A Convolution-based DCT Algorithm

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Abstract

In this paper, based on some theorems of Number theory, a new convolution-based algorithm for computing the DCT (with power of two length) is proposed. In terms of computational counts, the proposed algorithm computes a length-N DCT (with N a power of two) using only N multiplications.

1 Introduction

Since the discovery of the discrete cosine transform (DCT) [1], many new algorithms for computing the DCT have been developed. These algorithms are either indirect computations using fast Fourier transforms [1]-[2] or direct computations using matrix factorization (or recurisive computation) [3]. On the other hand,the convolution-based approach deals commonly with the prime length (Prime factors) DFTs [4]. These algorithms can be optimized using the Winograd's convolution algorithm [5], or be implemented using the number theoretical transform (NTT) which needs only order N multiplications.

In this paper, based on some theorems of Number theory, a new convolution-based algorithm for computing the DCT (with power of two length) is proposed. In terms of computational counts, the proposed algorithm computes a length-N DCT (with N a power of two) using only N multiplications.

2 Some Useful Theorems in Number Theory and the Properties of DCT

Theorem 1 *If $n > 2$, then*

$$\begin{aligned} 4k+1 &\equiv 5^{\beta_1} \pmod{2^n} \\ 4k+3 &\equiv -5^{\beta_2} \pmod{2^n} \end{aligned}$$

where $k \in Z$ (the set of integers) and $\beta_1, \beta_2 \in Z^+$ (the set of positive integers).

Theorem 2 *If $n > 2$, then*

$$5^{2^{n-3}} \equiv 1 + 2^{n-1} \pmod{2^n}$$

The proofs of Theorem 1 and 2 can be found in [6].

Theorem 1 implies that there is a one-to-one mapping between the following two subsets in Z_{2^n} (the integers modulo 2^n) that is

$$\{4t+1 \mid t = 0, 1, \cdots, 2^{n-2}-1\} \longleftrightarrow \{5^t \mid t = 0, 1, \cdots, 2^{n-2}-1\}.$$

Corollary 1 *For the matrix of index functions*

$$M = [f((4i+1)(4j+1) \pmod{4N})]_{i,j=0,1,\cdots,N-1}$$

there exist a circular convolution matrix C and two permutation matrix P_1 and P_2, such that

$$M = P_1 C P_2$$

[proof]: By theorem1

$$4i+1 \equiv 5^t \pmod{4N}$$

Therefore, we can reorder the rows and columns in M *i.e.*,

$$\begin{aligned} C &= \left[f((5^{N-t_1} \cdot 5^{t_2}) \pmod{4N})\right] \\ &= \left[f(5^{t_2-t_1} \pmod{4N})\right]_{t_1,t_2=0,1,\cdots,N-1} \end{aligned}$$

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Thus, C is a circular convolution matrix , and the input and output reordering processes can be achieved by two permutation matrices (say P_1 and P_2), respectively.

According to Wang [3] , there are four types of DCT definitions and the computation of the four types of DCT can be reduced to the computation of the type-IV DCT. Therefore , the fast algorithms for any type of the DCT depends only on the computation of type-IV DCT .

3 Proposed Algorithm for Computing Type-IV DCT

From [3] , the type-IV DCT can be rewritten as

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot \cos(\frac{2\pi(2n+1)(2k+1)}{8N})_{\ k=0,1,\cdots,N-1} \quad (1)$$

We will prove that the work for computing N-point type-IV DCT can be achieved by computing an N-point skew circular convolution and permutations using the following processes.

STEP 1. Extend $\left[C_N^{IV}\right]$ (the notations defined in [3] are adopted in the rest of this paper for simplicity) as follows:

$$Y(k) = \sum_{n=0}^{2N-1} y(n) \cdot \cos(\frac{2\pi(2n+1)(2k+1)}{8N})_{\ k=0,1,\cdots,2N-1}$$

where

$$y(n) = \begin{cases} x(n) & 0 \le n \le N-1 \\ 0 & N \le n \le 2N-1 \end{cases}$$

and then

$$X(k) = Y(k) \quad for\ k = 0,1,\cdots,N-1$$

STEP 2. Reorder the input and output sequence.

Similar to the previous work [2] ,the above 2N-point transform can be rewritten as :

$$\tilde{Y}(k) = \sum_{n=0}^{2N-1} \tilde{y}(n) \cos(\frac{2\pi(4n+1)(4k+1)}{8N}) \quad k = 0,1,\cdots,2N-1 \quad (2)$$

where

$$\begin{cases} \tilde{y}(n) = y(2n) \\ \tilde{y}(2N-n-1) = y(2n+1) \end{cases} \quad n = 0,1,\cdots,N-1$$

and

$$\begin{cases} \tilde{Y}(k) = Y(2k) \\ \tilde{Y}(2N-k-1) = Y(2k+1) \end{cases} \quad k = 0,1,\cdots,N-1$$

STEP 3. The matrix representation of (2) is

$$G_{2N} = \left[\cos(\frac{2\pi(4n+1)(4k+1)}{8N})\right]_{n,k=0,1,\cdots,2N-1}$$

From **corollary 1** , it follows that the following equation hold :

$$G_{2N} = P_{2N} C_{2N} Q_{2N}$$

where P_{2N} and Q_{2N} are two permutation matrices and C_{2N} is a 2N-point circular convolution matrix and can be represented as :

$$C_{2N} = \left[\cos(\frac{2\pi \cdot 5^{(j-i)}}{8N})\right]_{i,j=0,1,\cdots,2N-1}$$

STEP 4. Since

$$\cos(\frac{2\pi \cdot 5^{n+N}}{8N}) = -\cos(\frac{2\pi \cdot 5^n}{8N}) \quad (3)$$

(by **theorem 2** , details are in the Appendix)

$$\begin{aligned} C_{2N} &= \begin{bmatrix} H_N & -H_N \\ -H_N & H_N \end{bmatrix} \\ &= \begin{bmatrix} I_N \\ -I_N \end{bmatrix} \cdot [H_N] \cdot \begin{bmatrix} I_N & -I_N \end{bmatrix} \end{aligned} \quad (4)$$

where H_N is the so-called N-point skew circular convolution matrix.

By (4) , it follows that the computation of C_{2N} can be achieved by calculating an N-point skew circular convolution and additional N additions/substractions.

Remark 1: In step 1,we extend the input sequence with N zeros , therefore , the N additions/substractions in step 4 can be replaced by the "sign change" operations.

Remark 2: In step 1, we only need half of the output sequence. Therefore, the post-operations of (4) can be achieved by "sign change" operations.

According to the above discussion , we can conclude that the computation of an N-point type-IV DCT can be achieved by an N-point skew circular convolution with some permutations and sign changes of input and output sequences. From [7] , an N-point skew circular convolution (or the polynomial procudt modulo $z^n + 1$) can be computed by means of the generalized number theoretical transform (GNTT) with only N multiplications.

4 Algorithms for Discrete Sinusoidal Transforms

According to the previous works [3,8,9] , the relations between some well-known discrete sinusoidal transforms (DFT , DHT (discrete Hartley transform) , DCT and DST (discrete Sine transform)) are very clear , and listing as follows :

$$DFT(N) \implies \begin{cases} DFT(\frac{N}{2}) \\ two\ DCT^{II}(\frac{N}{4}) \end{cases} \quad (5)$$

$$DHT(N) \implies \begin{cases} DHT(\frac{N}{2}) \\ two\ DCT^{II}(\frac{N}{4}) \end{cases} \quad (6)$$

$$DCT^{II}(N) \implies \begin{cases} DCT^{II}(\frac{N}{2}) \\ DCT^{IV}(\frac{N}{2}) \end{cases} \quad (7)$$

$$DST^{II}(N) \implies \begin{cases} DST^{II}(\frac{N}{2}) \\ DCT^{IV}(\frac{N}{2}) \end{cases} \quad (8)$$

Based on the discussion of section 3 , we can compute the $DCT^{IV}(N)$ using N-point skew circular convolution ($SCC(N)$). Therefore, the following result can be derived by the recursive formulas (5)-(8).

$$\begin{aligned} DFT(N) &\implies two\ SCC(\tfrac{N}{4}), two\ SCC(\tfrac{N}{8}), \cdots \\ DHT(N) &\implies two\ SCC(\tfrac{N}{4}), two\ SCC(\tfrac{N}{8}), \cdots \\ DCT^{II}(N) &\implies SCC(\tfrac{N}{2}), SCC(\tfrac{N}{4}), \cdots \\ DST^{II}(N) &\implies SCC(\tfrac{N}{2}), SCC(\tfrac{N}{4}), \cdots \end{aligned}$$

with some interblock additions and sign changes.

Remark 3: Although the DFT is defined in the complex number system, we can still derive an algorithm using only real SCCs.

5 Conclusion

In this paper,we have developed an algorithm which transfers the problem of N-point type-IV DCT into the problem of N-point skew circular convolution.In theory, this algorithm can achieve the lower bound of the number of multiplications according to the minimum complexity polynomial algorithms. In practice , by means of the number theoretical transform,we can compute $\left[C_N^{IV}\right]$ using only N multiplications, or we can use a filter-type structure that is very suitable for the VLSI implementation.

According to the relations between type-IV DCT and other famous transforms , we have mentioned that the other discrete sinusoidal transforms can be computed by means of the combination of some SCCs of smaller size,and possess the same advantages in both theoretical and practical as the type-IV DCT.

Appendix

[proof of eq. 3]:

$$\begin{aligned} & \cos(\tfrac{2\pi \cdot 5^{(n+N)}}{8N}) \\ = \ & \cos(\tfrac{2\pi \cdot 5^n \cdot (1+4N)}{8N}) \quad \text{(by theorem 3)} \\ = \ & \cos(5^n \pi + \tfrac{2\pi \cdot 5^n}{8N}) \\ = \ & \cos(\pi + \tfrac{2\pi \cdot 5^n}{8N}) \\ = \ & -\cos(\tfrac{2\pi \cdot 5^n}{8N}) \end{aligned}$$

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