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A Block Representation for Products of Hyperbolic Householder Transform

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Abstract—In this paper, a block representation for products of hyperbolic Householder transform, which is rich in matrix-matrix multiplications, is presented. Not only the representation is derived by a rather straightforward way, but it also extends the previous results [1,2] to the complex domain.

Keywords—BLAS 3 operation, Block representation, Complex domain, Hyperbolic Householder transform, QR factorization.

1. INTRODUCTION

The Householder transform [3] is very useful in matrix computations and signal processing [4]. In order to increase the performance of the Householder transform for QR factorization on vector supercomputers like CRAY series, Bischof and Van Loan [5] presented the first block Householder transform in terms of WY representation, which is rich in matrix-matrix multiplications, i.e., BLAS 3 operation [6]. Later, Schreiber and Van Loan [1] proposed a compact WY representation. Puglisi [2] presented an improved algorithm for involving more BLAS 3 operations based on the Woodbury-Morrison formula. We refer the reader to [7,8] for numerical behaviors of the compact representation.

In this paper, a block representation for products of hyperbolic Householder transform, which is rich in matrix-matrix multiplications, is presented. Not only the representation is derived by a rather straightforward way instead of using the Woodbury-Morrison formula, but it also extends the previous results [1,2] to the complex domain.

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In Section 2, we first introduce the form for the complex Householder transform by Chung and Yan [9], then propose an alternative form and this form will be used to derive the block representation for the hyperbolic Householder transform.

2. THE COMPLEX HOUSEHOLDER TRANSFORM

In [10], the complex Hermitian transform has been developed. Recently, Venkaial, Krishna, and Paulraj [11] also extended the real Householder transform [3] to the complex domain C^n. They first guessed the transform H being $H = I - (1 + (\mathbf{a}^*\mathbf{z}/\mathbf{z}^*\mathbf{a}))(\mathbf{z}\mathbf{z}^*/\mathbf{z}^*\mathbf{z})$, where $\mathbf{a}, \mathbf{b} \in C^n$ and $\mathbf{z} = \mathbf{a} - \mathbf{b}$, then it was verified that $H\mathbf{a} = \mathbf{b}$ and H is unitary. Later, Xia and Suter [12] proved the necessary part of the Householder transform [11]. If $\mathbf{a}^*\mathbf{a} \neq \mathbf{a}^*\mathbf{b}$, they first guessed that $H = I - (1+y)(\mathbf{z}\mathbf{z}^*/\mathbf{z}^*\mathbf{z})$, where y is a complex number, then it was shown that $y = -(\mathbf{z}^*\mathbf{b}/\mathbf{z}^*\mathbf{a})$.

In the work of Chung and Yan [9], a complex Householder transform, $H = I - (\mathbf{z}\mathbf{z}^*/\mathbf{z}^*\mathbf{a})$, is given. This transform still satisfies the requirements $H\mathbf{a} = \mathbf{b}$ and H is unitary. Specifically, the transform is shown by a straightforward derivation although the two forms in [11,12] can be simplified to $I - (\mathbf{z}\mathbf{z}^*/\mathbf{z}^*\mathbf{a})$. Let Φ be a diagonal matrix with diagonal entries $+1$ and -1. Suppose it satisfies $\mathbf{a}^*\Phi\mathbf{a} = \mathbf{b}^*\Phi\mathbf{b}$ and $\Phi\mathbf{a} \neq \mathbf{b}$, where $\mathbf{a}$, $\mathbf{b} \in C^n$. In the hyperbolic Householder transform [13,14], we want to find a hypernormal matrix H such that $H\mathbf{a} = \mathbf{b}$ and $H^*\Phi H = \Phi$. According to the derivation in [9], we obtain

$$H = \hat{H}(\mathbf{a}, \mathbf{b}) = \Phi - \frac{\mathbf{z}\mathbf{z}^*}{\mathbf{z}^*\mathbf{a}}, \qquad \text{where } \mathbf{z} = \Phi\mathbf{a} - \mathbf{b}.$$

Note that the above hyperbolic Householder transform is equal to the complex Householder transform when $\Phi = I$.

For deriving the block representation of the hyperbolic Householder transform, we use an alternative form, $H = \Phi\hat{H}(\mathbf{a}, \Phi\mathbf{b}) = I - \Phi\mathbf{w}t\mathbf{w}^*$, for hyperbolic Householder transform, where $\mathbf{w} = \Phi\mathbf{a} - \Phi\mathbf{b}$ and $t = (1/\mathbf{w}^*\mathbf{a})$. This alternative form also satisfies $H\mathbf{a} = \mathbf{b}$ and $H^*\Phi H = \Phi$ (see the Appendix).

3. THE BLOCK HYPERBOLIC HOUSEHOLDER TRANSFORM

As what follows, some notations follow those used in [4]. Suppose $Q_m = H_1H_2\dots H_m$ is a product of these m $(< n)$ alternative $n \times n$ hyperbolic Householder matrices as described in Section 2. Let $Q_m = I - \Phi Y_m T_m Y_m^*$ and $H_{m+1} = I - \Phi\mathbf{y}_{m+1}t_{m+1}^{-1}\mathbf{y}_{m+1}^*$, where Y_m is a $n \times m$ matrix, T_m is a $m \times m$ matrix, $\mathbf{y}_{m+1} = \Phi\mathbf{a}_{m+1} - \Phi\mathbf{b}_{m+1}$ ($\mathbf{a}_{m+1}^*\Phi\mathbf{a}_{m+1} = \mathbf{b}_{m+1}^*\Phi\mathbf{b}_{m+1}$ and $\mathbf{a}_{m+1} \neq \mathbf{b}_{m+1}$), and $t_{m+1} = \mathbf{y}_{m+1}^*\mathbf{a}_{m+1}$. The derivation to the block representation of Q_mH_{m+1} is shown as follows.

Let $Q_{m+1} = Q_mH_{m+1}$, then we have

$$Q_{m+1} = (I - \Phi Y_m T_m Y_m^*)(I - \Phi\mathbf{y}_{m+1}t_{m+1}^{-1}\mathbf{y}_{m+1}^*) = I - \Phi E, \tag{1}$$

where $E = Y_mT_mY_m^* + \mathbf{y}_{m+1}t_{m+1}^{-1}\mathbf{y}_{m+1}^* - Y_mT_mY_m^*\Phi\mathbf{y}_{m+1}t_{m+1}^{-1}\mathbf{y}_{m+1}^*$. Since the leftmost side of each term of E is Y_m or $\mathbf{y}_{m+1}$; the rightmost side of each term of E is Y_m^* or $\mathbf{y}_{m+1}^*$, we let

$$E = Y_{m+1}T_{m+1}Y_{m+1}^*, \tag{2}$$

where

$$Y_{m+1} = \begin{pmatrix} Y_m & \mathbf{y}_{m+1} \end{pmatrix} \quad \text{and} \quad T_{m+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} Y_{m+1}T_{m+1}Y_{m+1}^* &= Y_mAY_m^* + Y_mB\mathbf{y}_{m+1}^* + \mathbf{y}_{m+1}CY_m^* + \mathbf{y}_{m+1}D\mathbf{y}_{m+1}^* \\ &= Y_mT_mY_m^* + \mathbf{y}_{m+1}t_{m+1}^{-1}\mathbf{y}_{m+1}^* - Y_mT_mY_m^*\Phi\mathbf{y}_{m+1}t_{m+1}^{-1}\mathbf{y}_{m+1}^*, \end{aligned}$$

where $A = T_m$, $B = -T_m Y_m^* \Phi t_{m+1}^{-1} \mathbf{y}_{m+1}$, $C = 0$, and $D = t_{m+1}^{-1}$. It follows that

$$T_{m+1} = \begin{pmatrix} T_m & -T_m Y_m^* \Phi t_{m+1}^{-1} \mathbf{y}_{m+1} \\ 0 & t_{m+1}^{-1} \end{pmatrix}. \tag{3}$$

By (1)–(3), we have

$$Q_{m+1} = I - \Phi Y_{m+1} T_{m+1} Y_{m+1}^*. \tag{4}$$

Equation (4) extends the previous result [1] to the complex domain.

By induction, we have $Q_k = I - \Phi Y_k T_k Y_k^*$ for $k = 1, 2, \dots,$ where

$$Y_k = (\mathbf{y}_1 \quad \mathbf{y}_2 \quad \cdots \quad \mathbf{y}_k),$$
$$T_k = \begin{pmatrix} T_{k-1} & -T_{k-1} Y_{k-1}^* \Phi t_k^{-1} \mathbf{y}_k \\ 0 & t_k^{-1} \end{pmatrix},$$

and $T_1 = t_1^{-1}$. By (3), we also have

$$T_{m+1}^{-1} = \begin{pmatrix} T_m^{-1} & Y_m^* \Phi \mathbf{y}_{m+1} \\ 0 & t_{m+1} \end{pmatrix}.$$

Let $S_k = T_k^{-1}$ for $k = 1, 2 \dots,$ then it follows that $Q_k = I - \Phi Y_k S_k^{-1} Y_k^*$, where

$$S_k = T_k^{-1} = \begin{pmatrix} S_{k-1} & Y_{k-1}^* \Phi \mathbf{y}_k \\ 0 & t_k \end{pmatrix}, \qquad \text{with } S_1 = t_1. \tag{5}$$

Now we consider $s_{k,ij}$, the ij entry of S_k, by (5), it is given by

$$\begin{aligned} s_{k,ij} &= s_{j,ij} = \mathbf{y}_i^* \Phi \mathbf{y}_j, && 1 < i < j \le k, \\ s_{k,ij} &= s_{i,ij} = 0, && 1 < j < i \le k, \\ s_{k,ii} &= s_{i,ii} = t_i. \end{aligned}$$

That is, we have

$$S_k = \operatorname{diag}(t_1, t_2, \dots, t_k) + A_k,$$

where $A_k = [a_{ij}]$ and $a_{ij} = \mathbf{y}_i^* \Phi \mathbf{y}_j$ for $1 < i < j \le k$; $a_{ij} = 0$ otherwise. The above block representation, $Q_k = I - \Phi Y_k S_k^{-1} Y_k^*$, is the same as the one [2] when $\Phi = I$. On the other hand, our block representation also extends the previous result [2] to the complex domain.

APPENDIX

From

$$\begin{aligned} \mathbf{w}^*\mathbf{a} + \mathbf{a}^*\mathbf{w} &= (\Phi(\mathbf{a}-\mathbf{b}))^*\mathbf{a} + \mathbf{a}^*(\Phi(\mathbf{a}-\mathbf{b})) = \mathbf{a}^*\Phi\mathbf{a} - \mathbf{b}^*\Phi\mathbf{a} + \mathbf{a}^*\Phi\mathbf{a} - \mathbf{a}^*\Phi\mathbf{b}, \\ \mathbf{w}^*\Phi\mathbf{w} &= (\Phi(\mathbf{a}-\mathbf{b}))^*\Phi(\Phi(\mathbf{a}-\mathbf{b})) = (\mathbf{a}-\mathbf{b})^*\Phi(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}^*\Phi\mathbf{a} - \mathbf{a}^*\Phi\mathbf{b} - \mathbf{b}^*\Phi\mathbf{a} + \mathbf{b}^*\Phi\mathbf{b} \end{aligned}$$

and

$$\mathbf{w}^*\mathbf{a} + \mathbf{a}^*\mathbf{w} - \mathbf{w}^*\Phi\mathbf{w} = \mathbf{a}^*\Phi\mathbf{a} - \mathbf{b}^*\Phi\mathbf{b} = 0,$$

it yields to

$$\begin{aligned} \mathbf{H}^*\Phi\mathbf{H} &= \left(I - \Phi\frac{\mathbf{w}\mathbf{w}^*}{\mathbf{w}^*\mathbf{a}}\right)^* \Phi \left(I - \Phi\frac{\mathbf{w}\mathbf{w}^*}{\mathbf{w}^*\mathbf{a}}\right) = \Phi - \left(\frac{1}{\mathbf{a}^*\mathbf{w}} + \frac{1}{\mathbf{w}^*\mathbf{a}} - \frac{\mathbf{w}^*\Phi\mathbf{w}}{(\mathbf{w}^*\mathbf{a})(\mathbf{a}^*\mathbf{w})}\right)\mathbf{w}\mathbf{w}^* \\ &= \Phi - \frac{\mathbf{w}^*\mathbf{a} + \mathbf{a}^*\mathbf{w} - \mathbf{w}^*\Phi\mathbf{w}}{(\mathbf{w}^*\mathbf{a})(\mathbf{a}^*\mathbf{w})}\mathbf{w}\mathbf{w}^* = \Phi. \end{aligned}$$

Finally, $\mathbf{Ha} = \mathbf{b}$ can be verified as follows:

$$\mathbf{Ha} = \left(I - \Phi\frac{\mathbf{w}\mathbf{w}^*}{\mathbf{w}^*\mathbf{a}}\right)\mathbf{a} = \mathbf{a} - \Phi\mathbf{w} = \mathbf{a} - (\mathbf{a}-\mathbf{b}) = \mathbf{b}.$$

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