Bipancyclicity of Hierarchical Hypercube Networks

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Abstract - *The hierarchical hypercube network is suitable for massively parallel systems. The number of links in the hierarchical hypercube network forms a compromise between those of the hypercube and the cube-connected cycles. Recently, some interesting properties of the hierarchical hypercube network were investigated. Since the hierarchical hypercube is bipartite. A bipartite graph is bipancyclic if it contains cycles of every even length from 4 to |V(G)| inclusively. In this paper, we show that the hierarchical hypercube network is bipancyclic.*

Keywords: Bipancyclic, bipartite graph, embedding, Gray code, Hamiltonian cycle, hierarchical hypercube networks, hypercube.

1 Introduction

In recent decades, many interconnection network topologies have been proposed in the literature (see [12], [16]) for purpose of connecting hundreds or thousands of processing elements. Among these topologies, the hypercube network is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [21].

Malluhi and Bayoumi were proposed the hierarchical hypercube network [19], which is an alternative to the hypercube. It owns many favorable topological properties for building massively parallel systems. An appealing property of this network is the low number of connections per processor which enhances the VLSI design and fabrication of the system. Other alluring features include regularity, symmetry and logarithmic diameter which imply easy and fast algorithms for communication. Besides, it can perform one-to-one communication, one-to-all communication and divide-and-conquer algorithms efficiently [17]-[19]. Moreover, the one-to-one disjoint paths algorithm was investigated in [24].

On the other hand, linear arrays and rings, which are two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. Many efficient algorithms designed on linear arrays and rings for solving a variety of algebraic problems and graph problems can be found in previous works [16]. The *pancyclicity* of a network represents its power of embedding cycles of all possible lengths. An n-node network (graph) is *pancyclic* if it contains all cycles of lengths from 3 to n [3]. It can embed rings of all possible lengths with dilation 1, congestion 1, load 1, and expansion 1. The *pancycle problem* on a network W asks, for every integer $3 \le l \le |W|$, whether or not W contains a cycle of length l, where $|W|$ is the number of nodes contained in W. Obviously, a pancyclic network is Hamiltonian because a cycle of length n corresponds to a Hamiltonian cycle. The pancycle problem was solved on many networks, e.g., the twisted cube [5], the butterfly graph [13], the arrangement graph [7], the hypercomplete network [6], the alternating group graph [15] the CCC network [9], and the hierarchical cubic network [8].

The hypercube network [16] and the hierarchical hypercube network [19] are bipartite graphs. *Bipancyclicity* is essentially a restriction of the concept of pancyclicity to bipartite graphs whose cycles are necessarily of even length. A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to the number of its vertices. In this paper we solve the pancycle problem on the hierarchical hypercube network, that is, we show that the hierarchical hypercube network is bipancyclic.

The rest of this paper is organized as follows: In the next section, the structure of the hierarchical hypercube network is first reviewed. And the cycle embedding problem in the network is solved in Section 3. Finally, this paper concludes in Section 4.

2 Preliminaries

A network is conveniently represented as an undirected graph whose vertices represent the nodes (i.e., processors) of the network and whose edges represent the communication links of the network. Throughout this paper,

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for the graph theoretical definitions and notations we follow [22].

Let $G=(V, E)$ be a connected graph, where the set of vertices $V(G)$ represent processors, and the set of edges $E(G)$ represent links between processors. We use network and graph, node and link (vertex and edge) interchangeably. A graph $G=(V_0 \cup V_1, E)$ is bipartite if $V(G)$ is the union of two disjoint sets V_0 and V_1, such that every edge joins V_0 with V_1. Two vertices, u and v, have the same color if and only if u and v are in the same partite set. If e_1 and e_2 are distinct edges that are incident to a common vertex, then e_1 and e_2 are adjacent edges.

The *degree* of a vertex in G is the number of edges incident to it. If all vertices have the same degree d, then G is called *regular* or *d-regular*. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path between u and v.

An n-dimensional hypercube, denoted by Q_n, is one of the most popular networks. There are 2^n nodes contained in a Q_n network, each is uniquely represented by a binary sequence $b_{n-1}b_{n-2}\ldots b_0$ of length n. Two nodes in a Q_n network are adjacent if and only if they differ at exactly one bit position. An edge of Q_n network is *dimension* k ($0 \le k \le n-1$) if its two end vertices differ at b_k. The hypercube network suffers from a practical limitation when it is used as the topology of a multiprocessor system. As n increases, it becomes more difficult to design and fabricate the nodes of Q_n because of the large fanout.

To remove the limitation, the cube-connected cycles (CCC) network [20] was designed as a substitute for the hypercube. The node degree of CCC is restricted to three. However, this restriction degrades the performance of CCC at the same time. For example, CCC has a larger diameter than the hypercube. Taking both the practical limitation and the performance into account, the hierarchical hypercube (HHC) network [19] was proposed as a compromise between the hypercube and CCC. An HHC network, which has a two-level structure, takes hypercube as basic modules and connects them in a hypercube manner. An HHC network has a logarithmic diameter, which is the same as a hypercube network. Since the topology of an HHC is closely related to the topology of a hypercube network, it inherits some favorable properties from the latter.

Recall that a CCC network can be obtained by replacing each node of a Q_k network with a cycle of k nodes so that these k nodes are connected to the k neighbors of the original node in the Q_k network. Actually, an HHC network is a modification of a CCC network in which the k–node cycle is replaced with a hypercube. Assume $k=2^m$. An HHC network can be constructed as follows: start with Q_{2^m} network and replace each node of it with a Q_m network.

Since there are a total of $2^{2^m} \times 2^m = 2^{2^m+m}$ nodes in the HHC network, each node of the HHC network can be uniquely represented by a binary sequence $b_{n-1}b_{n-2}\ldots b_0$, where $n=2^m+m$. Refer to Figure 1, where an example with $m=2$ is shown. For convenience, $b_{n-1}b_{n-2}\ldots b_0$ is expressed as a two-tuple (S, P), where $S=b_{n-1}b_{n-2}\ldots b_m$ tells which Q_m network the node is located in and $P=b_{m-1}b_{m-2}\ldots b_0$ gives the address of the node in the located Q_m network.

Let $P^{(l)} = b_{m-1}\ldots b_{l+1}\bar{b}_l b_{l-1}\ldots b_0$ ($S^{(m+l)} = b_{n-1}\ldots b_{m+l+1}\bar{b}_{m+1} b_{m+l-1}\ldots b_m$), where $\bar{b}_l$ denote the complement of b_l. An HHC network can be defined in terms of graph as follows.

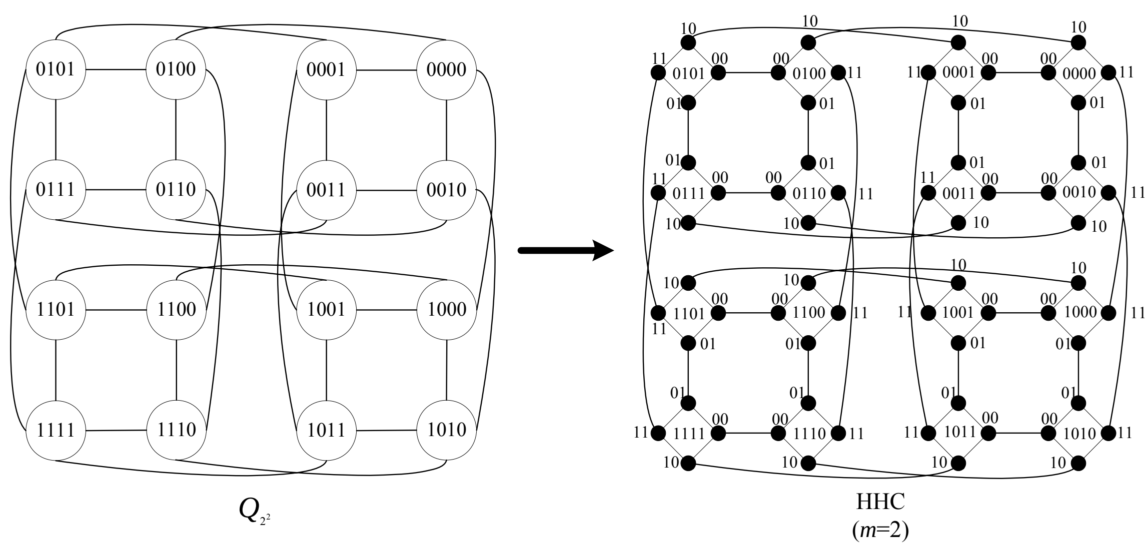

Figure 1: Construction of an HHC network from a Q_{2^2} network.

Definition 2.1 *The node set of an n-dimensional HHC (n-HHC for short) is* $\{(S, P) \mid S = b_{n-1}b_{n-2}\ldots b_m$*, and* $P = b_{m-1}b_{m-2}\ldots b_0$*, and* $b_i \in \{0, 1\}$ *for all* $0 \le i \le n-1\}$*, where* $n = 2^m + m$ *and* $m \ge 1$*. Node adjacency of an n-HHC network is defined as follows:* (S, P) *is adjacent to* (1) $(S, P^{(l)})$ *for all* $0 \le l \le m-1$ *and* (2) $(S^{(m+\mathrm{dec}(P))}, P)$*, where* $\mathrm{dec}(P)$ *is the decimal value of P.*

Edges defined by (1) are referred to as internal edges, and those defined by (2) are referred to as external edges. Internal edges are within Q_m networks and each of external edges connects two Q_m networks. As Figure 1, node (0000, 01) and node (0000, 11) are connected by an internal edge; node (0000, 01) and node (0010, 01) are connected by an external edge. Notice that an n-HHC network is $(m + 1)$-regular, symmetric, and has a diameter of 2^{m+1} (see [19]). In subsequent discussion, whenever a node A of an n-HHC network is mentioned, we use A_S and A_P to denote the S part and P part of A, respectively. For each b_i in S part, we decrease each index i by m, so that all the index i would follow $0 \le i \le 2^m - 1$ in the rest of paper.

In the following, we define Gray codes, which will be used in the next section.

Definition 2.2 [11] *An m-bit Gray code, denoted by* G_m*, defines an ordering among all the m-bit binary numbers.* G_1 *is defined as* $(0, 1)$*, and for* $m > 1$*,* G_m *is defined recursively in terms of* G_{m-1} *as* $(0G_{m-1}, 1G_{m-1}^{\ r})$*, where* $G_{m-1}^{\ r}$ *stands for the reverse ordering of* G_{m-1} *and* $0G_{m-1}$ $(1G_{m-1}^{\ r})$ *stands for prefixing each binary number in* G_{m-1} $(G_{m-1}^{\ r})$ *with* 0 (1).

For example, G_2 can be (00, 01, 11, 10) and G_3 can be (000, 001, 011, 010, 110, 111, 101, 100). Notice that every two adjacent binary numbers, including the first one and the last one, in G_m differ in exactly one bit position.

3 Cycles Embedding in HHC Networks

In this section, we embed cycles of all possible lengths into an n-HHC network. Since an n-HHC network is bipartite (see [19]), only cycles of even lengths, ranging from 4 to 2^{2^m}, can be embedded.

We used $d_H(V_0, V_1)$ to denoted the *Hamming distance* between V_0 and V_1, which is the number of different bits between V_0 and V_1. A path from V_0 to V_m is denoted $V_0 \to V_1 \to V_2 \to \ldots \to V_m$. It can be also abbreviated to a V_0-V_m *path*. A cycle c_l is denoted $V_0 \to V_1 \to V_2 \to \ldots \to V_m \to V_0$, where l is the length of the cycle. Obeying the convention of most graph books, every path (or cycle) in this paper contains no repeated node.

For example, $A = (00000000, 000) \to (00000001, 000) \ {*\!\!\to}\ (00000001, 010) \to (00000101, 010) \ {*\!\!\to}\ (00000101, 000) \to (00000100, 000) \ {*\!\!\to}\ (00000100, 010) \to (00000000, 010) = B$ expresses a path, denoted by A-B path, from A = (00000000, 000) to B =(00000000, 010), where $*\!\!\to$ denotes a shortest path within a Q_3 network. The path in a 11-HHC contains internal edges and external edges alternately. Each subpath of it within a Q_3 network is maintained shortest. It is easy to obtain a shortest path between any two distinct nodes in a Q_m [21]. So, if the subpaths within Q_m networks are ignored, then a path in an n-HHC network can be simply represented by a sequence of external edges, called an *external edge sequence* (EES). In this example, the path contains four external edges that can be represented by their P parts, i.e., 000, 010, 000 and 010 in sequence. Hence, the path can be simply represented by an EES (000, 010, 000, 010).

Lemma 3.1 [8] *Suppose* $d_H(X, Y) = d \ge 1$*. There are X-Y path in a* Q_m *whose length are* $d+2, d+4, \ldots, c$*, where* $m \ge 1$*,* $c = 2^m - 1$ *if d is odd, and* $c = 2^m - 2$ *if d is even.*

Let $d_H(X, Y) = 1$. By lemma 3.1, there are X-Y path in Q_m whose length is ranging from 3 to $2^m - 1$ connect these adjacent nodes, X and Y. Then, there are cycles in Q_m whose length is even and ranging from 4 to 2^m. In the other word, we have following corollary.

Corollary 3.2 *An m-cube* (Q_m) *is bipancyclic, where* $m>1$.

Lemma 3.3 *Suppose* $4 \le l \le 2^{2^m+m}$ *and* c_l *is a cycle or path within an n-HHC. Let A and B are two arbitrary adjacent vertices in* c_l *such that no external edges in* c_l *incident to them. Then, we can replace the link* (A, B) *by a path which obtained according to the EES* (A_P, B_P, A_P, B_P)*. Then, the cycle* (*or path*) *is extended to length* $l + 6$.

For example, by corollary 3.2, we can construct cycles c_6 = (00000000, 000) $\to$ (00000000, 001) $\to$ (00000000, 011) $\to$ (00000000, 111) $\to$ (00000000, 110) $\to$ (00000000, 010) $\to$ (00000000, 000) within a Q_3 of an 11-HHC ($m = 3$). Arbitrarily select two adjacent nodes A = (00000000, 111) and B = (00000000, 110) from c_6. By lemma 3.3, the link (A, B) can be replaced by the path which obtained according to the EES (111, 110, 111, 110). Then, the cycle is extended to length 12. The extended cycle is described as follow and the path obtained by the EES (111, 110, 111, 110) is underlined.

c_{12}: (00000000, 000)$\to$(00000000, 001)$\to$ (00000000, 011)$\to$<u>(00000000, 111)$\to$(10000000, 111)$\to$(10000000, 110)$\to$(11000000, 110)$\to$ (11000000, 111)$\to$(01000000, 111)$\to$(01000000, 110)$\to$(00000000, 110)</u>$\to$(00000000, 010)$\to$ (00000000, 000).

Some observations on above example, the original cycle c_6 is located within a Q_3 of an 11-HHC. The cycle c_{12} is the result of extending cycle c_6 and the cycle c_{12} pass through 4 different Q_3's. Clearly, the result of applying lemma 3.2 one time can increase 3 Q_m's to the cycle. By lemma 3.1, we can extend the cycle c_l repeatedly until $l \le 32$ $(= 4 \times |V(Q_3)|)$ and l is even. Then, we will describe the cycle construction in an n-HHC in next theorem.

Theorem 3.4 An n-HHC network contains cycles of lengths ranging from 4 to 2^n ($= 2^{2^m+m}$), where $m > 2$.

Proof. In the proof, we assume l is even. To construct a cycle of length l, two cases should be considered as follows.

Case 1. ($4 \le l \le 2^m$): Without losing generality, we apply corollary 3.2 in a Q_m which is located in 0^{2^m}, where 0^{2^m} represents 2^m consecutive 0's. By corollary 3.2, we can construct cycles in a Q_m of an n-HHC whose length is even and ranging from 4 to 2^m.

Case 2. ($2^m + 2 \le l \le 2^{2^m+m}$): Let node X and node Y be two adjacent vertices in c_{2^m-2} such that no external edges in c_{l-4} incident to them. Without loss of generality, assume $X_P = x_{m-1}x_{m-2}\ldots x_1 0$ and $Y_P = X_P^{(1)} = x_{m-1}x_{m-2}\ldots x_1 1$. There are at most 2^{m-1} can be selected. We sort them by Gray code ordering. When $l = k2^m + 2$ and $1 \le k \le 2^{2^m-m}$, the cycle is extended by adding new Q_m's. We describe how to add new Q_m's in two parts: (A) $k=1$ or $k=2^t$, where $2 \le t \le 2^m - m$; (B) otherwise.

(A) First, we select a new link (X, Y) by Gary code ordering. Then, we replace the link (X, Y) of Q_m 0^{2^m} by a path which is obtained according to the EES (A_P, B_P, A_P, B_P) by applying lemma 3.3. Clearly, we add three Q_m to the cycle, and therefore the length of the cycle is $l+2$ $(= l-4+6)$.

(B) We can find a Q_m in c_{l-4} where Q_m's link (X, Y) is not replaced. And we apply lemma 3.3 to extend the cycle. Clearly, we add three Q_m to the cycle and the length of the cycle is $l+2$ $(= l-4+6)$.

Then, we can apply corollary 3.2 to extend the cycle c_l, where $k2^m+4 \le l \le (k+3)2^m$ and $1 \le k \le 2^{2^m} - 1$. Apply the method describe above repeatedly until all Q_m's of an n-HHC are added to the cycle. We can extend the cycles with all even length from $4 \times 2^m + 2$ to 2^{2^m+m}. There are 2^{2^m} Q_m's in an n-HHC. Each time we apply lemma 3.3, we can add three Q_m's to the cycle. After $(2^{2^m} - 1)/3$ times, we can add all Q_m's of an n-HHC to the cycle. Note that $(2^{2^m} - 1)/3$ is an integer since $2^{2^m} - 1 = 2^{2^{m-1} \times 2} - 1 = (2^{2^{m-1}})^2 - 1 = (2^{2^{m-1}} + 1)(2^{2^{m-1}} - 1) = 3\prod_{i=1}^{m-1}(2^{2^i} + 1)$. As a result, all Q_m's can be added to the cycle.
■

To consider the $m \le 2$, these cases are special. When m=1, obviously a 3-HHC is also a cycle with length 8. When $m = 2$, there are 16 Q_2's in a 6-HHC. Clearly, a Q_2 is also a cycle c_4. We use the construction method of theorem 3.4 which repeatedly applies lemma 3.3 five $(=(2^{2^2} - 1)/3)$ times to add all Q_2's to the cycle. Then, we have cycles c_l, where $10 \le l \le 2^6$ and l is even. So, a 6-HHC network contains cycles of all possible even lengths, except 6.

4 Conclusions

The hierarchical hypercube network was originally proposed in [17]-[19] for building massively parallel systems. It uses logarithmic links of a comparable hypercube and owns many favorable topological properties include regularity, symmetry and logarithmic diameter which imply easy and fast algorithms for communication. And an appealing property of this network is the low number of connections per processor which enhances the VLSI design and fabrication of the system. Besides, it can perform one-to-one communication, one-to-all communication and divide-and-conquer algorithms efficiently [17]-[19]. Moreover, the one-to-one disjoint paths algorithm was investigated in [24]. The hierarchical hypercube network was originally proposed in [9-11] for building massively parallel systems. It uses logarithmic links of a comparable hypercube and owns many favorable topological properties include regularity, symmetry and logarithmic diameter which imply easy and fast

In this paper, we solved the cycle embedding problem by showing that there are cycles of all possible even length in the n-dimensional hierarchical hypercube network, where $m > 2$. Consequently, the hierarchical hypercube network can efficiently execute all algorithms that are executable on linear arrays or rings. Many of such algorithms can be found in [1].

Finally, further research problems on the hierarchical hypercube network are suggested. For instance, Hamiltonian-laceability [10], [14], [23] and conditional faults [2], [4] problems were proposed. It still includes research issues in the hierarchical hypercube network.

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