

Projected Gradient Methods for Non-negative Matrix Factorization

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Abstract

Non-negative matrix factorization (NMF) minimizes a bound-constrained problem. While in both theory and practice bound-constrained optimization is well studied, so far no study formally applies its techniques to NMF. In this paper, we propose two projected-gradient methods for NMF. The new methods have sound optimization properties. We discuss efficient implementations and show that one proposed method converges faster than the popular multiplicative update approach. A simple MATLAB code is provided.

1 Introduction

Non-negative matrix factorization (NMF) (Paatero and Tapper, 1994; Lee and Seung, 1999) is useful to find a representation of non-negative data. Given an $n \times m$ data matrix V with $V_{ij} \geq 0$ and a pre-specified positive integer $r < \min(n, m)$, NMF finds two non-negative matrices $W \in R^{n \times r}$ and $H \in R^{r \times m}$ so that

$$V \approx WH.$$

If each column of V represents an object, NMF approximates it by a linear combination of r “basis” columns in W . NMF has been applied to many areas such as finding basis vectors of images (Lee and Seung, 1999), document clustering (Xu et al., 2003), and molecular pattern discovery (Brunet et al., 2004), etc. Donoho and Stodden (2004) addressed theoretical issues of the NMF approach.

The usual approach to find W and H is by minimizing the difference between V and WH :

$$\begin{aligned} \min_{W,H} \quad & f(W, H) \equiv \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m (V_{ij} - (WH)_{ij})^2 \\ \text{subject to} \quad & W_{ia} \geq 0, H_{bj} \geq 0, \quad \forall i, a, b, j. \end{aligned} \quad (1)$$

In optimization, inequalities upper- and lower-bounding variables are referred to as bound constraints. Hence (1) is a standard bound-constrained optimization problem. We also note that

$$\sum_{i=1}^n \sum_{j=1}^m (V_{ij} - (WH)_{ij})^2 = \|V - WH\|_F^2,$$

where $\|\cdot\|_F$ is the Frobenius norm.

The most popular approach to solve (1) is a multiplicative update algorithm by Lee and Seung (2001). It is simple to implement and often gives good results. At each iteration this method multiplies elements of W and H by some factors. As zero elements are not updated, all components in W and H are kept strictly positive throughout iterations. Such a strategy is contrary to traditional bound-constrained optimization methods, which usually allow iterations to have bounded elements (i.e., zero elements here). Up to now no study has formally applied bound-constrained optimization techniques to NMF. This paper will investigate such methods in detail. Some earlier NMF work requires W 's column sums are all ones: $\sum_{i=1}^n W_{ia} = 1, \forall a = 1, \dots, r$. The function value does not change as $f(WD, D^{-1}H) = f(W, H)$ for any $r \times r$ positive diagonal matrix D . With such additional constraints, (1) is not a bounded problem. As adding them may complicate the optimization procedures, here we do not consider this modification.

Among existing bound-constrained optimization techniques, projected gradients are a simple and effective one. Though several papers used this method for NMF (Hoyer, 2002; Chu et al., 2004; Shepherd, 2004), there is neither a systematic study nor an easy implementation competitive with the multiplicative update. This paper gives a comprehensive study on using projected gradient methods for NMF. Several useful modifications lead to efficient implementations. While the

multiplicative update method still lacks convergence results, our proposed methods have sound optimization properties. We experimentally show that one proposed method converges faster than the multiplicative update. This new method is thus an appealing approach to solve NMF. We provide a complete MATLAB implementation.

Another NMF optimization formula minimizes the (generalized) Kullback-Leibler divergence between V and WH (Lee and Seung, 1999):

$$\begin{aligned} \min_{W,H} \quad & \sum_{i=1}^n \sum_{j=1}^m \left(V_{ij} \log \frac{V_{ij}}{(WH)_{ij}} - V_{ij} + (WH)_{ij} \right) \\ \text{subject to} \quad & W_{ia} \geq 0, H_{bj} \geq 0, \forall i, a, b, j. \end{aligned}$$

Strictly speaking, this formula is not a bound-constrained problem, which requires the objective function to be well-defined at any point of the bounded region. Now the log function is not well-defined if $V_{ij} = 0$ or $(WH)_{ij} = 0$. Hence we do not consider this formulation.

This paper is organized as follows. Section 2 discusses existing approaches for NMF problem (1), and presents several new properties not mentioned before. Section 3 introduces projected gradient methods in bound-constrained optimization. Section 4 investigates specific but essential modifications on applying projected gradients to NMF. Stopping conditions in an NMF code are discussed in Section 5. Experiments on synthetic and real data sets are in Section 6. Discussion and conclusions are in Section 7. Appendix B gives the MATLAB code of one proposed approach. All sources used in this paper are available at <http://www.csie.ntu.edu.tw/~cjlin/nmf>.

2 Existing Methods and New Properties

Many methods are available for NMF. Earlier Paatero (1999) had some discussions, but bound constraints are not rigorously handled. A more recent and complete survey is Chu et al. (2004). This section briefly discusses some existing methods and presents several observations not mentioned before.

To begin, we need certain properties of the NMF problem (1). The gradient

of the function $f(W, H)$ consists of two parts:

$$\nabla_W f(W, H) = (WH - V)H^T \text{ and } \nabla_H f(W, H) = W^T(WH - V), \quad (2)$$

which are respectively partial derivatives to elements in W and H . From the Karush-Kuhn-Tucker (KKT) optimality condition (e.g., (Bertsekas, 1999)), (W, H) is a stationary point of (1) if and only if

$$\begin{aligned} W_{ia} &\geq 0, H_{bj} \geq 0, \\ \nabla_W f(W, H)_{ia} &\geq 0, \nabla_H f(W, H)_{bj} \geq 0, \\ W_{ia} \cdot \nabla_W f(W, H)_{ia} &= 0, \text{ and } H_{bj} \cdot \nabla_H f(W, H)_{bj} = 0, \forall i, a, b, j. \end{aligned} \quad (3)$$

Optimization methods for NMF produce a sequence $\{W^k, H^k\}_{k=1}^\infty$ of iterations. Now (1) is non-convex and may possess several local minima. A common misunderstanding is that easily limit points of the sequence are local minima. In fact, most non-convex optimization methods guarantee only the stationarity of limit points. Such a property is still useful as any local minimum must be a stationary point.

2.1 Multiplicative Update

The most used approach to minimize (1) is a simple multiplicative update by Lee and Seung (2001):

Algorithm 1 Multiplicative Update

1. Initialize $W_{ia}^1 > 0, H_{bj}^1 > 0, \forall i, a, b, j$.
2. For $k = 1, 2, \dots$

$$W_{ia}^{k+1} = W_{ia}^k \frac{(V(H^k)^T)_{ia}}{(W^k H^k (H^k)^T)_{ia}}, \quad \forall i, a, \quad (4)$$

$$H_{bj}^{k+1} = H_{bj}^k \frac{((W^{k+1})^T V)_{bj}}{((W^{k+1})^T W^{k+1} H^k)_{bj}}, \quad \forall b, j. \quad (5)$$

This algorithm is a fixed-point type method: If $(W^k H^k (H^k)^T)_{ia} \neq 0$ and $W_{ia}^{k+1} = W_{ia}^k > 0$, then

$$(V(H^k)^T)_{ia} = (W^k H^k (H^k)^T)_{ia} \quad \text{implies} \quad \nabla_W f(W^k, H^k)_{ia} = 0,$$

which is part of the KKT condition (3). Lee and Seung (2001) shown that the function value is non-increasing after every update:

$$f(W^{k+1}, H^k) \leq f(W^k, H^k) \quad \text{and} \quad f(W^{k+1}, H^{k+1}) \leq f(W^{k+1}, H^k). \quad (6)$$

They claimed that the limit of the sequence $\{W^k, H^k\}_{k=1}^{\infty}$ is a stationary point (i.e., a point satisfying the KKT condition (3)). However, Gonzales and Zhang (2005) indicated that this claim is wrong as having (6) may not imply the convergence. Therefore, this multiplicative update method still lacks sound optimization properties.

To have Algorithm 1 well-defined, one must ensure that denominators in (4) and (5) are strictly positive. Moreover, if $W_{ia}^k = 0$ at the k th iteration, then $W_{ia} = 0$ at all subsequent iterations. Thus one should keep $W_{ia}^k > 0$ and $H_{bj}^k > 0$, $\forall k$. The following theorem discusses when this property holds:

Theorem 1 *If V has neither zero column nor row, and $W_{ia}^1 > 0$ and $H_{bj}^1 > 0$, $\forall i, a, b, j$, then*

$$W_{ia}^k > 0 \text{ and } H_{bj}^k > 0, \forall i, a, b, j, \forall k \geq 1. \quad (7)$$

The proof is straightforward, and is left in Appendix A.

If V has zero columns or rows, a division by zero may occur. Even if Theorem 1 holds, denominators close to zero may still cause numerical problems. Some work such as (Piper et al., 2004) proposed adding a small positive number in the denominators of (4)-(5). We observed numerical difficulties in few situations and give more discussions in Section 6.3.

Regarding the computational complexity, $V(H^k)^T$ and $(W^{k+1})^T V$ in (4) and (5) are both $O(nmr)$ operations. One can calculate the denominator in (4) by either

$$(WH)H^T \quad \text{or} \quad W(HH^T). \quad (8)$$

The former takes $O(nmr)$ operations, but the latter costs $O(mr^2)$. As $r < n$, the latter is better. Similarly for (5), $(W^T W)H$ should be used. This discussion indicates the importance of having fewer $O(nmr)$ operations (i.e., WH , $W^T V$, or VH^T) in any NMF code.

In summary, the overall cost of Algorithm 1 is

$$\#iterations \times O(nmr).$$

All time complexity analysis in this paper assumes that V, W , and H are implemented as dense matrices.

2.2 Alternating Non-negative Least Squares

From the non-increasing property (6), Algorithm 1 is a special case of a general framework, which alternatively fixes one matrix and improves the other:

Find W^{k+1} such that $f(W^{k+1}, H^k) \leq f(W^k, H^k)$, and

Find H^{k+1} such that $f(W^{k+1}, H^{k+1}) \leq f(W^{k+1}, H^k)$.

The extreme situation is to obtain the best point (Paatero, 1999; Chu et al., 2004):

Algorithm 2 Alternating non-negative least squares

1. Initialize $W_{ia}^1 \geq 0, H_{bj}^1 \geq 0, \forall i, a, b, j$.
2. For $k = 1, 2, \dots$

$$W^{k+1} = \arg \min_{W \geq 0} f(W, H^k), \quad (9)$$

$$H^{k+1} = \arg \min_{H \geq 0} f(W^{k+1}, H). \quad (10)$$

This approach is the “block coordinate descent” method in bound-constrained optimization (Bertsekas, 1999), where sequentially one block of variables is minimized under corresponding constraints and the remaining blocks are fixed. Here we have the simplest case of only two block variables W and H .

We refer to (9) or (10) as a sub-problem in Algorithm 2. When one block of variables is fixed, a sub-problem is indeed the collection of several non-negative least square problems: From (10),

$$H^{k+1}\text{'s } j\text{th column} = \min_{\mathbf{h} \geq 0} \|\mathbf{v} - W^{k+1}\mathbf{h}\|^2, \quad (11)$$

where \mathbf{v} is the j th column of V and \mathbf{h} is a vector variable. Chu et al. (2004) suggest projected Newton’s methods such as Lawson and Hanson (1974) to solve

each problem (11). Clearly, solving sub-problems (9) and (10) per iteration could be more expensive than the simple update in Algorithm 1. Then Algorithm 2 may be slower even though we expect it better decreases the function value at each iteration. Efficient methods to solve sub-problems are thus essential. Section 4.1 proposes using project gradients and discusses why they are suitable for solving sub-problems in Algorithm 2.

Regarding the convergence, one may think that it is a trivial result. For example, Paatero (1999) directly stated that for the alternating non-negative least square approach, no matter how many blocks of variables we have, the convergence is guaranteed. However, this issue deserves some attention. Past convergence analysis for “block coordinate descent” methods requires sub-problems to have unique solutions (Powell, 1973; Bertsekas, 1999), but this property does not hold here: Sub-problems (9) and (10) are convex, but not strictly convex. Hence they may have multiple optimal solutions. For example, when H^k is the zero matrix, any W is optimal for (9). Fortunately, for the case of *two blocks*, Grippo and Sciandrone (2000) showed that this uniqueness condition is not needed. Directly from (Grippo and Sciandrone, 2000, Corollary 2), we have the following convergence result:

Theorem 2 *Any limit point of the sequence $\{W^k, H^k\}$ generated by Algorithm 2 is a stationary point of (1).*

The remaining issue is whether the sequence $\{W^k, H^k\}$ possesses at least one limit point (i.e., there is at least one convergent subsequence). In optimization analysis, this property often comes from the boundedness of the feasible region, but our region under constraints $W_{ia} \geq 0$ and $H_{bj} \geq 0$ is unbounded. One can easily add a large upper bound to all variables in (1). As the modification still leads to a bound-constrained problem, Algorithm 2 can be applied and Theorem 2 holds. In contrast, it is unclear how to easily modify the multiplicative update rules if there are upper bounds in (1).

In summary, contrary to Algorithm 1, which still lacks convergence results, Algorithm 2 possesses nice optimization properties.

2.3 Gradient Approaches

In (Chu et al., 2004, Section 3.3), several gradient-type approaches were mentioned. Here we briefly discuss those selecting the step size along the negative gradient direction. By defining

$$W_{ia} = E_{ia}^2 \text{ and } H_{bj} = F_{bj}^2,$$

Chu et al. (2004) reformulate (1) as an unconstrained optimization problem of variables E_{ia} and F_{bj} . Then standard gradient descent methods can be applied. The same authors also mention that Shepherd (2004) uses

$$\begin{aligned} W^{k+1} &= \max(0, W^k - \alpha_k \nabla_W f(W^k, H^k)), \\ H^{k+1} &= \max(0, H^k - \alpha_k \nabla_H f(W^k, H^k)), \end{aligned}$$

where α_k is the step size. This approach is already a projected gradient method. However, in the above references, details are not discussed.

3 Projected Gradients for Bound-constrained Optimization

We consider the following standard form of bound-constrained optimization:

$$\begin{aligned} \min_{\mathbf{x} \in R^n} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & l_i \leq x_i \leq u_i, \quad i = 1, \dots, n, \end{aligned}$$

where $f(\mathbf{x}) : R^n \rightarrow R$ is a continuously differentiable function, and \mathbf{l} and \mathbf{u} are lower and upper bounds, respectively. Projected gradient methods update the current solution \mathbf{x}^k to \mathbf{x}^{k+1} by the following rule:

$$\mathbf{x}^{k+1} = P[\mathbf{x}^k - \alpha^k \nabla f(\mathbf{x}^k)],$$

where

$$P[x_i] = \begin{cases} x_i & \text{if } l_i < x_i < u_i, \\ u_i & \text{if } x_i \geq u_i, \\ l_i & \text{if } x_i \leq l_i, \end{cases}$$

maps a point back to the bounded feasible region. Variants of projected gradient methods differ on selecting the step size α^k . We consider a simple and effective

one called “Armijo rule along the projection arc” in Bertsekas (1999), which is originated from Bertsekas (1976). It is illustrated in Algorithm 3.

Algorithm 3 Projected gradient for bound-constrained optimization

1. Given $0 < \beta < 1, 0 < \sigma < 1$. Initialize any feasible \mathbf{x}^1 .

2. For $k = 1, 2, \dots$

$$\mathbf{x}^{k+1} = P[\mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k)],$$

where $\alpha_k = \beta^{t_k}$, and t_k is the first non-negative integer t for which

$$f(\mathbf{x}^{k+1}) - f(\mathbf{x}^k) \leq \sigma \nabla f(\mathbf{x}^k)^T (\mathbf{x}^{k+1} - \mathbf{x}^k). \quad (12)$$

The condition (12), used in most proofs of projected gradient methods, ensures the sufficient decrease of the function value per iteration. By trying the step sizes $1, \beta, \beta^2, \dots$, Bertsekas (1976) proved that $\alpha_k > 0$ satisfying (12) always exists and every limit point of $\{\mathbf{x}^k\}_{k=1}^\infty$ is a stationary point. A common choice of σ is 0.01, and we consider $\beta = 1/10$ in this paper. Searching α_k is the most time consuming operation in Algorithm 3, so one should check as few step sizes as possible. Since α_{k-1} and α_k may be similar, a trick in (Lin and Moré, 1999) uses α_{k-1} as the initial guess and then either increases or decreases it in order to find the largest β^{t_k} satisfying (12). Moreover, with non-negative t_k , Algorithm 4 may be too conservative by restricting $\alpha_k \leq 1$. Sometimes, a larger step more effectively projects variables to bounds at one iteration. Algorithm 4 implements a better initial guess of α at each iteration and allows α to be larger than one.

Algorithm 4 An improved projected gradient method

1. Given $0 < \beta < 1, 0 < \sigma < 1$. Initialize any feasible \mathbf{x}^1 . Set $\alpha_0 = 1$.

2. For $k = 1, 2, \dots$

(a) Assign $\alpha_k \leftarrow \alpha_{k-1}$

(b) If α_k satisfies (12), repeatedly increase it by

$$\alpha_k \leftarrow \alpha_k / \beta$$

until either α_k satisfies (12) or $\mathbf{x}(\alpha_k / \beta) = \mathbf{x}(\alpha_k)$.

Else repeatedly decrease α_k by

$$\alpha_k \leftarrow \alpha_k \cdot \beta$$

until α_k satisfies (12).

(c) Set

$$\mathbf{x}^{k+1} = P[\mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k)],$$

The convergence was proved in, for example, Calamai and Moré (1987). One may think that finding α with the largest function reduction may lead to faster convergence:

$$\alpha_k \equiv \arg \min_{\alpha \geq 0} f(P[\mathbf{x}^k - \alpha \nabla f(\mathbf{x}^k)]). \quad (13)$$

The convergence of selecting such α_k was proved in McCormick and Tapia (1972). However, (13) is a piecewise function of α , which is difficult to be minimized.

A major obstacle for minimizing bounded problems is to identify free (i.e., $l_i < x_i < u_i$) and active (i.e., $x_i = l_i$ or u_i) components at the convergent stationary point. Projected gradients are considered effective for doing so since they are able to add several active variables at one single iteration. However, once these sets have been (almost) identified, in a sense the problem reduces to unconstrained and the slow convergence of gradient-type methods may occur. We will explain in Section 4.1 that for NMF problems, this issue may not be serious.

4 Projected Gradient Methods for NMF

We apply projected gradient methods to NMF in two situations. The first solves non-negative least square problems discussed in Section 2.2. The second directly minimizes (1). Both approaches possess convergence properties following from Theorem 2 and Calamai and Moré (1987), respectively. Several modifications specific to NMF will be presented.

4.1 Alternating Non-negative Least Squares Using Projected Gradients

Section 2.2 indicated that Algorithm 2 relies on efficiently solving sub-problems (9) and (10), each of which is a bound-constrained problem. We propose using project gradients to solve them.

The sub-problem (10) consists of m independent non-negative least square problems (11), so one could solve them separately, a situation suitable for parallel environments. However, in a serial setting, we think that treating them together is better:

1. These problems are closely related as they share the same constant matrices V and W^{k+1} in (11).
2. Working on the whole H but not its individual columns implies all operations are matrix-based. Since finely tuned numerical linear algebra code has better speed-up on matrix than on vector operations, we can thus save computational time.

For an easier description of our method, we focus on (10) and rewrite it as

$$\begin{aligned} \min_H \quad & \bar{f}(H) \equiv \frac{1}{2} \|V - WH\|_F^2 \\ \text{subject to} \quad & H_{bj} \geq 0, \quad \forall b, j. \end{aligned} \tag{14}$$

Both V and W are constant matrices in (14). If we concatenate H 's columns to

a vector $\text{vec}(H)$, then

$$\begin{aligned}\bar{f}(H) &= \frac{1}{2}\|V - WH\|_F^2 \\ &= \frac{1}{2}\text{vec}(H)^T \begin{bmatrix} W^T W & & \\ & \ddots & \\ & & W^T W \end{bmatrix} \text{vec}(H) + H\text{'s linear terms.}\end{aligned}$$

The Hessian matrix (i.e., second derivative) of $\bar{f}(H)$ is block diagonal and each block $W^T W$ is an $r \times r$ positive semi-definite matrix. As $W \in R^{n \times r}$ and $r \ll n$, $W^T W$ and hence the Hessian matrix tend to be well-conditioned, a good property for optimization algorithms. Thus gradient-based methods may converge fast enough. A further investigation of this conjecture is in the experiment section 6.2.

The high cost of solving the two sub-problems (9) and (10) at each iteration is a concern. It is thus essential to analyze the time complexity and find efficient implementations. Each sub-problem now requires an iterative procedure, whose iterations are referred to as sub-iterations. When using Algorithm 4 to solve (14), we must maintain the gradient

$$\nabla \bar{f}(H) = W^T(WH - V)$$

at each sub-iteration. Following the discussion near Eq. (8), one should calculate it by $(W^T W)H - W^T V$. Constant matrices $W^T W$ and $W^T V$ can be computed respectively in $O(nr^2)$ and $O(nmr)$ before sub-iterations.

The main computational task per sub-iteration is to find a step size α such that the sufficient decrease condition (12) is satisfied. Assume \bar{H} is the current solution. To check if

$$\tilde{H} \equiv P[\bar{H} - \alpha \nabla \bar{f}(\bar{H})],$$

satisfies (12), calculating $\bar{f}(\tilde{H})$ takes $O(nmr)$ operations. If there are t such \tilde{H} 's, the cost $O(tnmr)$ is huge. We propose the following strategy to reduce the cost: For a quadratic function $f(\mathbf{x})$ and any vector \mathbf{d} ,

$$f(\mathbf{x} + \mathbf{d}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T \mathbf{d} + \frac{1}{2} \mathbf{d}^T \nabla^2 f(\mathbf{x}) \mathbf{d}. \quad (15)$$

Hence for two consecutive iterations $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$, (12) can be written as

$$(1 - \sigma) \nabla f(\bar{\mathbf{x}})^T (\tilde{\mathbf{x}} - \bar{\mathbf{x}}) + \frac{1}{2} (\tilde{\mathbf{x}} - \bar{\mathbf{x}})^T \nabla^2 f(\bar{\mathbf{x}}) (\tilde{\mathbf{x}} - \bar{\mathbf{x}}) \leq 0.$$

Now $\bar{f}(H)$ defined in (14) is quadratic, so (12) becomes

$$(1 - \sigma)\langle \nabla \bar{f}(\bar{H}), \tilde{H} - \bar{H} \rangle + \frac{1}{2}\langle \tilde{H} - \bar{H}, (W^T W)(\tilde{H} - \bar{H}) \rangle \leq 0, \quad (16)$$

where $\langle \cdot, \cdot \rangle$ is the sum of the component-wise product of two matrices. The major operation in (16) is the matrix product $(W^T W) \cdot (\tilde{H} - \bar{H})$, which takes $O(mr^2)$. Thus the cost $O(tnmr)$ of checking (12) is significantly reduced to $O(tmr^2)$. With the cost $O(nmr)$ for calculating $W^T V$ in the beginning, the complexity of using Algorithm 4 to solve the sub-problem (14) is

$$O(nmr) + \#\text{sub-iterations} \times O(tmr^2),$$

where t is the average number of checking (12) at each sub-iteration.

The pseudo code for optimizing (14) is in Appendix B.2. We can use the same procedure to obtain W^{k+1} by rewriting (10) as a form similar to (14):

$$\bar{f}(W) \equiv \frac{1}{2}\|V^T - H^T W^T\|_F^2,$$

where V^T and H^T are constant matrices.

The overall cost to solve (1) is

$$\#\text{iterations} \times (O(nmr) + \#\text{sub-iterations} \times O(tmr^2 + tnr^2)). \quad (17)$$

At each iteration there are two $O(nmr)$ operations: $V(H^k)^T$ and $(W^{k+1})^T V$, the same as those in the multiplicative update. If t and $\#\text{sub-iterations}$ are small, this method is competitive.

To reduce the number of sub-iterations, a simple but useful technique is to warm start the solution procedure of each sub-problem. (W^k, H^k) may not change much near the convergent point, so W^k is an effective initial point for solving (9).

4.2 Directly Applying Projected Gradients to NMF

We may directly apply Algorithm 4 to solve (1). Similar to solving non-negative least square problems in Section 4.1, the most expensive operation is on checking the sufficient decrease condition (12). From the current solution (\bar{W}, \bar{H}) , we simultaneously update both matrices to (\tilde{W}, \tilde{H}) :

$$(\tilde{W}, \tilde{H}) \equiv P [(\bar{W}, \bar{H}) - \alpha (\nabla_W f(\bar{W}, \bar{H}), \nabla_H f(\bar{W}, \bar{H}))].$$

As $f(W, H)$ is not a quadratic function, (15) does not hold. Hence the trick (16) cannot be applied to save the computational time. Then calculating $f(\tilde{W}, \tilde{H}) = \frac{1}{2}\|V - \tilde{W}\tilde{H}\|_F^2$ takes $O(nmr)$ operations. The total computational cost is

$$\#\text{iterations} \times O(tnmr),$$

where t is the average number of (12) checked per iteration.

Given any random initial (W^1, H^1) , if $\|V - W^1 H^1\|_F^2 > \|V\|_F^2$, very often after the first iteration $W^2 = 0$ and $H^2 = 0$ cause the algorithm to stop. The solution $(0, 0)$ is a useless stationary point of (1). A simple remedy is to find a new initial point (W^1, \bar{H}^1) so that $f(W^1, \bar{H}^1) < f(0, 0)$. By solving $\bar{H}^1 = \arg \min_{H \geq 0} f(W^1, H)$ using the procedure described in Section 4.1, we have

$$\|V - W^1 \bar{H}^1\|_F^2 \leq \|V - W^1 \cdot 0\|_F^2 = \|V\|_F^2.$$

In general the strict inequality holds, so $f(W^1, \bar{H}^1) < f(0, 0)$.

5 Stopping Conditions

In all algorithms mentioned so far, we did not specify when the procedure should stop. Several implementations of the multiplicative update (e.g., (Hoyer, 2004)) have an infinite loop, which must be interrupted by users after a time or iteration limit. Some (e.g., Brunet (2004)) check the difference between recent iterations. If the difference is small enough, then the procedure stops. However, such a stopping condition does not reveal whether a solution is close to a stationary point or not. In addition to a time or iteration limit, standard conditions to check the stationarity should also be included in NMF software. Moreover, in alternating least squares, each sub-problem involves an optimization procedure, which needs a stopping condition as well.

In bound-constrained optimization, the following common condition checks if a point \mathbf{x}^k is close to a stationary point (Lin and Moré, 1999):

$$\|\nabla^P f(\mathbf{x}^k)\| \leq \epsilon \|\nabla f(\mathbf{x}^1)\|, \quad (18)$$

where $\nabla^P f(\mathbf{x}^k)$ is the *projected gradient* defined as

$$\nabla^P f(\mathbf{x})_i \equiv \begin{cases} \nabla f(\mathbf{x})_i & \text{if } l_i < x_i < u_i, \\ \min(0, \nabla f(\mathbf{x})_i) & \text{if } x_i = l_i, \\ \max(0, \nabla f(\mathbf{x})_i) & \text{if } x_i = u_i. \end{cases} \quad (19)$$

This condition follows from an equivalent form of the KKT condition for bounded problems: $l_i \leq x_i \leq u_i, \forall i$, and

$$\|\nabla^P f(\mathbf{x})\| = \mathbf{0}.$$

For NMF, (18) becomes

$$\|\nabla^P f(W^k, H^k)\|_F \leq \epsilon \|\nabla f(W^1, H^1)\|_F. \quad (20)$$

For alternating least squares, each sub-problem (9) or (10) requires a stopping condition as well. Ideally, the condition for them should be related to the “global” one for (1), but a suitable condition is not obvious. For example, we cannot use the same stopping tolerance in (20) for sub-problems: A user may specify $\epsilon = 0$ and terminate the code after a certain time or iteration limit. Then the same $\epsilon = 0$ in solving the first sub-problem will cause Algorithm 2 to keep running at the first iteration. We thus use the following stopping conditions for sub-problems: The returned matrices W^{k+1} and H^{k+1} from the iterative procedures of solving (9) and (10) should respectively satisfy

$$\begin{aligned} \|\nabla_W^P f(W^{k+1}, H^k)\|_F &\leq \bar{\epsilon}_W, \text{ and} \\ \|\nabla_H^P f(W^{k+1}, H^{k+1})\|_F &\leq \bar{\epsilon}_H, \end{aligned}$$

where we set

$$\bar{\epsilon}_W = \bar{\epsilon}_H \equiv \max(10^{-3}, \epsilon) \|\nabla f(W^1, H^1)\|_F$$

in the beginning and ϵ is the tolerance in (20). If the projected gradient method for solving (9) stops without any iterations, we decrease the stopping tolerance by

$$\bar{\epsilon}_W \leftarrow \bar{\epsilon}_W / 10. \quad (21)$$

For (10), $\bar{\epsilon}_H$ is reduced by a similar way.

Table 1: Results of running synthetic data sets (from small to large) under various stopping tolerances. We present the average time (in seconds), number of iterations, and objective values of using 30 initial points. Approaches with the smallest time or objective values are bold-faced. Note that when $\epsilon = 10^{-5}$, `mult` and `pgrad` often exceed the iteration limit of 8,000.

ϵ	Time				#iterations				Objective values			
	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
<code>mult</code>	0.10	1.22	2.60		698	4651	7639		390.4	389.3	389.3	
<code>alspgrad</code>	0.03	0.10	0.45	0.97	6	26	100	203	412.9	392.8	389.2	389.1
<code>pgrad</code>	0.05	0.24	0.68		53	351	1082		401.6	389.9	389.1	
<code>lsqnonneg</code>	6.32	27.76	57.57		23	96	198		391.1	389.1	389.0	

(a) $m = 25, r = 5, n = 125$.

ϵ	Time				#iterations				Objective values			
	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
<code>mult</code>	0.16	14.73	21.53		349	6508	8000		1562.1	1545.7	1545.6	
<code>alspgrad</code>	0.03	0.13	0.99	5.51	4	14	76	352	1866.6	1597.1	1547.8	1543.5
<code>pgrad</code>	0.38	3.17	10.29		47	1331	4686		1789.4	1558.4	1545.5	

(b) $m = 50, r = 10, n = 250$.

ϵ	Time				#iterations				Objective values			
	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
<code>mult</code>	0.41	8.28	175.55		170	2687	8000		6535.2	6355.7	6342.3	
<code>alspgrad</code>	0.02	0.21	1.09	10.02	2	8	31	234	9198.7	6958.6	6436.7	6332.9
<code>pgrad</code>	0.60	2.88	35.20		2	200	3061		8141.1	6838.7	6375.0	

(c) $m = 100, r = 20, n = 500$.

6 Experiments

We compare four methods discussed in this paper and refer to them as the following

1. `mult`: the multiplicative update method described in Section 2.1.
2. `alspgrad`: alternating non-negative least squares using projected gradients for each sub-problem (Section 4.1).
3. `pgrad`: a direct use of projected gradients on (1) (Section 4.2).
4. `lsqnonneg`: Using MATLAB command `lsqnonneg` to solve m problems (11) in alternating least squares.

All implementations were in MATLAB (<http://www.mathworks.com>). Experiments were conducted on an Intel Xeon 2.8GHz computer. Results of using synthetic and real data are in the following subsections. All sources for experiments are available at <http://www.csie.ntu.edu.tw/~cjlin/nmf>.

6.1 Synthetic Data

We consider three problem sizes:

$$(m, r, n) = (25, 5, 25), (50, 10, 250), \text{ and } (100, 20, 500).$$

The matrix V is randomly generated by the normal distribution (mean 0 and standard deviation 1)

$$V_{ij} = |N(0, 1)|.$$

The initial (W^1, H^1) is constructed by the same way, and all four methods share the same initial point. These methods may converge to different points due to the non-convexity. To have a fair comparison, for the same V we try 30 different initial (W^1, H^1) , and report the average results. As synthetic data may not resemble practical problems, we leave detailed analysis of the proposed algorithms in Section 6.2, which considers real data.

We set ϵ in (20) to be 10^{-3} , 10^{-4} , 10^{-5} , and 10^{-6} in order to investigate the convergence speed to a stationary point. We also impose a time limit of 1,000 seconds and a maximal number of 8,000 iterations on each method. As `lsqnonneg` takes long computing time, we run it only on the smallest data. Due to the slow convergence of `mult` and `pgrad`, for $\epsilon = 10^{-6}$ we run only `alspgrad`.

Results of average time, number of iterations, and objective values are in Tables 2a-2c. For small problems, Table 2a shows that all four methods give similar objective values as $\epsilon \rightarrow 0$. The method `lsqnonneg` is rather slow, a result supporting our argument in Section 4.1 that a matrix-based approach is better than a vector-based one. For larger problems, when $\epsilon = 10^{-5}$, `mult` and `pgrad` often exceed the maximal number of iterations. Clearly, `mult` quickly decreases the objective value in the beginning, but slows down in the end. In contrast, `alspgrad` has the fastest final convergence. For the two larger problems, it gives the smallest objective value under $\epsilon = 10^{-6}$, but takes less time than that by `mult`

under $\epsilon = 10^{-5}$. Due to the poor performance of `pgrad` and `lsqnonneg`, subsequently we focus on comparing `mult` and `alspgrad`.

6.2 Image Data

We consider three image problems used in Hoyer (2004):

1. CBCL face image database.
<http://cbcl.mit.edu/cbcl/software-datasets/FaceData2.html>.
2. ORL face image database.
<http://www.uk.research.att.com/facedatabase.html>.
3. Natural image data set (Hoyer, 2002).

All settings are the same as those in Hoyer (2004). We compare objective values and project-gradient norms of `mult` and `alspgrad` after running 25 and 50 seconds. Table 2 presents average results of using 30 random initial points. For all three problems, `alspgrad` gives smaller objective values. While `mult` may quickly lower the objective value in the beginning, `alspgrad` catches up very soon and has faster final convergence. Results here are consistent with the findings in Section 6.1. Regarding the projected-gradient norms, those by `alspgrad` are much smaller. Hence solutions by `alspgrad` are closer to stationary points.

To further illustrate the slow final convergence of `mult`, Figure 1 checks the relation between the running time and the objective value. The CBCL set with the first of the 30 initial (W^1, H^1) is used. The figure clearly demonstrates that `mult` very slowly decreases the objective value at final iterations.

The number of sub-iterations for solving (9) and (10) in `alspgrad` is an important issue. First, it is related to the time complexity analysis. Second, Section 4.1 conjectures that the number should be small as $W^T W$ and HH^T are generally well-conditioned. Table 3 presents the number of sub-iterations and the condition numbers of $W^T W$ and HH^T . Compared to usual gradient-based methods, the number of sub-iterations is relatively small. Another projected gradient method `pgrad` discussed in Table 6 easily takes hundreds or thousands of iterations. For condition numbers, both CBCL and Natural sets have $r < n < m$, so HH^T tends

Table 2: Image data: Average objective values and projected-gradient norms of using 30 initial points under specified time limits. Smaller values are bold-faced.

Problem	CBCL			ORL			Natural		
Size (n r m)	361	49	2,429	10,304	25	400	288	72	10,000
Time limit (in seconds)	25			50			25		
Objective Value mult	963.81	914.09	16.14	14.31	370797.31	353709.28			
alspgrad	923.77	870.18	14.34	13.82	377167.27	352355.64			
$\ \nabla f^P(W, H)\ _F$ mult	488.72	327.28	19.37	9.30	54534.09	21985.99			
alspgrad	230.67	142.13	4.82	4.33	19357.03	4974.84			

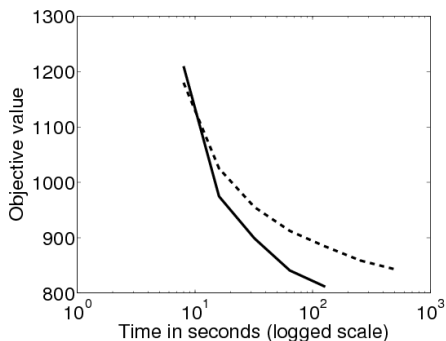


Figure 1: Time (seconds in log scale) vs. objective values for **mult** (dashed line) and **alspgrad** (solid line).

to be better conditioned than $W^T W$. ORL has the opposite as $r < m < n$. All condition numbers are small, and this result confirms our earlier conjecture. For ORL, $\text{cond}(W^T W) > \text{cond}(H H^T)$, but the number of sub-iterations on solving W is more. One possible reason is the different stopping tolerances for solving (9) and (10).

6.3 Text Data

NMF is useful for document clustering, so we next consider a text set RCV1 (Lewis et al., 2004). This set is an archive of manually categorized newswire stories from Reuters Ltd. The collection has been fully pre-processed, including removing stop words, stemming, and transforming into vector space models. Each vector, cosine normalized, contains features of logged TF-IDF (term frequency, inverse document frequency). Training/testing splits have been defined. We remove documents in

Table 3: Number of sub-iterations and condition numbers in solving (9) and (10) of `alspgrad`. For sub-iterations, we calculate (total sub-iterations)/(total iterations) under each initial point, and report the average of 30 values. For condition numbers, we find the median at all iterations, and report the average. Note that HH^T (W^TW) corresponds to the Hessian of minimizing $W(H)$.

Problem Time limit (in seconds)	CBCL		ORL		Natural	
	25	50	25	50	25	50
W : # sub-iterations	34.51	47.81	9.93	11.27	21.94	27.54
$\text{cond}(HH^T)$	224.88	231.33	76.44	71.75	93.88	103.64
H : # sub-iterations	11.93	18.15	6.84	7.70	3.13	4.39
$\text{cond}(W^TW)$	150.89	124.27	478.35	129.00	38.49	17.19

the training set which are associated with more than one class and obtain a set of 15,933 instances in 101 classes. We further remove classes which have less than five documents. Using $r = 3$ and 6, we then randomly select r classes of documents to construct the $n \times m$ matrix V , where n is the number of the vocabulary set and m is the number of documents. Some words never appear in the selected documents and cause zero rows in V . We remove them before experiments. The parameter r is the number of clusters that we intend to assign documents to. Results of running the two approaches by 25 and 50 seconds are in Table 4. We again have that `alspgrad` gives smaller objective values. In addition, projected-gradient norms by `alspgrad` are smaller.

Section 2.1 stated that `mult` is well-defined if Theorem 1 holds. Now V is a sparse matrix with many zero elements since words appeared in a document are only a small subset of the whole vocabulary set. Thus some columns of V are close to zero vectors, and for few situations, numerical difficulties occur. In contrast, projected gradient methods do not have such problems.

7 Discussion and Conclusions

We discuss some future issues and draw conclusions.

Table 4: Text data: Average objective values and projected-gradient norms of using 30 initial points under specified time limits. Smaller values are bold-faced. Due to the unbalanced class distribution, interestingly the random selection of six classes results in less documents (i.e., m) than that of selecting three classes.

Size (n r m)		5,412	3	1,588	5,737	6	1,401
Time limit (in seconds)		25		50		50	
Objective Value	mult	710.160		710.135	595.245		594.869
	alspgrad	710.128		710.128	594.631		594.520
$\ \nabla f^P(W, H)\ _F$	mult	4.646		1.963	13.633		11.268
	alspgrad	0.016		0.000	2.250		0.328

7.1 Future Issues

As resulting W and H usually have many zero components, NMF is said to produce a sparse representation of the data. To achieve better sparseness, some work such as (Hoyer, 2002; Piper et al., 2004) added penalty terms to the NMF objective function:

$$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m (V_{ij} - (WH)_{ij})^2 + \alpha \|W\|_F^2 + \beta \|H\|_F^2, \quad (22)$$

where α and β are positive numbers. Besides the Frobenius norm, which is quadratic, we can also use a linear penalty function

$$\alpha \sum_{i,a} W_{ia} + \beta \sum_{b,j} H_{bj}. \quad (23)$$

Our proposed methods can be used for such formulations. As penalty parameters α and β only indirectly control the sparseness, Hoyer (2004) proposes a scheme to directly specify the desired sparsity. It is interesting to investigate how to incorporate projected gradient methods in such frameworks.

7.2 Conclusions

This paper proposes two projected gradient methods for NMF. The one solving least square sub-problems in Algorithm 2 leads to faster convergence than the popular multiplicative update method. Its success is due to our following findings:

1. Sub-problems in Algorithm 2 for NMF generally have well-conditioned Hessian matrices (i.e., second derivative) due to the property $r \ll \min(n, m)$.

Hence projected gradients converge quickly though they use only the first order information.

2. The cost on step size selection in projected gradients is significantly reduced by some reformulations which again use the property $r \ll \min(n, m)$.

Therefore, taking special NMF properties is crucial when applying an optimization method to NMF.

Roughly speaking, optimization methods have the following two extremes:

Low cost per iteration	\longleftrightarrow	High cost per iteration
slow convergence		fast convergence

For example, Newton's methods are expensive per iteration, but have very fast final convergence. Approaches with low cost per iteration usually decrease the objective value more quickly in the beginning, a nice property enjoyed by the multiplicative update method for NMF. Based on our analysis, we feel that the multiplicative update is very close to the first extreme. The proposed method of alternating least squares using projected gradients tends to be more in the between. With faster convergence and sound optimization properties, it is an appealing approach for NMF.

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A Proof of Theorem 1

When $k = 1$, (7) holds by the assumption of this theorem. Using induction, if (7) is correct at k , then at $(k + 1)$, clearly denominators of (4) and (5) are strictly positive. Moreover, as V has neither zero column nor row, both numerators are strictly positive as well. Thus (7) holds at $(k + 1)$, and the proof is complete.

B MATLAB Code

B.1 Main Code for `alspgrad` (Alternating Non-negative Least Squares Using Projected Gradients)

```
function [W,H] = nmf(V,Winit,Hinit,tol,timelimit,maxiter)

% NMF by alternative non-negative least squares using projected gradients
% Author: Chih-Jen Lin, National Taiwan University

% W,H: output solution
% Winit,Hinit: initial solution
% tol: tolerance for a relative stopping condition
% timelimit, maxiter: limit of time and iterations

W = Winit; H = Hinit; initt = cputime;

gradW = W*(H*H') - V*H'; gradH = (W'*W)*H - W'*V;
initgrad = norm([gradW; gradH'],'fro');
fprintf('Init gradient norm %f\n', initgrad);
tolW = max(0.001,tol)*initgrad; tolH = tolW;

for iter=1:maxiter,
    % stopping condition
    projnorm = norm([gradW(gradW<0 | W>0); gradH(gradH<0 | H>0)]);
    if projnorm < tol*initgrad | cputime-initt > timelimit,
        break;
    end

    [W,gradW,iterW] = nlssubprob(V',H',W',tolW,1000); W = W'; gradW = gradW';
    if iterW==1,
        tolW = 0.1 * tolW;
    end

    [H,gradH,iterH] = nlssubprob(V,W,H,tolH,1000);
    if iterH==1,
        tolH = 0.1 * tolH;
    end
end
```

```

    if rem(iter,10)==0, fprintf(' '); end
end
fprintf('\nIter = %d Final proj-grad norm %f\n', iter, projnorm);

```

B.2 Solving the Sub-problem (14) by the Projected Gradient Algorithm 4

```

function [H,grad,iter] = nlssubprob(V,W,Hinit,tol,maxiter)

% H, grad: output solution and gradient
% iter: #iterations used
% V, W: constant matrices
% Hinit: initial solution
% tol: stopping tolerance
% maxiter: limit of iterations

H = Hinit; WtV = W'*V; WtW = W'*W;

alpha = 1; beta = 0.1;
for iter=1:maxiter,
    grad = WtW*H - WtV;
    projgrad = norm(grad(grad < 0 | H >0));
    if projgrad < tol,
        break
    end

    % search step size
    for inner_iter=1:20,
        Hn = max(H - alpha*grad, 0); d = Hn-H;
        gradd=sum(sum(grad.*d)); dQd = sum(sum((WtW*d).*d));
        suff_decr = 0.99*gradd + 0.5*dQd < 0;
        if inner_iter==1,
            decr_alpha = ~suff_decr; Hp = H;
        end
        if decr_alpha,
            if suff_decr,
                H = Hn; break;
            else
                alpha = alpha * beta;
            end
        else
            if ~suff_decr | Hp == Hn,
                H = Hp; break;
            else
                alpha = alpha/beta; Hp = Hn;
            end
        end
    end
end

```

```
    end
end

if iter==maxiter,
    fprintf('Max iter in nlssubprob\n');
end
```