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THE DETERMINATION OF MODAL DAMPING RATIOS AND NATURAL FREQUENCIES FROM BISPECTRUM MODELING

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We present a new signal processing and testing technique by using a higher statistical moment, the bispectrum, to determine the damping ratio and natural frequency of offshore structures excited by both unexpected Gaussian forces and known non-Gaussian driving forces. Due to unexpected exciting forces, such as turbulence, in the ocean, environment, the transfer functions of offshore structures are not determined through operating a known driving force and measuring its response. In order to overcome this problem, some of the existing techniques try to model the unexpected forces as white Gaussian forces or almost white Gaussian forces and determine the modal parameters from the response only. Others try to average the input and output to suppress unexpected parts. Our method uses third-order moments to keep the influence of the unexpected Gaussian forces away from the determination of the transfer function of the structure which has linear properties. We model the third-order moment property of the response function with a bispectral model. The modal parameters can be calculated from the estimated model's coefficients. The method has been proven by a number of simulations.

1. INTRODUCTION

In this paper we discuss the time series corresponding to a uniformally sampled vibration record of offshore structural systems with stationary random force. We assume that a structural system can be represented by a set of linear differential equations derived by cutting the continuous structure to lumped mass-spring damper systems. These systems can then be modeled as autoregressive moving-average models. Many methods have been developed to deal with this problem. We use third-order moment and bispectrum analysis instead of conventional second-order moment and power spectrum analysis. In Section 2, we introduce the third-order moment and bispectrum. In Section 3, the derivation of the autoregressive moving-average (ARMA) model from structural system is discussed briefly. In Section 4, some examples with both numerical analysis and graphs are illustrated.

2. THIRD-ORDER MOMENT AND BISPECTRUM ANALYSIS

For a real discrete process $X(t)$ that is zero mean and third-order stationary, define

$$R(m, n) = E\{X(k)X(k+m)X(k+n)\} \tag{1}$$

as the third-order moment. Where m, n are integers and $[E]$ denotes expectation.

Using this definition, we have

$$R(m, n) = R(n, m) = R(-n, m-n) = R(n-m, -m). \tag{2}$$

Define the bispectrum of this process as the double Fourier transform of equation (1), i.e.

$$B(\omega_1, \omega_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m, n) \exp[-j(m\omega_1 + n\omega_2)]. \tag{3}$$

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Using the relations in equation (2), we have

$$B(\omega_1, \omega_2) = B(\omega_2, \omega_1) = B^*(-\omega_2, -\omega_1) = B(-\omega_1 - \omega_2, \omega_2)$$
$$= B(\omega_1, \omega_1 - \omega_2) \quad (4)$$

where (*) indicates the complex conjugate.

From the definition of equation (3), we can see that $B(\omega_1, \omega_2)$ is periodic both in ω_1 and ω_2 with period 2π. Thus, only the values of $B(\omega_1, \omega_2)$ in the triangular region $\omega_2 \geq 0$, $\omega_1 \geq \omega_2$, $\omega_1 + \omega_2 \leq \pi$ is enough for a complete description of the bispectrum. In Fig. 1, we illustrate this region and also the symmetric relations.

In this paper we use a parametric approached bispectrum instead of a directly double Fourier transform.

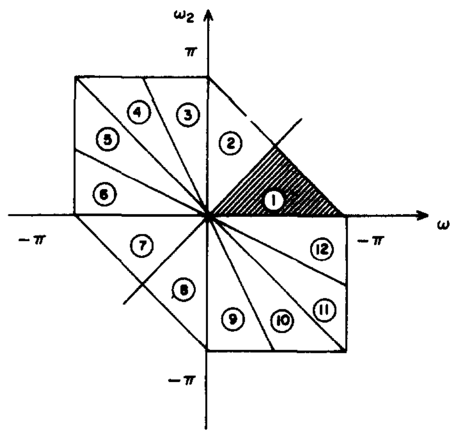

Figure 1. The triangle bounded by $\omega_2 \geq 0$, $\omega_1 \geq \omega_2$ and $\omega_1 + \omega_2 \leq \pi$ are shown as the shaded area. All other regions have the same values as this region.

3. STRUCTURAL SYSTEM MODEL AND ARMA REPRESENTATION

Consider a simple vibration system

$$\mathbf{M}\ddot{\mathbf{Z}}(t) + \mathbf{C}\dot{\mathbf{Z}}(t) + \mathbf{K}\mathbf{Z}(t) = \mathbf{f}(t) \quad (5)$$

with n d.o.f. where **M**, **C**, **K** are $n \times n$ square matrices. And **f** is the n dimensional input random force with

$$\mathbf{E}\{\mathbf{f}(t)\mathbf{f}^T(t)\} = D\delta(t-s). \quad (6)$$

Equation (5) can be simplified to:

$$\ddot{\mathbf{X}}(t) + \mathbf{C}'\dot{\mathbf{X}}(t) + \mathbf{K}'\mathbf{X}(t) = \mathbf{f}'(t) \quad (7)$$

where $\mathbf{C}' = M^{-1/2}\mathbf{C}M^{-1/2}$, $\mathbf{K}' = M^{-1/2}\mathbf{K}M^{-1/2}$, $\mathbf{f}' = M^{-1/2}\mathbf{f}$. This is the general form of vibration system with n d.o.f. Equation (7) can be modified to a set of $2n$ first-order differential equations. Let

$$q_i = X_i, \qquad q_{i+n} = \dot{X}_i.$$

Equation (7) can be represented as

$$\frac{\mathrm{d}}{\mathrm{d}t} q(t) = Aq(t) + Bf(t) \quad (8)$$

where

$$A=\begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{K}' & -\mathbf{C}' \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1/2} \end{bmatrix}.$$

Equation (8) is then a continuous Marcov process and can be discretised to a discrete type as

$$q(t+1)=\exp(\mathbf{A}Ts)q(t)+\int_0^{Ts}\exp[\mathbf{A}(Ts-S)]\boldsymbol{B}f(tTs+S)\,\mathrm{d}s \tag{9}$$

t is a discrete time index as nTs. Ts is the sampling interval and must satisfy $Ts(\lambda i-\lambda j)\neq 0$ (MOD 2π), λi, λj are the distinct eigenvalues of equation (5). If it happens that $Ts(\lambda i-\lambda j)=0$ (MOD 2π), then the output signal for these two modes becomes indistinguishable.

The integral in the right-hand side of equation (9) can be regarded as a set of new random forces with mean zero and covariance matrices

$$\mathbf{D}'=\int_0^{\mathrm{Ts}}\exp(\mathbf{A}s)\mathbf{BDB}^T[\exp(\mathbf{A}s)]^T\,\mathrm{d}s. \tag{10}$$

Equation (9) can be rewritten as

$$q(\mathrm{t}+1)=\exp(\mathbf{A}Ts)q(t)+g(t) \tag{11}$$

which is a discrete Marcov process.

Now, from [3], equation (11) can be represented as an ARMA $(2n, 2n)$ model as follows. Let

$$\mathbf{A}=\exp(\mathbf{A}Ts).$$

We assume that the discrete system is observable, i.e., the $2n\times 2n$ observation matrix L is non-singular, where

$$\mathbf{L}=\begin{bmatrix} \mathbf{S}^T \\ \mathbf{S}^T\mathbf{A} \\ \mathbf{S}^r\mathbf{A}^2 \\ \vdots \\ \mathbf{S}^T\mathbf{A}^{n-1} \end{bmatrix}$$

$$\mathbf{S}'=(0,0,\ldots,1,0,\ldots,0).$$

Depending upon which d.o.f. is observed.

Now apply $\mathbf{L}$ to equation (11), as in [3], equation (11) changes to a similar form

$$\mathbf{r}(t+1)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{2n} & -\alpha_{2n-1} & & \cdots & -\alpha_1 \end{bmatrix}\mathbf{r}(t)+\mathbf{L}g(t) \tag{12}$$

where a_i, $i=1,2,\ldots,2n$ are the coefficients of the AR part of the ARMA model

$$\sum_{i=0}^{2n}\alpha_i Y(t-i)=\sum_{j=1}^{2n}\beta_j X(t-j).$$

Consider the polynomial

$$\sum_{i=0}^{2n}\alpha_i\mu^{2n-i} \tag{13}$$

equation (13) can be represented by the products of its roots

$$\sum_{i=0}^{2n} \alpha_i \mu^{2n-i} = \sum_{j=1}^{n} (\mu - \mu_j)(\mu - \mu_j^*). \tag{14}$$

The damping ratio ξ and the damped frequency ω_d for each mode can be computed by the inverse of

$$\mu_j, \mu_j^* = \exp\left(-\xi_j \omega_j Ts \pm \omega_j Ts\sqrt{1-\xi_j^2}\right) \tag{15}$$

$$j = 1, 2, \ldots, n, \qquad \text{for } \xi_j < 1.$$

Consider a real pth order AR process $X(n)$

$$X(n) + \sum_{i=1}^{p} a_i (X(n-i) = W(n) \tag{16}$$

where $W(n)s$ are white noise with $E\{W(n)\} = 0$, $E\{W(n)W(n)\} = Q$, $E\{W(n)W(n)\} = \beta \neq 0$. We also assume that the $X(m)$ is independent with $W(n)$ for $m < n$. $W(n)$ is non-Gaussian as $\beta \neq 0$.

Because $W(n)$ is third-order stationary, it follows that $X(n)$ is third-order stationary if it is derived from the stable model. Now multiply $X(n-k)X(n-l)$ to both sides of equation (16)

$$R(-k, -l) + \sum_{i=1}^{p} a_i R(i-k, i-l) = \beta\delta(k, l) \tag{17}$$

where

$$\delta(k, l) = \begin{cases} 1, & \text{for } k = 0 \text{ and } l = 0 \\ 0, & \text{else} \end{cases}$$

$R(m, n)$ is the third-order moment of the AR process.

In equation (17), let $k = l$ and varying k, l from 0 to p we get

$$\begin{bmatrix} R(0,0) & R(1,1) & \cdots & R(p,p) \\ R(-1,-1) & R(0,0) & & R(p-1,p-1) \\ \vdots & \vdots & \ddots & \vdots \\ R(-p,-p) & R(-p+1,-p+1) & \cdots & R(0,0) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{18}$$

Equation (18) can be represented as

$$\mathbf{R}a = b$$

R is a Toeplitz form and in general not symmetric. The essential condition for equation (18) to exist is that the polynomial

$$A(Z) = 1 + \sum_{i=1}^{p} a_i Z^{-i} \tag{19}$$

has all its roots inside the unit circle.

Define

$$H(Z) = 1/A(Z) \tag{20}$$

then the bispectrum of this AR process $X(n)$ is given by

$$B(\omega_1, \omega_2) = \beta H(\omega_1) H(\omega_2) H^*(\omega_1 + \omega_2) \tag{21}$$

$H(\omega)$ is $H(z)$ in equation (30) with $z = \exp(i\omega)$. It is obvious that $B(\omega_1, \omega_2)$ satisfies equation (3). Now for a structural system with n d.o.f. we may modify it to a ARMA (p, p) model with $p = 2n$

$$X(n) + \sum_{i=1}^{p} a_i X(n-i) = W(n) + \sum_{j=1}^{p} b_j W(n-j). \tag{22}$$

Equation (18) can be modified for this ARMA model as follows: multiply $X(n-m)X(n-1)$ to both sides of equation (22) we then get equation (17). By varying k, 1 from p to $2p$, which skips the effect of the pth order MA part of equation (22), we get

$$R(-k, -l) + \sum_{i=1}^{p} a_i R(i-k, i-l) = b_p \beta$$

and

$$\mathbf{R}a = b \tag{23}$$

where

$$\mathbf{R} = \begin{bmatrix} R(-p,-p) & R(-p+1,-p+1) & \cdots & R(0,0) \\ R(-p-1,-p-1) & R(-p,-p) & \cdots & R(-1,-1) \\ \vdots & & & \vdots \\ R(-2p,-2p) & R(-2p+1,-2p+1) & \cdots & R(-p,-p) \end{bmatrix}, \quad a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix},$$

$$b = \begin{bmatrix} bp\beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We use the relations $E\{W(n)W(n)W(n)\} = \beta$, $E\{W(n)W(n)W(m)\} = 0$ for $m \neq n$ and $E\{X(n)X(n)W(m)\} = 0$ for $n < m$ in the derivation (23). $\mathbf{R}$ is also a Toeplitz form in equation (23). a_i, $i = 1, 2n$, are the coefficients of the AR part. Then by solving the roots the polynomial

$$\sum_{i=0}^{2n} a_i X^{2n-i}$$

we can evaluate the damped natural frequencies and damping ratios of this system.

Many methods have been developed for solving this problem. Most of them have made an assumption that the input random force is white. In most situations this assumption is right but in some cases this may be wrong. In these cases those methods will fail to work. But if the input force is Gaussian then we can still solve this kind of problem by using the higher order moment method as mentioned earlier. We illustrate this by the definition in equation (1)

$$R(m, n) = E\{X(k)X(k+m)X(k+n)\}$$

if the input force is Gaussian it follows that $X(k)$ is also Gaussian then for the basic property of Gaussian sequence all the third-order moment $R(m, n)s$ are equal to zero. If we apply another set of non-Gaussian white input to this vibration system, then only the effect of this set of input force exists even if the original input force is not white. So we can solve this problem without the problem that coloured input force might have on the system's response. It is also very important that this applied non-Gaussian input should be statistically independent of the original input force.

4. EXAMPLES

We now give some examples.

Consider first a single d.o.f. system

$$X(t)+2\omega\xi X(t)+\omega^2 X(t)=f(t)+g(t) \tag{25}$$

where ω, ξ are the undamped natural frequency and the damping ratio of the system respectively, $f(t)$ is the unexpected input Gaussian force but not white, and $g(t)$ independent with $f(t)$ is the known applied white non-Gaussian noise with $E[g(t)]=0$, $E[g(t)g(t)]=Q$, and $E[g(t)g(t)g(t)]=\beta$.

In this example we set $\omega^2=10{\cdot}0$ and $2\omega\xi=0{\cdot}2$, the coefficient derived from A matrix is (1·0-(1·5816, 0·9608), the damping ratio and damped natural frequency are 0·03163 and 0·5033 Hz respectively.

We use equation (11) for the simulation set sampling time interval equal to 0·2 sec and truncate the first 2000 data so as to avoid the transient effect. $R(m, n)$ is computed as its expected value

$$R(m,n)=\frac{1}{N}\sum_{i=1}^{N-\max(m,n)} X(i)X(i+m)X(i+n). \tag{26}$$

In equation (26) $X(k)$ was subtracted from its mean value and is a zero mean sequence. Equation (23) can be solved with an effective method as in [5] as **R** is a Toeplitz form. We use the second program because matrix **R** in this equation is non-symmetric. The applied force $g(t)$ was produced by calling a Gaussian sequence and taking its absolute value then minimising its mean value. After this procedure $g(t)$ becomes a non-Gaussian white noise with zero mean.

The results are listed in Table 1 and Figs 2 and 3.

Now consider a multi d.o.f. system.

$$\mathbf{M}\ddot{X}(t)+\mathbf{C}\dot{X}(t)+\mathbf{K}X(t)=\mathbf{f}(t)+g(t). \tag{27}$$

We use a three d.o.f. system for illustration, and

$$\mathbf{M}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \mathbf{C}=\begin{bmatrix}0{\cdot}4&-0{\cdot}4&0\\-0{\cdot}4&0{\cdot}8&-0{\cdot}4\\0&-0{\cdot}4&0{\cdot}8\end{bmatrix}$$

$$\mathbf{K}=\begin{bmatrix}49&-49&0\\-49&85&-36\\0&-36&61\end{bmatrix},\qquad \boldsymbol{\xi}=\begin{bmatrix}0{\cdot}0171\\0{\cdot}0435\\0{\cdot}0548\end{bmatrix},\qquad \mathbf{f}=\begin{bmatrix}0{\cdot}385\\1{\cdot}202\\1{\cdot}829\end{bmatrix}.$$

The coefficients of AR derived from **A** matrix are $a=(1{\cdot}0, -2{\cdot}394, 3{\cdot}497, -3{\cdot}764, 3{\cdot}030, -1{\cdot}812, 0{\cdot}741)$, the damping ratios and damped natural frequencies are (0·0171, 0·0435, 0·0548) and (0·385, 1·202, 1·829) respectively. In this case we use an 0·15 sec sampling time interval and the first 2000 data were truncated. The estimated coefficients

TABLE 1

ξ and ω_d for a d.o.f. system

	A	ξ	ω_d
Matrix **A**	(1·0, −1·5816, 0·9608)	0·03163	0·5033
Estimated	(1·0, −1·5741, 0·9595)	0·03227	0·5077

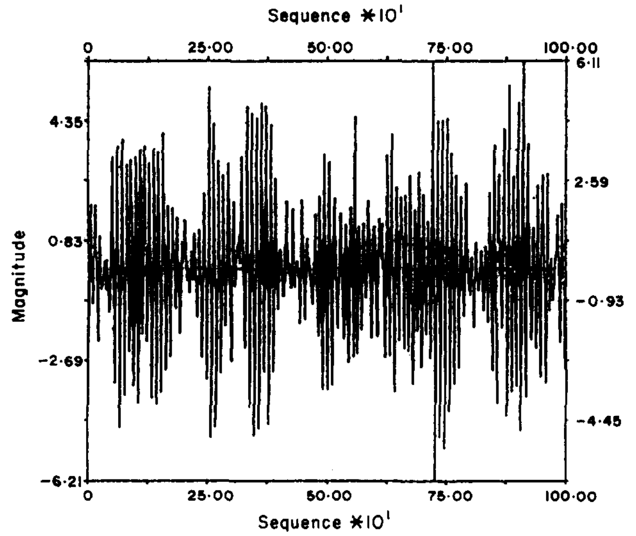

Figure 2. Output signal of one d.o.f. system with $2\xi\omega = 0{\cdot}2$ and $\omega^2 = 10$. Total of 1000 data were plotted after truncating 2000 data.

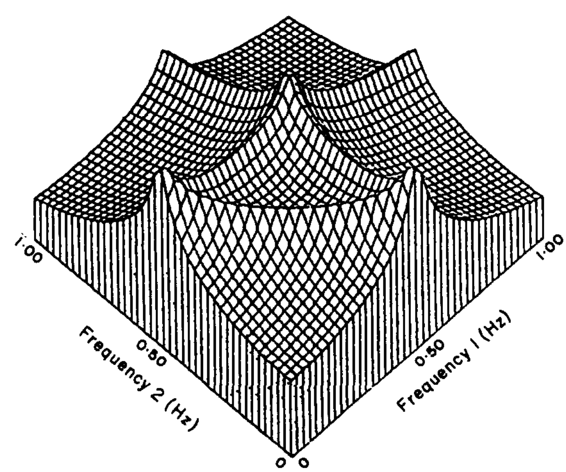

Figure 3. Bispectrum of one d.o.f. system plotted in log scale with sampling frequency 5 Hz. Peak value at about 0.5 Hz.

TABLE 2

ξ_s and f_s parameters for three d.o.f. system

	ξ_1	ξ_2	ξ_3	f_1	f_2	f_3
Matrix A	0·0171	0·0435	0·0548	0·385	1·202	1·829
Estimated	0·0185	0·0467	0·0547	0·395	1·203	1·824

are AR are $a = (1{\cdot}0, -2{\cdot}39, 3{\cdot}501, -3{\cdot}705, 3{\cdot}033, -1{\cdot}801, 0{\cdot}735)$. The results are listed in Table 2 and Figs 4 and 5.

We use only the coefficient of the AR part in evaluating the bispectrums as the effect of the MA part is not so important. The difference between this bispectrum and true bispectrum is only at the valley. The locations of the peaks are still the same. The graphs are plotted with the log scale in magnitude.

If the coefficients of the MA part are also required, we can evaluate them with a similar method. This is discussed in Appendix A.

Both examples illustrated here give satisfactory results because we have an approximate estimate for both damping ratios and natural frequencies. If these results are acceptable, then there remains one problem: how can we determine the d.o.f. of the vibration system? In other words, what value should we assign to n as in equation (5) if we have measured

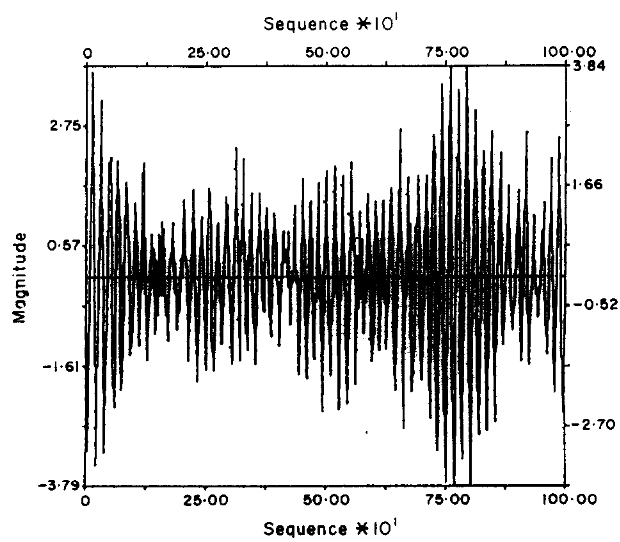

Figure 4. Output signal of three d.o.f. system as in equation (27). Total of 1000 data were plotted after truncating 2000 data.

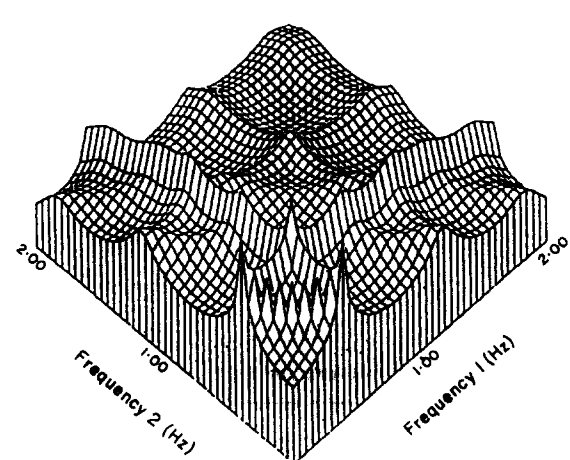

Figure 5. Bispectrum of three d.o.f. system with sampling frequency 6·67 Hz. Respective peak values at about 0·40, 1·20 and 1·80 Hz.

a set of vibration data from a continuous offshore structural system. This is very important. In general, the d.o.f. for a continuous system is infinite. It is impossible for us to solve this vibration problem using an infinite d.o.f. A suitable d.o.f. should be chosen before we estimate the parameters of the vibration system. Many authors have discussed this problem [7]-[9], optimum d.o.f. are chosen with different criteria. In this paper, we have not discussed this problem because the criteria are not as suitable for our estimating method.

In this paper we use third-order moments for the estimation of system parameters because the input force is often seen to be not white. If in some special cases the input forces are known to be white, all the above equations will work with some modification. We discuss this in Appendix B.

REFERENCES

1. D. R. BRILLINGER, 1985 *Annals of Mathematical Statistics* **36**, 1351-1374 An introduction to polyspectra.
2. M. HUZII 1981 *Journal of Time Series Analogs*, **2**, 87-93. Estimation of coefficient of an autoregressive process by using a higher order moment.
3. R. C. K. LEE, 1964 *Optimal Estimation Identification and Control.* Cambridge: M.I.T. Press.
4. M. R. RAGBUVEER and C. L. NIKIAS 1985 *IEEE Transactions*, **ASSP33**, 1213-1230. Bispectrum estimation: a parametric approach.
5. S. ZOHAR, 1979 *IEEE Transactions*, **ASSP27**, 656-658. Fortran subroutines for the solution of Toeplitz sets of linear equations.
6. W. GERSCH and R. S. Z. LIU 1976 *Journal of Applied Mechanics*, 159-165. Time series methods for the synthesis of random vibration systems.
7. W. GERSCH and D. R. SHARPE 1973 *IEEE Transactions on Automatic Control*, 367-369. Estimate of power spectra with finite-order autoregressive models.
8. D. BURSHTEIN and E. WEINSTEIN 1985 *IEEE Transactions*, **ASSP33**, 1017-1019. Some relations between the various criteria for autoregressive model order determination.
9. PIET M. and T. BROERSEN, 1985 *IEEE Transactions*, **ASSP33**, 874-879. Selecting the order of autoregressive models from small samples.

APPENDIX A

If the coefficients of both AR and MA parts are needed, we can solve the coefficients of the MA part b_j after the coefficients of the AR part a have been solved. Now a_i can be solved by equation (23). We solve b_j as follows, rewrite equation (22)

$$X(n)+\sum_{i=1}^{p} a_i X(n-i)=W(n)+\sum_{j=1}^{p} b_i \omega(n-j). \tag{A1}$$

Multiply both sides of equation (A1) by $X(t-p)\times X(t-p)$

$$R(-p,-p)+\sum_{i=1}^{p} a_i R(i-p, i-p)=b_p\beta\equiv H_0. \tag{A2}$$

In this equation we have used the fact that

$$E\{X(m)X(m)W(n)\}=\begin{cases}0 & \text{for } m<n\\ \beta & \text{for } m=n\end{cases}.$$

Now multiply both sides of equation (A1) by $X(t-p)\times X(t-p+k)$, change k from 1 to p, to get a set of recursive equations

$$R(-p,-p+k)+\sum_{i=1}^{p} a_i R(i-p, i-p+k)=a_1H_{k-1}+a_2H_{k-2}+\cdots+a_kH_0+b_kH_0\equiv H_k. \tag{A3}$$

In this equation we use equation (A1) for $X(t-p+k)$. If $(a_i)s$ have been solved then all the $(H_k)s$ can be computed. Now, we get

$$H_0 = b_1 \beta$$
$$H_1 = a_1 H_0 + b_1 H_0$$
$$H_2 = a_1 H_1 + a_2 H_0 + b_2 H_0$$
$$\vdots$$
$$H_p = a_1 H_{P-1} + a_2 H_{P-2} + \cdots + a_p H_0 + b_p H_0$$

and then

$$b_1 = (H_1 - a_1 H_0)/H_0$$
$$b_2 = (H_2 - a_1 H_1 - a_2 H_0)/H_0$$
$$\vdots$$
$$b_p = (H_p - aH_{P-1} - a_2 H_{p-2} - \cdots - a_p H)/H_0$$

which can be written as a formula

$$b_k = \left(H_k - \sum_{i=1}^{p} a_i H_{k-1}\right) H_0 \tag{A4}$$

then all the (b_j)s can be computed.

APPENDIX B

If the original input force is known to be white, then second-order moments $R(m)$s are enough for computing the parameters of the structural vibration system. All the above equations can still work with some modification as follows. Multiply both side of equation (22) with $X(n-k)$

$$R(-k) + \sum_{i=1}^{p} R(i-k) = E\{W(n)X(n-k)\} + \sum_{i=1}^{p} E\{W(n-j)X(n-k)\}.$$

By varying k from p to $2p$ we have the same equation as equation (23)

$$\mathbf{Ra} = \mathbf{b} \tag{A5}$$

but with

$$\mathbf{R} = \begin{bmatrix} R(-p) & R(-p+1) & \cdots & R(0) \\ R(-p-1) & R(-p) & \cdots & R(-1) \\ \vdots & \vdots & & \vdots \\ R(-2p) & R(-2p+1) & \cdots & R(-p) \end{bmatrix} \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} \mathbf{b} = \begin{bmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$E\{W(m)X(n)\} = \begin{cases} \sigma & \text{for } m = n \\ 0 & \text{for } m > n \end{cases}$$
$$\sigma = E\{W^2(m)\}.$$

We define the powers spectrum as

$$P(\omega) = \sigma H(\omega) H^*(\omega) \tag{A6}$$

where $H(\omega)$ is defined as in equation (20) with $z = \exp(i\omega)$.

TABLE B1

ξ and f parameters for a three d.o.f. system

	ξ_1	ξ_2	ξ_3	f_1	f_2	f_3
Matrix A	0·0171	0·0435	0·0548	0·385	1·202	1·829
Estimated	0·0185	0·0422	0·0553	0·385	1·203	1·829

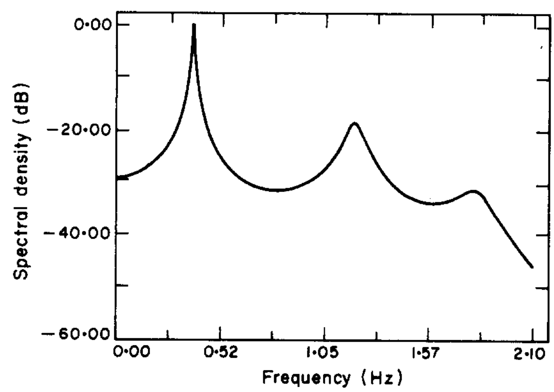

Figure B1. Power spectrum density of three d.o.f. system using second-order moments as in example 2.

As in example 2 with the same system parameters but where the original input force is white, we solve this problem with all the third-order moments replaced by second-order moments. The results are given in Table B1 and Fig. B1. A total of 30 000 data are used with 30 records each having 1000 data. We also truncate the first 2000 data to avoid the transient effect.

The true coefficients of the AR part are

$$\mathbf{a} = (1{\cdot}0, -2{\cdot}394, 3{\cdot}497, -3{\cdot}764, 3{\cdot}030, -1{\cdot}812, 0{\cdot}741)$$

and the estimated coefficients of the AR part are

$$\mathbf{a} = (1{\cdot}0, -2{\cdot}393, 3{\cdot}497, -3{\cdot}762, 3{\cdot}026, -1{\cdot}809, 0{\cdot}741)$$

We also use the AR part for plotting the power spectrum density function as in Fig. B1. It can be seen from the above that we have a better estimate than the one in example 2. The reason is that $R(m)s$ has a better statistic character than $R(m, m)s$.

CONCLUSION

From the above results as in Tables 1 and 2, we can get an approximate estimate of both damped natural frequency and damping ratio although they are not quite correct. This is due to the bad statistic characteristics of $R(n, n)$ as it is a cubic term and very close to zero and is easily affected by the characteristic of the input force. This phenomenon can be improved by taking more data for better averages. In fact, we have made many assumptions in our simulation and these assumptions, in general, were not true, proving that it is not necessary to ask for higher precision. We are continuing our work in this field and trying to improve our methods by using third-order moments $R(m, n)$ instead of $R(n, n)$.