A is relatively more competitive in the dollar market, and B the yen market.
- Figure next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5%. B borrows dollars at 11.5%.
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.



As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 389 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the \$/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on the term structures of interest rates in the currencies involved and the spot exchange rate.
- It has zero value when $SP_{\mathrm{Y}} = P_{\mathrm{D}}$.

Example

- Take a two-year swap on p. 389 with principal amounts of US$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}\right) - \left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}\right) = 0.074.$$

As a Package of Forward Contracts

- From Eq. (33) on p. 360, the forward contract maturing i years from now has a dollar value of

$$f_i \equiv (SY_i)\, e^{-qi} - D_i e^{-ri}. \tag{38}$$

  - Y_i is the yen inflow at year i.
  - S is the \$/yen spot exchange rate.
  - q is the yen interest rate.
  - D_i is the dollar outflow at year i.
  - r is the dollar interest rate.

As a Package of Forward Contracts (concluded)

- This formulation may be preferred to the cash market approach in cases involving costs of carry and convenience yields because forward prices already incorporate them.
- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

Example

- Take the swap in the example on p. 392.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.