行政院國家科學委員會專題研究計畫 期中進度報告

平面圖之「簡潔編碼」與「簡潔呈現」演算法(2/3)
期中進度報告(精簡版)

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中文摘要

本計畫研究如何以簡潔的方式呈現一張圖形，獲得目前四連通平面圖的可視性呈現中，高度最佳的表示法。

中文關鍵字

演算法，平面圖，簡潔呈現，可視性呈現

英文摘要

We investigate how to draw a four-connected planar graph using visibility representation. We obtain a drawing algorithm whose height is currently the best and achives the optimum.

英文關鍵字

algorithms, planar graphs, compact drawing, visibility representation.

研究成果自評

感謝國科會的經費支持，讓本計畫可以順利的進行，目前所獲得的研究成果，與當初當初計畫書當中所規劃的略有差異，原本希望利用 separator-based 的方法，在簡潔呈現上面獲得最佳的成果，不過後來發現可以使用新的工具，獲得本報告當中的結果，算是意外的驚喜。從報告的正文中，可以看到這個問題已經經過非常長時間的競爭，我們很幸運地能夠在激烈的競爭中得到最好的結果，覺得非常榮幸！

Reference	*Plane Graph G*	*4-connected Plane Graph G*
[5][6]	Width of VR $\leq 2n-5$	Height of VR $\leq n-1$
[2]	Width of VR$\leq \lfloor \frac{3n-6}{2} \rfloor$	
[4]	Width of VR$\leq \lfloor \frac{22n-24}{15} \rfloor$	
[3]		Width of VR $\leq n-1$
[7]	Height of VR $\leq \lceil \frac{15n}{16} \rceil$	
[8]	Width of VR$\leq \lfloor \frac{13n-24}{9} \rfloor$	Height of VR $\leq \lceil \frac{3n}{4} \rceil$
[1]	Width of VR $\leq \frac{3n}{4} + 2\lceil \sqrt{n} \rceil$, Height of VR $\leq \frac{2n}{3} + 2\lceil \sqrt{\frac{n}{2}} \rceil$	
[11]	Height of VR $\leq \frac{2n}{3} + O(n)$	
ours		Height of VR $\leq \lceil \frac{n}{2} \rceil$

1 introduction

A visibility representation (VR for short) of a plane graph G is that the vertices of G are represented by non-overlapping horizontal line segments and the vertex segment must be visible vertically to each other for any two vertices that are adjacent in G. We summarize previous results as follows.

In this paper, we show that every 4 connected plane triangulation with n vertices has a VR with height at most$\frac{n}{2} + 4$, which is obtainable in O(n) time. The remainder of the paper is organized as follows. Section 2 gives the preliminaries. Section 3 describes and analyzes our algorithm. Section 4 discusses the tightness of our algorithm.

2 preliminaries

Let $G = (V, E)$ be a 2 connected plane graph and (s, t) an external edge of G. An st numbering of G is a one-to-one mapping ξ : V $\longrightarrow \{1, 2, \ldots, n\}$, such that $\xi(s) = 1, \xi(t) = n$, and each vertex $v \neq s, t$ has two neighbors u, w with $\xi(u) < \xi(v) < \xi(w)$, where u is called a smaller neighbor of v and w is called a bigger neighbor of v. Given an st numbering ξ of G, we can orient G by directing each edge in E from its lower numbered end vertex to its higher numbered end vertex. The resulting orientation is called the orientation derived from ξ which is an st-orientation of G. All directed path in G start with $\xi(s) = 1$, which is called source. . All directed path in G end with $\xi(t) = n$, which is called sink. We denote the words "counterclockwise" and "clockwise" as ccw and cw.

G is a plane triangulation with three exterior vertices v_1, v_2, v_n in ccw order. A realizer R of G is a partition of the interior edges of G into three sets T_1, T_2, T_n. The edges in T_1, T_2, T_n are directed edges such that the following statements hold:

1. For each $i \in \{1, 2, n\}$, the interior edges incident to v_i are in T_i and directed toward v_i.
2. For each interior vertex v of G, v has only one edge leaving v in each of T_1, T_2, T_n. The ccw order of the edges incident to v is: leaving in T_1, entering in T_n, leaving in T_2, entering in T_1, leaving in T_n and entering in T_2. Each entering block could be empty.

An ordered list O consisting of elements $a_1, a_2, \ldots, a_k$ is written as $O = \langle a_1, a_2, \ldots, a_k \rangle$. The reverse of an ordered list O is $\langle a_k, \ldots, a_2, a_1 \rangle$, which is denoted by O^r.

G is a 4-connected plane triangulation with three exterior vertices V_N, V_W, V_E in ccw order. We delete the edge (V_W, V_E), and G has a new exterior vertex V_S. This graph is called G'. Let G' be a plane graph with four vertices on its exterior face. A graph satisfying the following two conditions is a *proper triangulated plane* (PTP for short)[5]. Every interior face of G' is a triangle and the exterior face of G' is a quadrangle [6]; G' has no separating triangles.

Fact 1 (see [5, 6]) *Let G be a 2-connected plane graph with an st-orientation O. VR of G can be obtained from O and the height of the VR equals the length of the longest directed path in O, which can be obtainable in linear time.*

Fact 2 (Theorem 3 of Zhang and He [11]) *Let $G = (V, E)$ be a plane triangulation with n vertices. Let v_1, v_2, v_n be three external vertices in counterclockwise order. Let $R = \{T_1, T_2, T_n\}$ be a realizer of G. If there is a path in any of $T_i, i = 1, 2, n$ with length at least k, then G has an st orientation O, constructible in linear time with length(O)$\leq n - k + O(1)$.*

Zhang and He [11] did not write ub this form. They proved that if there is a path in any of $T_i, i = 1, 2, n$ with length at least $\frac{n}{3}$, then G has an *st*-orientation O with $length(O) \leq \frac{2n}{3} + O(1)$. By the proof Zhang and He [11], we can rewrite their theorem in this form.

G' is a PTP. A regular edge labeling (REL for short), Zhang and He defined in [9], of G' is a partition of the interior edges into two subsets S_1, S_2 of directed edges and the follows hold:

1. For each interior vertex v, the edges incident to v in ccw order around v as follows: a set of edges in S_1 leaving v; a set of edges in S_2 entering v; a set of edges in S_1 entering v; a set of edges in S_2 leaving v. All sets must be nonempty.

2. All interior edges incident to V_N are in S_1 and the direction of edges is entering V_N. All interior edges incident to V_W are in S2 and the direction of edges is leaving V_W. All interior edges incident to V_S are in S1 and the direction of edges is leaving V_S. All interior edges incident to V_E are in S_2 and the direction of edges is entering V_E. All blocks must be nonempty.

G_1 is the directed subgraph of G' induced by S_1 and four exterior edges directed as $V_S \longrightarrow V_W, V_W \longrightarrow V_N, V_S \longrightarrow V_E, V_E \longrightarrow V_N$. Then G_1 is an st-plane graph with source V_S and sink V_N. G_2 is the directed subgraph of G' induced by S_2 and four exterior edges directed as $V_W \longrightarrow V_S, V_S \longrightarrow V_E, V_W \longrightarrow V_N, V_N \longrightarrow VE$. Then G_2 is an st-plane graph with source V_W and sink V_E. Then we call G_1 the S-N net and G_2 the W-E net of G' derived from the REL (S_1, S_2).G' has a REL if and the only if G' is a PTP. [9]

Property 1 *G' is a PTP. G' has an REL separating G' into two subgraphs G_1 and G_2. We need a property of an st-numbering O_p for G' in our algorithm. For all directed path p in G' with an st-orientation Op, p do not pass all paths in $G1$ twice.*

3 our algorithm

With fact 1, we konw that we can find the height of VR of G by the way to find the longest path of an *st*-orientation of G. Our algorithm is as follows. G' is a PTP. First, we find an *st*-numbering

O_p of G' with property 1. Then, we compute the length of the longest path PL of G' with st-orientation O. If the length of PL is more than $\frac{n}{2}$, we create a new st -numbering O to pass the vertices in PL by only two vertices. If the length of PL is no more than $\frac{n}{2}$, Op is the st-numbering we need.

Let G_1 has k_1 faces and G_2 has k_2 faces. For each edge in G_1 (G_2, resp.), $left(e)$ denotes the face on the left of edge e in G_1(G_2, resp.). Let $right(e)$ denote the face on the right of edge e. We give each face of G_1 (G_2, resp.) a number. The left most face ,with V_S at the bottom and V_N on the top, is numbered 0. For each edge in G_1 (G_2, resp.), the $left(e)$ is smaller than $right(e)$. By the way, the right most face is k_1 in G_1 and the right most face is k_2 in G_2 with V_S at the bottom and V_N on the top. We define the i-th S-N separation path SN_i to be the directed path,which the faces numbered by $0, \ldots, i-1$ are on its left and other faces are on its right, in G_1. And we define the ith W-E separation path WE_i to be the directed path, which the faces numbered by $0, \ldots, i-1$ are on its left and other faces are on its right, in G_2.

Fact 3 (Zhang and He [9]) *Let G' be a PTP. Let (S_1, S_2) be an REL of G'. Let G_1 be the S-N net and let G_2 be the W-E net derived from (S_1, S_2). Then the following statements hold:*

1. *For each vertex $v \neq V_N, V_S$ in G_1, select an outgoing edge in G_1. For V_S, select an outgoing edge not leading to V_W or V_E. Then the set T_1 of the selected edges is a tree of G.*

2. *For each vertex $v \neq V_W, V_E$ in G_2, select an outgoing edge in G_2. For V_W, select the edge (V_W, V_E). Then the set T_2 of the selected edges is a tree of G.*

Lemma 1 *G' is a PTP with n vertices in G'. G' has four exterior nodes V_N, V_W, V_S, V_E in ccw order. If there is a path PL with property 1 and there are k vertices in PL, G' has an st-Orientation O and the longest path with O is at most $n - k + 4$.*

proof.

This lemma is similar to Fact 2. The proof is modified from the lemma of 4[11]. First, we separate G' into three parts. The area at the left of PL is part A. Path PL is part B. The area at the right of PL is part C. Let the REL of G' that separates G' into G_1 and G_2. We reverse the direction of all edges in G_1 and called the reversed graph G_3. We select edges in G_1 by Fact 2 to build a tree, which is called T_n, with root V_N.T_e is a tree with root V_E in G_2 by t Fact 2. And we select T_s from G_3 with root V_S , by Fact 2. For all $K \in \{A, C\}$, T_{iK} is the subtree of T_i at area K.

V_N is numbered 1 in O. V_S is numbered n in O. V_N is in area A. For all vertices in area A, O_a is the cw postordering for T_{NA} with V_N being the root of T_{NA}. For the vertices in B, they are all in path PL besides V_W, V_S, V_E and V_N. Then we travel the path from V_W to the node before V_N,or the node before V_S,or the node before V_E by inserting the first vertex to the very end, and the second vertex to the very front. Recursively, insert the remaining vertices into the next available end or next available front of the ordered list until all nodes in PL, besides V_W, V_S, V_E and V_N, has been numbered. Denote this order by O_b. V_S is in area C. All of the vertices in area C, let O_c be the ccw postordering from T_{SC}. The numbering O in G' is $(O_a)^r$ O_b O_c. (see Fig1., Fig2. and Fig.3)

We prove that O is an st-orientation. V_N is the first vertex in O, and V_S is the last vertex in O. For any vertex $v \neq V_N$ in A, the parent of v in T_{NA} precedes v in O. The parent u in T_e precedes v in the cw postordering in T_{NA}. If u in T_{NA}, u precedes v in O_a. Thus, u succeeds v in $(O_a)^r$.

And u succeeds v in O, too. On the other hand, if u is not in T_{NA} , then u is either in area B or area C. Whether u is in area B or area C, u succeeds v in O. For any vertex v in B, the parent of v is u_N in T_n at area A. And u_N precedes v in O. The parent is u_S in T_s at area C. And u_S succeeds v in O. For any vertex $v \neq V_E$ and V_S in C, the parent of v in T_{SC} succeeds v in O. The parent u in T_e precedes v in the ccw postordering in T_s. If u in T_{SC}, u precedes v in O_c. Thus, u precedes v in O_c, and hence in O. On the other hand, if u is not in T_{SC}, u is in area A or area B. Whether u is in area B or area C, u precedes v in O. V_W precedes V_N, and V_E succeeds V_N. V_W precedes V_S, and V_E succeeds V_S. O is an st-orientation for G'. In O, the increasing path can only pass area B at most two nodes. There are $k-2$ nodes in area B. Thus, the length of the longest path in G' with O is at most $n-k+4$.

Lemma 2 *The time complexity of our algorithm is $O(n)$.*

proof.

We construct the numbering O_p of the nodes in G' with property 1. First, we number the nodes in $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path, we number from V_W to the node before V_E by $1, 2, \ldots, j-1$, if there are j nodes in $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path. We number the paths on the area at left of $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path and separate by $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path, first. We number the unnumbered nodes by path in the order $SN_2, SN_3, \ldots, SN_{k_1-1}$ and with increasing order from the node near $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path to the node near V_N in G_1. Then we number the other unnumbered nodes in the order $SN_2, SN_3, \ldots, SN_{k_1-1}$ and with increasing order from the node near $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path to the node near V_S in G_1. At last, we number V_S to be $(n-2)$, V_N to be $(n-1)$, and V_E to be n. We prove O_p is an st-orientation as follow.

For all nodes v in $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path , the nodes at the right or left of $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path succeed v in O_p. For all nodes $v \neq V_W$ in $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path, the node directing to v precedes v in O_p. For all other nodes $v \neq V_N, V_W, V_S$, and V_E, the node connecting v near $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path in G_1 precedes v. If v connects $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path, the node in $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path precedes v. For the other nodes v at the area at left of $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path and $v \neq V_N$, the node which v directing to in G_1 succeeds v in O_p. For the other nodes $v \neq V_S$, and at the area at right of $WE_{\lceil \frac{k_2+1}{2} \rceil}$ path, the node directing to v in G_1 succeeds v in O_p. And V_W precedes V_N and V_S. V_E succeeds V_N and V_S. V_W is 1, and V_E is n. Thus, O_p is an st-orientation.

This numbering Op can be finished in linear time. The orientation derived from O_p of G' is a directed acyclic graph. Finding the longest path in directed acyclic graph takes linear time. If the length of he longest path is more than $\frac{n}{2}$, we create a new st-numbering O. And we take linear time to create a new st-numbering. We can find the height of VR $\leq \frac{n}{2}$ in linear time by our algorithm.

Theorem 1 *Let $G = (V, E)$ be a 4-connected plane triangulation with n vertices. There exists a VR of G with height no more than $\frac{n}{2} + 4$, which is obtainable in $O(n)$ time.*

proof.

In algorithm 1, if the length of PL is more than $\frac{n}{2}$, we create a new st-numbering O to pass the vertices in PL by only two vertices. Thus, the length of the longest path with st-orientation O in G' is no more than $\frac{n}{2} + 4$ by Lemma 1. If the length of PL is no more than $\frac{n}{2}$, Op is the st-orientation we want to make the height of VR no more than $\frac{n}{2}$ by Fact 1. Algorithm 1 can be done in $O(n)$ time by Lemma 2. Thus, the theorem is proved.

4 nearly tightness

Zhang, and X. He [10] and Lin, Lu, and Sun [4] used nested triangles to find the bound of the height and width of VR in 2-connected plane triangulation. We use similar techniques to find the bound of the height of VR in 4-connected plane triangulation. Let G_k be k nested 4-connected triangulation with $n = 4k$. We want to show that VR of G_k requires a height of $\frac{n}{2} = 2k$.

1. When $k = 1$, the height of VR of G_1 is no less than 2.
2. Assume that it is true when $k = t$. It means that the height of VR of G_t no less than $2t$.
3. Then we consider the case of $k = t + 1$. Each time G has one more nested level, the height of VR increases by two units. When $k = t + 1$, the height of VR is $2t + 2 = 2(t + 1)$. (See Fig 4.)

Thus, the VR for 4-connected plane triangulation requires a size of height tat least $\frac{n}{2}$.

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