微分樹與模型校正

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一、中文摘要

本計劃探討一個一般性的模型校正演算法叫微分樹法。微分樹法的概念雖十分簡單，應用則十分廣，尤其是基本計算爲樹狀圖時。我們用三個重要應用來說明微分樹法的效率及一般性：無套利利率模型之校正、利率差與選擇權利率差、與美式選擇權之隱含波動。大量的實驗顯示微分樹法在以上三個應用皆十分有效率。

英文摘要

This project investigates a general computational paradigm for model calibration called the *differential tree method.* The idea is very simple, and the method is particularly applicable when the model under consideration has a tree (lattice) structure. We illustrate its wide applicability with three canonical problems: no-arbitrage interest rate model calibration, spread and option-adjusted spread of non-benchmark bonds, and implied volatility of American-style options. Comprehensive computer experiments show the differential tree method to be highly efficient in all three case studies.

二、計劃緣由與目的

Many computational problems in derivative pricing involve trees. The binomial option pricing model and various discrete-time interest rate models are familiar examples. Computation on such structures conceptually proceeds via backward induction on the tree with local calculation at each node taking inputs from its successor nodes. Efficient algorithms are generally obtainable for pricing purposes if the tree recombines.

Our main concern, *model calibration*, is the inverse of the pricing problem: Given price information, it finds parameter values on the tree that generate *model prices* consistent with the data. Calibration, therefore, can be perceived as a root-finding problem.

In this project, we propose a general iterative procedure, the *differential tree method* (Lyuu [1999]), for the calibration of models based on trees. The basic idea behind the differential tree method is simple. It takes advantage of the inductive structure of the tree to calculate not only the model price but also its derivatives with respect to the parameters whose values we desire. This extra effort adds only a small amount of overhead to the overall computation, yet it pays off immensely in terms of faster convergence and clean software design.

We illustrate the wide-ranging applications of the method with three canonical problems: calibration of no-arbitrage interest rate models (in this paper, the Black-Derman-Toy model [1990]), the calculation of spread and option-adjusted spread (OAS) of non-benchmark bonds, and the computation of the implied volatility of American-style options (all listed stock options in the U.S. are American, for example). In each case, we obtain efficient algorithms in terms of speed, computer memory, and convergence rates. This claim will be substantiated with extensive computer experiments.

三、實驗方法

BDT interest rate model

Consider the following term structures in Hull and White [1990]: r* + 0.05 ln t for t-year zero-coupon bond yield and 1.4 (1-e^(-0.1 t)) for t-year zero-coupon bond yield volatility. We run our differential tree algorithm using forward induction for various time partitions and various maturities.

Spread and option-adjusted spread

Model prices calculated from trees calibrated off the benchmark bonds as a rule do not match market prices of non-benchmark bonds. To gauge the incremental return, or *yield spread*, over the benchmark bonds, one looks for the spread over the short rates in the tree that equates the model price and the market price. When the underlying bond contains embedded options, the spread is called *option-adjusted spread.* The running time of the differential tree method depends on the convergence rate of the Newton-Raphson method. In practice, only a small number of iterations is needed to bring the algorithm to convergence. Hence the empirical running time is O(n^2).

Volitility implied by American options

The computational problem here is structurally similar to the option-adjusted spread. We solve the Black-Scholes formula for the implied volatility as the initial guess.

四、結論與討論

BDT interest rate model

The differential tree method converges very fast. The experiments show that with one partition per year, it takes an average of less than 3.5 iterations to achieve a relative error of 10^(-13) with r*=0.06. We also compare forward induction with backward induction. The running time of backward induction with the differential tree is O(n^3) versus O(n^2) for forward induction. We benchmark backward induction with differential trees and forward induction. The result is as expected: The forward induction implementation is far more efficient. The running time is quadratic at 0.000020 n^2 seconds. Furthermore, we can calibrate the tree up to 270,000 years. We also look at the problem from a different perspective by fixing the time span at 30 years and look at how forward induction performs by increasing the number of partitions. The conclusion is it can go as fine as 70 periods per year. The running time is quadratic, at about 0.000098 n^2 seconds. The experiment is then repeated for 10-year trees. The running time grows roughly as 0.000051 n^2 seconds, and we can go as fine as 1,850 periods per year.

Spread and option-adjusted spread

The experiments apply the differential tree method to zero-coupon bonds under the BDT model and flat term structures: 8% for t-year zero-coupon bond yield and 10% for t-year zero-coupon bond yield volatility. The differential tree method converges very fast. For example, for the spread computation, it takes about 7.85 seconds for a tree with 500 periods on a 75MHz Sun SPARCstation 20 with 64MB of DRAM. In general, the running time is about 0.000031 n^2 seconds. Similar conclusions hold for option-adjusted spreads. We use the 8% callable coupon bond under the BDT model and flat term structures as above. The results show that the differential tree method converges very fast. For example, it takes about 8.11 seconds for a tree with 500 periods on a 75MHz Sun SPARCstation 20 with 64MB of DRAM. In general, the running time is quadratic, at about 0.0000065 n^2 seconds per iteration.

Volitility implied by American options

The fast convergence of the differential tree method is validated again. The running time is quadratic: about 0.00000081 n^2 seconds for the American call and 0.00000045 n^2 seconds per iteration for the American put. Note that American calls will not be exercised early in our case of non-dividend-paying stock. Increasing the number of partitions may sometimes decrease the running time due to the reduction of the number of iterations.

Conclusions

The differential tree method was shown to be extremely efficient. In terms of software development, the method results in a program structure which almost parallels that of its dual, the pricing module; only the derivatives calculation and adjustments to the approximate root at the end of each iteration need to be added.

Other interesting applications are easy to think of. Take the calibration of interest rate models again. For finite-dimensional models, the limited number of parameters makes matching exactly the term structures impossible. For them, one may aim to minimize the mean square error | P(x)-p |^2 between the market and the model-implied term structures. By providing the derivatives efficiently, the differential tree method can easily solve this optimization problem. We conclude that, as a general computing paradigm, the differential tree method has potential for efficient calibration of any tree-based models.

五、參考文獻

[Black, Scholes, 1973] Black, F. and Scholes, M. " The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81, 1973, 637–659.

[Black, Derman, Toy, 1990] Black, F., Derman, E., and Toy, W. "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *Financial Analysts Journal*, Jan–Feb, 1990, 33–39.

[Cox, Ingersoll, Ross, 1985] Cox, J., Ingersoll, E., and Ross, S. A. "A Theory of Term Structure of Interest Rate." *Econometrica*, Vol. 53, 1985, 385–467.

[Hull, White, 1990] Hull, J., and White, A. "Pricing Interest-Rate-Derivative Securities." *The Review of Financial Studies*, Vol. 3, No. 4, 573–592, 1990.

[Hull, White, 1994] Hull, J., and White, A. "One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities." *Journal of Financial and Quantitative Analysis*, Vol. 28, No. 2, 1994, 235–254.

[Hull, 1999] Hull, J. *Options, Futures and Other Derivative Securities*. 4th ed. Englewood Cliffs, New Jersey: Prentice-Hall, 1999.

[Lyuu, 1999] Lyuu, Yuh-Dauh. "A General Computational Method for Calibration Based on Differential Trees. " *The Journal of Derivatives*, Vol. 7, No. 1 (Fall 1999), 79–90.

[Lyuu, 2000] Lyuu, Yuh-Dauh. *Financial Engineering and Computation: Principles, Mathemtics, Algorithms.* To be published by the Cambridge University Press, 2000.

[Vasicek, 1977] Vasicek, O. A. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, Vol. 5, 1977, 177–188.