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Normal Bases Expansion of the Discrete Cosine Transform

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***Abstract*—In this correspondence, we show that the discrete cosine transform (DCT) can be obtained by projecting the discrete Fourier transform from the extension field to the basefield. Applying the framework of projection operator, a fast fully recursive algorithm for computing the DCT is also presented.**

I. Introduction

In their recent work [1], Hong *et al.* have drawn out the relationship between the discrete Hartley transform (DHT) and the discrete Fourier transform (DFT) from the viewpoints of field extension and projection. This approach not only demonstrates the intimate connection among various transforms but also presents a powerful framework for developing fast transformation algorithms based on the standard FFT algorithms.

In this correspondence, the result of [1] is extended to derive the discrete cosine transform (DCT), which is an important discrete orthogonal transform in the area of data compression, from the normal bases expansion of the DFT. A new interpretation of the forward and the inverse DCT transforms is then presented. Based on the proposed technique, a fully recursive fast DCT algorithm can be derived more easily in a systematic way.

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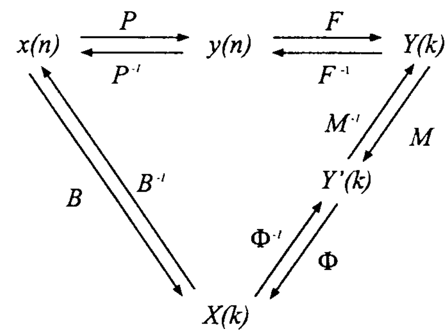

Fig. 1. Relationship among $x(n), y(n), Y(k), Y'(k)$, and $X(k)$.

II. A New Interpretation of the Discrete Cosine Transform

The DCT of an N-point real sequence $x(n)$ is given by [2], [3]

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{k(2n+1)\pi}{2N}\right), \quad k = 0, 1, \ldots, N-1 \tag{1}$$

where, for convenience, the $1/\sqrt{2}$ normalization factor for $X(0)$ is not included. For simplicity of expression, assume that N is even. Define a new sequence $y(n)$ by permuting $x(n)$ as follows:

$$y(n) = \begin{cases} x(2n), & n = 0, 1, \ldots, N/2 - 1 \\ x(2N - 2n - 1), & n = N/2, \ldots, N - 1. \end{cases} \tag{2}$$

The DFT of $y(n)$ is defined as

$$Y(k) = \sum_{n=0}^{N-1} y(n) \omega_N^{nk}, \quad k = 0, 1, \ldots, N-1 \tag{3}$$

where $\omega_N = e^{\frac{-j2\pi}{N}}$, and $j = \sqrt{-1}$. Equations (2) and (3) can, respectively, be represented in a more compact form as

$$\{y(n)\} = P\{x(n)\} \qquad n = 0, 1, \ldots, N-1 \tag{4}$$

$$\{Y(k)\} = \mathcal{F}\{y(n)\}, \quad n,\ k = 0, 1, \ldots, N-1. \tag{5}$$

It follows that P and $\mathcal{F}$ are invertible. In order to have a fixed basis, $Y(k)$ must be multiplied by an index k dependent weight factor to form $Y'(k)$. That is, the k-variable dependent weight factor will be embedded in $Y'(K)$ (the so-called weighted Fourier coefficient). In other words, $Y'(k)$ can be represented in a more compact form as

$$\{Y'(k)\} = M\{Y(k)\} = Y(k) \cdot \omega_{4N}^{k - \frac{N}{2}}, \quad k = 0, 1, \ldots, N-1. \tag{6}$$

In order to have a better understanding and for the ease of explanation, the relations among $x(n), y(n), X(k), Y(k)$, and $Y'(k)$ are illustrated in Fig. 1. In the figure, the map ϕ between $Y'(k)$ and $X(k)$ is what we are looking for. By composition of maps, we will show that the so-called basefield transform B describes exactly the relation presented in (1). Note that $x(n)$ and $X(k)$ reside in the basefield K (real field), whereas $Y(k)$ and $Y'(k)$ are in the finite extension field F (complex field) of K.

Since ϕ is a linear functional on F_K, then there exists an $\alpha \in F$ such that [4]

$$\phi(\xi) = \mathrm{Tr}(\alpha\xi), \quad \forall \xi \in F. \tag{7}$$

By applying the relation of (7) to the weighted Fourier coefficients $Y'(k)$, it follows that

$$\phi(Y'(k)) = \mathrm{Tr}(\alpha Y'(k)) = \alpha Y'(k) + \alpha^* Y'^*(k), \quad 0 \le k \le N-1. \tag{8}$$

As it was shown in [4], ϕ is a linear functional, and the map

$$\phi : Y'(k) \mapsto X(k) = \mathrm{Tr}(\alpha Y'(k)) \tag{9}$$

defines a one-to-one correspondence between $Y'(k)$ and $X(k)$. Since $Y(k)$ is the Fourier transform of a real sequence, it satisfies the conjugacy relation, i.e., $Y^*(k) = Y(N-k)$. In addition, it is also satisfied with $Y'^*(k) = Y'(N-k)$ because

$$\begin{aligned} Y'(N-k) &= Y(N-k) \cdot \omega_{4N}^{(N-k)-\frac{N}{2}} \\ &= Y^*(k) \cdot \left(\omega_{4N}^{k-\frac{N}{2}}\right)^* = Y'^*(k). \end{aligned} \tag{10}$$

Consider the action of ϕ on a conjugacy class $\{Y'(k), Y'(N-k)\}$.

$$X(k) = \phi(Y'(k)) = \mathrm{Tr}(\alpha Y'(k)) = \alpha Y'(k) + \alpha^* Y'^*(k). \tag{11}$$

$$\begin{aligned} X(N-k) &= \phi(Y'(N-k)) = \mathrm{Tr}(\alpha Y'(N-k)) \\ &= \alpha^* Y'(k) + \alpha Y'^*(k). \end{aligned} \tag{12}$$

Notice that (11) and (12) show that the DCT coefficients $X(k)$ and $X(N-k)$ can be interpreted as the linear combination of the conjugacy class of the weighted Fourier coefficients with respect to the normal basis generated by α. The corresponding matrix M for the projection of conjugacy class can then be obtained

$$\begin{aligned} \begin{pmatrix} X(k) \\ X(N-k) \end{pmatrix} &\triangleq M \cdot \begin{pmatrix} Y'(k) \\ Y'(N-k) \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \alpha^* \\ \alpha^* & \alpha \end{pmatrix} \cdot \begin{pmatrix} Y'(k) \\ Y'(N-k) \end{pmatrix} \end{aligned} \tag{13}$$

where α is equal to $\frac{1}{2}\omega_{4N}^{\frac{N}{2}}$. To have an invertible basefield transform, it must guarantee that M^{-1} exists. In addition, it is easy to derive that

$$M^{-1} \triangleq \begin{pmatrix} \alpha & \alpha^* \\ \alpha^* & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} \beta & \beta^* \\ \beta^* & \beta \end{pmatrix} \tag{14}$$

where $\beta = 2\alpha^* = \omega_{4N}^{-\frac{N}{2}}$. In other words, the weighted Fourier coefficients can be obtained from the DCT coefficients as follows:

$$\begin{pmatrix} Y'(k) \\ Y'(N-k) \end{pmatrix} = \begin{pmatrix} \beta & \beta^* \\ \beta^* & \beta \end{pmatrix} \cdot \begin{pmatrix} X(k) \\ X(N-k) \end{pmatrix}. \tag{15}$$

Since $\phi, M, \mathcal{F}$, and P are bijective maps, the basefield transform B can be obtained as

$$\begin{aligned} X(k) &= B\{x(n)\} \\ &= \phi \circ M \circ \mathcal{F} \circ P\{x(n)\} = \phi \circ M \circ \mathcal{F}\{y(n)\} = \phi\{Y'(k)\} \\ &= \mathrm{Tr}(\alpha Y'(k)) = \alpha Y'(k) + \alpha^* Y'^*(k) \\ &= \mathrm{Real}(2\alpha Y'(k)) \\ &= \mathrm{Real}\left(2\alpha\omega_{4N}^{k-\frac{N}{2}} \sum_{n=0}^{N-1} y(n)\omega_N^{nk}\right) \\ &= \mathrm{Real}\left[\omega_{4N}^{k}\left(\sum_{n=0}^{N/2-1} x(2n)\omega_N^{nk} \right.\right. \\ &\qquad \left.\left. + \sum_{n=N/2}^{N-1} x(2N-2n-1)\omega_N^{nk}\right)\right] \\ &= \mathrm{Real}\left[\omega_{4N}^{k}\left(\sum_{n:\mathrm{even}} x(n)\omega_N^{nk/2} \right.\right. \\ &\qquad \left.\left. + \sum_{n:\mathrm{odd}} x(n)\omega_N^{(N-(n+1)/2)k}\right)\right] \\ &= \mathrm{Real}\left(\sum_{n:\mathrm{even}} x(n)\omega_N^{(n/2+1/4)k} + \sum_{n:\mathrm{odd}} x(n)\omega_N^{-(n/2+1/4)k}\right) \\ &= \sum_{n=0}^{N-1} x(n)\cos\frac{\pi(2n+1)k}{N} \end{aligned} \tag{16}$$

where "$\circ$" denotes the operation of composition. Equation (16) shows that the DCT coefficients can be obtained as a projection of the weighted Fourier coefficients from the extension field to the basefield. In addition, the corresponding inverse relation can be obtained by applying B^{-1} $(= P^{-1} \circ \mathcal{F}^{-1} \circ M^{-1} \circ \phi^{-1})$ to $X(k)$, that is, applying $\mathcal{F}^{-1} \circ M^{-1} \circ \phi^{-1}$ to $X(k)$ to derive $y(n)$ and then applying P^{-1} to $y(n)$ to obtain $x(n)$.

$$\begin{aligned} y(n) &= \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\omega_N^{-nk} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} Y'(k)\omega_{4N}^{\frac{N}{2}-k}\omega_N^{-nk} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} (\beta X(k) + \beta^* X(N-k))\omega_{4N}^{\frac{N}{2}-k}\omega_N^{-nk} \\ &= \frac{1}{N}\left[X(0) + \sum_{k=1}^{N-1}\left(X(k)\omega_N^{\left(-n+\frac{1}{4}\right)k} + \omega_{4N}^{N-k} X(N-k)\omega_N^{-nk}\right)\right] \\ &= \frac{1}{N}\left[X(0) + \sum_{k=1}^{N-1}\left(X(k)\omega_N^{\left(-n+\frac{1}{4}\right)k} + X(k)\omega_N^{n+\frac{1}{4}k}\right)\right] \\ &= \frac{1}{N}\left[X(0) + \sum_{k=1}^{N-1}\left(2X(k)\cos\frac{\pi(4n+1)k}{N}\right)\right]. \end{aligned}$$

Applying P^{-1} to $y(n)$, it follows that

$$\begin{aligned} x(n) &= \frac{1}{N}\left[X(0) + \sum_{k=1}^{N-1}\left(2X(k)\cos\frac{\pi(2n+1)k}{N}\right)\right] \\ &= c_n \sum_{k=0}^{N-1} X(k)\cos\frac{\pi(2n+1)k}{N} \end{aligned} \tag{17}$$

where

$$c_n = \begin{cases} \frac{1}{N}, & \text{for } n = 0 \\ \frac{2}{N}, & \text{otherwise.} \end{cases} \tag{18}$$

Equation (17) shows that the inverse DCT can be obtained by applying the same technique as the forward DCT. In addition, this result coincides with the one given in [2].

III. A New Fully Recursive Fast DCT Algorithm

In this section, a fast algorithm for the DCT is derived from the viewpoints of field projection. If we assume the input is a real sequence, then only the Fourier kernel part and the twiddle factors contain complex numbers in $Y'(k)$. Therefore, only the above two factors need to be projected from $Y'(k)$ to $X(k)$. Based on the idea of [1], the coefficients of the kernel part and the twiddle factors are expanded with respect to different bases $\{\beta_0, \beta_1\}$ and $\{\gamma_0, \gamma_1\} = \{1, -i\}$, respectively. As pointed out previously, the linear functionals ϕ and ϕ^{-1} can be expanded by using the normal bases $\{\alpha_0, \alpha_1\} = \{\alpha, \alpha^*\} = \{\frac{1}{2}\omega_{4N}^{N/2}, \frac{1}{2}\omega_{4N}^{-N/2}\}$

and $\{\beta_0, \beta_1\} = \{\beta, \beta^*\} = \{\omega_{4N}^{-N/2}, \omega_{4N}^{N/2}\}$, respectively. Thus, the fast DCT algorithm can be derived as follows:

$$\begin{aligned} Y(k) &= \sum_{n=0}^{N-1} y(n)\omega_N^{nk} \\ &= \sum_{n=0}^{N/2-1} y(n)\omega_N^{nk} + \sum_{n=0}^{N/2-1} y(N-n-1)\omega_N^{(N-n-1)k} \\ &= \sum_{n=0}^{N/2-1} x(2n)\omega_N^{nk} + \sum_{n=0}^{N/2-1} x(2n+1)\omega_N^{-(n+1)k}. \end{aligned}$$

By applying the M operator, we can map $Y(k)$ into $Y'(k)$ as follows:

$$\begin{aligned} &Y'(k) \\ &= \sum_{n=0}^{N/2-1} x(2n)\omega_N^{nk+\frac{k}{4}-\frac{N}{8}} + \sum_{n=0}^{N/2-1} x(2n+1)\omega_N^{-(n+1)k+\frac{k}{4}-\frac{N}{8}} \\ &= \omega_{4N}^{-k}\left(\sum_{n=0}^{N/2-1} x(2n)\omega_N^{nk+\frac{k}{2}-\frac{N}{8}} + \sum_{n=0}^{N/2-1} x(2n+1)\omega_N^{-nk-\frac{k}{2}-\frac{N}{8}}\right) \\ &= \sum_{l=0}^{1}\omega_{-k}^{(l)}\gamma_l\left(\sum_{n=0}^{N/2-1} x(2n)\sum_{j=0}^{1}\omega_{nk+\frac{k}{2}-\frac{N}{8}}^{(j)}\beta_j \right. \\ &\quad \left. + \sum_{n=0}^{N/2-1} x(2n+1)\sum_{j=0}^{1}\omega_{-nk-\frac{k}{2}-\frac{N}{8}}^{(j)}\beta_j\right) \\ &= \sum_{l,j=0}^{1}\sum_{n=0}^{N/2-1} x(2n)\omega_{-k}^{(l)}\omega_{nk+\frac{k}{2}-\frac{N}{8}}^{(j)}\gamma_l\beta_j \\ &\quad + \sum_{l,j=0}^{1}\sum_{n=0}^{N/2-1} x(2n+1)\omega_{-k}^{(l)}\omega_{-nk-\frac{k}{2}-\frac{N}{8}}^{(j)}\gamma_l\beta_j. \end{aligned}$$

Projecting the above equation onto the real axis via ϕ yields

$$\begin{aligned} X(k) &= \sum_{l,j=0}^{1}\sum_{n=0}^{N/2-1} x(2n)\omega_{-k}^{(l)}\omega_{nk+\frac{k}{2}-\frac{N}{8}}^{(j)}\mathrm{Tr}(\alpha\gamma_l\beta_j) \\ &\quad + \sum_{l,j=0}^{1}\sum_{n=0}^{N/2-1} x(2n+1)\omega_{-k}^{(l)}\omega_{-nk-\frac{k}{2}-\frac{N}{8}}^{(j)}\mathrm{Tr}(\alpha\gamma_l\beta_j) \\ &= \sum_{n=0}^{N/2-1} x(2n)\left(\cos\frac{k\pi}{2N}\cos\frac{\pi(4n+2)k}{2N}\right. \\ &\quad \left. + \sin\frac{k\pi}{2N}\sin\frac{\pi(4n+2)k}{2N}\right) \\ &\quad + \sum_{n=0}^{N/2-1} x(2n+1)\left(\cos\frac{k\pi}{2N}\cos\frac{\pi(-4n-2)k}{2N}\right. \\ &\quad \left. + \sin\frac{k\pi}{2N}\sin\frac{\pi(-4n-2)k}{2N}\right) \\ &= \sum_{n=0}^{N/2-1}(x(2n)+x(2n+1))\cos\frac{\pi(2n+1)k}{2\cdot N/2}\cos\frac{k\pi}{2N} \\ &\quad + \sum_{n=0}^{N/2-1}(x(2n)-x(2n+1))\sin\frac{\pi(2n+1)k}{2\cdot N/2}\sin\frac{k\pi}{2N}. \end{aligned} \tag{19}$$

Equation (19) shows that the N-point DCT can be decomposed into a $N/2$-point type-II DCT and $N/2$-point type-II discrete sine transform (DST) [5]. As shown in [5], the N-point type-II DCT and type-II DST require only $O(N\log_2 N)$ multiplications and additions. Thus, (19) can be obtained in $O(N\log_2 N)$ multiplications and additions as well. If we set

$$G(k) = \sum_{n=0}^{N/2-1}(x(2n)+x(2n+1))\cos\frac{\pi(2n+1)k}{2\cdot N/2}$$

and

$$H(k) = \sum_{n=0}^{N/2-1}(x(2n)-x(2n+1))\sin\frac{\pi(2n+1)k}{2\cdot N/2}$$

then

$$\begin{aligned} &X(N-k) \\ &\quad = \phi(Y'(N-k)) = -G(k)\sin\frac{k\pi}{2N} + H(k)\cos\frac{k\pi}{2N}, \\ &\qquad k = 1, 2, \ldots, N/2-1. \end{aligned} \tag{20}$$

Thus, an $O(N\log_2 N)$ fully recursive fast DCT algorithm can be obtained as

$$\begin{pmatrix} X(k) \\ X(N-k) \end{pmatrix} = \begin{pmatrix} \cos\frac{k\pi}{2N} & \sin\frac{k\pi}{2N} \\ -\sin\frac{k\pi}{2N} & \cos\frac{k\pi}{2N} \end{pmatrix}\begin{pmatrix} G(k) \\ H(k) \end{pmatrix}, \quad k = 1, 2, \ldots, N/2-1 \tag{21}$$

where

$$X(0) = H(0) \quad \text{and} \quad X(N/2) = \cos\frac{\pi}{4}\cdot H(N/2).$$

IV. Conclusion

Normal bases expansion provides a general framework for constructing the isomorphic mapping between the basefield and the extension field transforms. Based on this technique, in this correspondence, the DCT can be treated as the projection of the weighted DFT. Moreover, a new fast DCT algorithm can be obtained in a systematic way, and the convolution property of the basefield transform can be derived accordingly.

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