行政院國家科學委員會專題研究計畫成果報告

小波轉換於多載波系統之應用(III)
Discrete Multitone Modulation Using Wavelet(III)
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1 中文摘要

這是研究離散多載波系統三年期計畫之第三年.本計畫第一年我們成功地推導出能完全消除ISI的轉化多工器.第二年我們用這理論來設計具有好的頻率選擇性的DMT系統.在本計畫中,我們將進一步運用這些理論來設計在受有色雜訊干擾之衰減通道環境底下的最佳轉化多工器.
關鍵詞：轉化多工器,多調變系統,多載波系統

摘要

This is the third year part of a three-year project. In the first year of the project, we have successfully derived the ISI-free transceivers for frequency distorted channel. The formulation has been applied to the design of transceivers with good frequency separation in the second year of the project. In this project, we will further apply the technique developed in the first two years to the design of optimal transceivers for distorted channel with colored noise.
Keywords: transceiver, multitone modulation.

2 緣由與目的

There has been great interest in the design of DMT systems recently. Fig. 1 shows an example of an M-band DMT transceiver over channel $C(z)$ with additive noise $\nu(n)$. The example is the so-called *block based* DMT (BDMT), where the transmitter and the receiver consist of constant matrices. The encoding at the transmitter end and the decoding at the receiver end are done blockwise. The DMT is called *orthogonal* if the transmitter $\mathbf{G}_0$ in Fig. 1 is an orthogonal matrix, i.e., $\mathbf{G}_0^T\mathbf{G}_0$ is a diagonal matrix. We call it *biorthogonal* if $\mathbf{G}_0$ is not orthogonal. BDMT transceivers have been studied extensively. In the commonly used DFT based DMT, the transmitter and the receiver are DFT matrices [1]. In [2], more general orthogonal matrices are proposed. It is shown therein that, for AWGN (additive white Gaussian noise) frequency selective channels the optimal orthogonal transmitter consists of eigen vectors associated with the channel. In this project, we will show that for any given transmission rate and probability of error, the optimal biorthogonal transceiver that minimizes the transmission power is orthogonal.

3 結果與討論:

Consider Fig. 1. Assume that $\nu(n)$ is a zero-mean WSS process and $C(z)$ is an FIR filter of order L. So the length of redundant samples is chosen to be L and $N = M + L$. The transmitter is of the form,

$$\mathbf{G}_0 = \begin{pmatrix}\mathbf{G}\\ \mathbf{0}\end{pmatrix},$$

where $\mathbf{G}$ is of dimensions M by M. Using this choice of $\mathbf{G}_0$ and some multirate identities, Fig. 1 can be redrawn as Fig. 2. The N by M matrix $\mathbf{C}$ is a lower triangular Toeplitz matrix. Using SVD, we decompose the channel matrix $\mathbf{C}$ as

$$\mathbf{C} = \underbrace{\begin{pmatrix}\mathbf{U}_0 & \mathbf{U}_1\end{pmatrix}}_{\mathbf{U}}\begin{pmatrix}\mathbf{\Lambda}\\ \mathbf{0}\end{pmatrix}\mathbf{V} = \mathbf{U}\begin{pmatrix}\mathbf{\Lambda}\\ \mathbf{0}\end{pmatrix}\mathbf{V}, \quad (1)$$

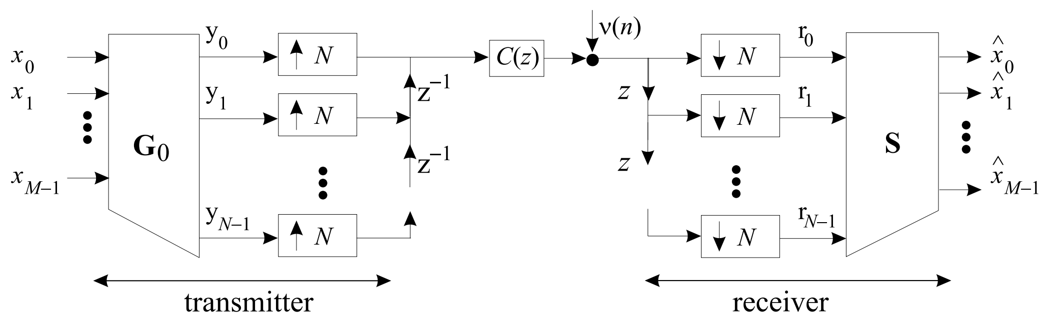

圖 1: An M-band BDMT transceiver over channel $C(z)$ with noise $\nu(n)$.

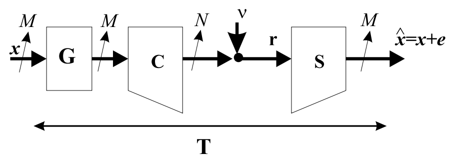

圖 2: Matrix representation of the zero padded B-DMT transceiver.

where $\mathbf{U}$ and $\mathbf{V}$ are orthonormal matrices of dimensions respectively $N \times N$ and $M \times M$, and $\mathbf{\Lambda}$ is an $M \times M$ diagonal matrix. The biorthogonal DMT transceiver is ISI free property if and only if the zero-padded transceiver satisfies the followings [3]:

(i) $\mathbf{G}$ is an $M \times M$ nonsingular matrix;

(ii) $\mathbf{S} = \mathbf{G}^{-1}\mathbf{B}$, where $\mathbf{B} = \mathbf{V}^T\mathbf{\Lambda}^{-1}[\mathbf{I}\ \mathbf{A}]\mathbf{U}^T$, for arbitrary $M \times L$ matrix $\mathbf{A}$.

Assume that the inputs x_k are PAM symbols of b_k bits. WLOG, we further assume that x_k have zero mean and they are uncorrelated with each other. The average bit rate per symbol in this case becomes $b = \frac{1}{M}\sum_{k=0}^{M-1} b_k$. The transmission power P is the average energy of the vector $\mathbf{y} = \begin{pmatrix} y_0 & y_1 & \cdots & y_{N-1}\end{pmatrix}^T$ as shown in Fig. 1, $P = \frac{1}{N}\sum_{k=0}^{M-1}\sigma_{y_k}^2$. Using this expression, we can write the transmission power as

$$P = \frac{1}{N}\sum_{k=0}^{M-1}\sigma_{x_k}^2 ||\mathbf{g}_k||_2^2. \qquad (2)$$

where $||\mathbf{g}_k||_2^2 = \sum_{\ell=0}^{M-1}[\mathbf{G}]_{\ell k}^2$ is the energy of the k-th column of $\mathbf{G}$. Under the ISI free condition, for a fixed bit rate per input symbol b and a fixed probability of error P_e, we will find the transceiver that minimizes the transmission power. The optimization process involves 2 steps.

Optimal Bit Allocation

At the receiver end, the output of the k-th band is $\widehat{x}_k = x_k + e_k$, where e_k comes entirely from channel noise as the transceiver achieves zero ISI. Define the $M \times 1$ output noise vector as $\mathbf{e} = \begin{pmatrix} e_0 & e_1 & \cdots & e_{M-1}\end{pmatrix}^T$, then $\mathbf{e} = \mathbf{S}\boldsymbol{\nu} = \mathbf{G}^{-1}\mathbf{B}\boldsymbol{\nu}$. Assuming the PAM symbols of the k-th band carry b_k bits, the probability of error for the k-th band is given by $P_e(k) = 2(1-2^{-b_k})Q\left(\sqrt{\frac{3\sigma_{x_k}^2}{(2^{2b_k}-1)\sigma_{e_k}^2}}\right)$. For a fixed probability of error P_e across all bands, we need to have $P_e(0) = P_e(1) = \cdots = P_e(M-1) = P_e$. Under the high bit rate assumption $2^{b_k} - 1 \approx 2^{b_k}$, we can see that $\sigma_{x_k}^2$ and $\sigma_{e_k}^2$ satisfy

$$\sigma_{x_k}^2 = c2^{2b_k}\sigma_{e_k}^2, \quad \text{where} \quad c = \frac{1}{3}\left(Q^{-1}\left\{P_e/2\right\}\right)^2. \qquad (3)$$

Using this relation and applying the AM-GM inequality, the transmission power in (2) satisfies

$$P \geq \frac{cM}{N}2^{2b}\prod_{k=0}^{M-1}\left(\sigma_{e_k}^2||\mathbf{g}_k||_2^2\right)^{1/M} \triangleq P_{opt,bit}. \qquad (4)$$

The equality holds if and only if $2^{2b_k}\sigma_{e_k}^2||\mathbf{g}_k||_2^2$ are the same for all k. Solving for the optimal b_k, we have

$$b_k = b - \log_2(\sigma_{e_k}||\mathbf{g}_k||_2) + \frac{1}{M}\log_2\left(\Pi_{\ell=0}^{M-1}\sigma_{e_\ell}||\mathbf{g}_\ell||_2\right). \qquad (5)$$

Optimal Transceivers

Note that the energy of the k-th column of $\mathbf{G}$ is $||\mathbf{g}_k||_2^2 = \left[\mathbf{G}^T\mathbf{G}\right]_{kk}$. Let $\mathbf{q} = \mathbf{B}\boldsymbol{\nu}$, then $\mathbf{e} = \mathbf{G}^{-1}\mathbf{q}$. The $M \times M$ autocorrelation matrix $\mathbf{R}_e$ of the noise vector $\mathbf{e}$ is given by $\mathbf{R}_e = \mathbf{S}\mathbf{R}_\nu\mathbf{S}^T = \mathbf{G}^{-1}\mathbf{R}_q\mathbf{G}^{-T}$, where $\mathbf{R}_q$ is the autocorrelation matrix of the vector $\mathbf{q}$. The output noise $\sigma_{e_k}^2$ of the k-th band is equal to $[\mathbf{R}_e]_{kk}$ or $\left[\mathbf{G}^{-1}\mathbf{R}_q\mathbf{G}^{-T}\right]_{kk}$. So (4) can be rewritten as

$$P_{opt,bit} = c\frac{M}{N}2^{2b}\left(\prod_{k=0}^{M-1}\left[\mathbf{G}^T\mathbf{G}\right]_{kk}\left[\mathbf{G}^{-1}\mathbf{R}_q\mathbf{G}^{-T}\right]_{kk}\right)^{1/M}.$$

Apply the Hadamard inequality, we have

$$\begin{aligned} P_{opt,bit} &\geq c\frac{M}{N}2^{2b}\left(\det\left(\mathbf{G}^T\mathbf{G}\right)\det\left(\mathbf{G}^{-1}\mathbf{R}_q\mathbf{G}^{-T}\right)\right)^{1/M} \\ &= c\frac{M}{N}2^{2b}(\det\mathbf{R}_q)^{1/M} \triangleq P_{opt,bit,G} \end{aligned} \tag{6}$$

The equality holds if and only if (i) $\mathbf{G}^T\mathbf{G}$ is diagonal and (ii) $\mathbf{G}^{-1}\mathbf{R}_q\mathbf{G}^{-T}$ is diagonal. The lower bound $P_{opt,bit,G}$ does not depend on the transmitter $\mathbf{G}$ and it is achieved if and only if $\mathbf{G}$ satisfy both conditions (i) and (ii). Let the Schur decomposition of $\mathbf{R}_q$ be

$$\mathbf{R}_q = \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T.$$

Then these 2 conditions can be satisfied by choosing $\mathbf{G} = \mathbf{Q}$. The optimal transmitter $\mathbf{G}$ is orthogonal. In this case the receiver that achieves ISI is given $\mathbf{S} = \mathbf{Q}^T\mathbf{B}$, where $\mathbf{B}$ is given in Theorem 1.

From (6), we see that given any $\mathbf{A}$, the achievable lower bound $P_{opt,bit,G} = c\frac{M}{N}2^{2b}(\det\mathbf{R}_q)^{1/M}$. The matrix $\mathbf{A}$ should be chosen such that $\det(\mathbf{R}_q)$ is minimized. Using the facts that $\mathbf{R}_q = \mathbf{B}\mathbf{R}_\nu\mathbf{B}^T$ and $\mathbf{B} = \mathbf{V}^T\mathbf{\Lambda}^{-1}[\mathbf{I}\ \mathbf{A}]\mathbf{U}^T$, we get

$$\det(\mathbf{R}_q) = \det(\mathbf{\Lambda}^{-2})\det\left([\mathbf{I}\ \mathbf{A}]\mathbf{U}^T\mathbf{R}_\nu\mathbf{U}\begin{pmatrix}\mathbf{I}\\ \mathbf{A}^T\end{pmatrix}\right). \tag{7}$$

The optimal $\mathbf{A}$ is such that $\det\left([\mathbf{I}\ \mathbf{A}]\mathbf{U}^T\mathbf{R}_\nu\mathbf{U}[\mathbf{I}\ \mathbf{A}]^T\right)$ is minimized. The optimal $\mathbf{A}$ has the following closed form expression [4]:

$$\mathbf{A} = -\mathbf{U}_0^T\mathbf{R}_\nu\mathbf{U}_1\left(\mathbf{U}_1^T\mathbf{R}_\nu\mathbf{U}_1\right)^{-1}, \tag{8}$$

where the matrix $\mathbf{U}_1$ is defined in (1). The minimum achievable $\det(\mathbf{R}_q)$ is

$$\det(\mathbf{\Lambda}^{-2})\det(\mathbf{R}_\nu)/\det(\mathbf{U}_1^T\mathbf{R}_\nu\mathbf{U}_1).$$

Using this expression and (6), the minimum transmission power for the optimal transceiver is

$$P_{biortho} = c\,2^{2b}\frac{M}{N}\left[\frac{\det(\mathbf{\Lambda}^{-2})\det(\mathbf{R}_\nu)}{\det(\mathbf{U}_1^T\mathbf{R}_\nu\mathbf{U}_1)}\right]^{1/M}.$$

Summarizing the results, we have

Theorem 1 *Consider the zero padded M-band DMT system in Fig. 2. Assume that the inputs are PAM symbols of b_k bits. For any fixed probability of error P_e and any fixed transmission bit rate per symbol b, the biorthogonal transceiver is ISI free and minimizes the transmission power P in (2) if and only if the following are true:*

(i) *The matrix $\mathbf{A}$ is given by $\mathbf{A} = -\mathbf{U}_0^T\mathbf{R}_\nu\mathbf{U}_1\left(\mathbf{U}_1^T\mathbf{R}_\nu\mathbf{U}_1\right)^{-1}$, where $\mathbf{R}_\nu$ is the autocorrelation matrix of the noise vector $\boldsymbol{\nu}$ and, $\mathbf{U}_0$ and $\mathbf{U}_1$ are as defined in (1).*

(ii) *The transmitter $\mathbf{G} = \mathbf{Q}$, where $\mathbf{Q}$ is the orthonormal matrix such that $\mathbf{Q}^T\mathbf{R}_q\mathbf{Q}$ is diagonal. The matrix $\mathbf{R}_q$ is given by $\mathbf{R}_q = \mathbf{B}\mathbf{R}_\nu\mathbf{B}^T$, where $\mathbf{B} = \mathbf{V}^T\mathbf{\Lambda}^{-1}[\mathbf{I}\ \mathbf{A}]\mathbf{U}^T$. The receiver is given by $\mathbf{S} = \mathbf{Q}^T\mathbf{B}$.*

(iii) *The bits b_k are allocated as $b_k = b - \log_2(\sigma_{e_k}\|\mathbf{g}_k\|_2) + \frac{1}{M}\log_2\left(\Pi_{\ell=0}^{M-1}\sigma_{e_\ell}\|\mathbf{g}_\ell\|_2\right)$.*

The minimum transmission power is

$$P_{biortho} = c\,2^{2b}\frac{M}{N}\left[\det(\mathbf{\Lambda}^{-2})\frac{\det(\mathbf{R}_\nu)}{\det(\mathbf{U}_1^T\mathbf{R}_\nu\mathbf{U}_1)}\right]^{1/M}.$$

Example. In this example we compare the transmission power of the derived optimal transceiver $P_{biortho}$, that of the orthogonal transceiver derived in [3] $P_{ortho,[3]}$, and that of the vector coding transceiver derived in [2] $P_{vc,[2]}$. The closed form expressions of $P_{ortho,[3]}$ and $P_{vc,[2]}$ are derived in citelin. The channel $C(z)$ and power spectrum for the colored noise $\nu(n)$ used in this example are showed in Fig. 3(a) and (b). These parameters are obtained from a typical ADSL environment. The channel $C(z)$ in this case has order $L = 4$. The bit error rate $P_e = 10^{-6}$ and average bit rate per sample is $R_b = \frac{M}{N}b = 2$. The results are plotted for $M = 10$ to 50 and they are showed in Fig. 4. One can see that the improvement of $P_{biortho}$ over $P_{ortho,[3]}$ is more significant when M is small. When M is large, the two curves converge. Also $P_{biortho}$ is approximately 5 dB smaller than $P_{vc,[2]}$. In the plot we see that $P_{biortho}$ and $P_{ortho,[3]}$ appear to be monotone decreasing.

4 計畫成果自評：

The result of this project is very satisfactory. We have successfully derived optimal transceivers for distorted channel with colored noise. As demonstrated in the numerical example, for a typical ADSL environment, the optimal biorthogonal transceivers outperform the orthogonal one and they are significantly better than other well-known systems.

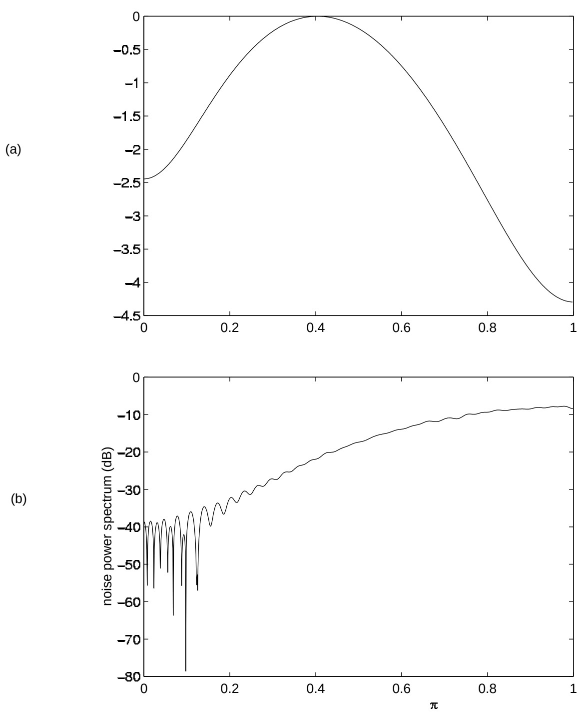

圖 3: (a) The frequency response of the channel $|C(e^{j\omega})|$. (b) The power spectrum of the channel noise $\nu(n)$.

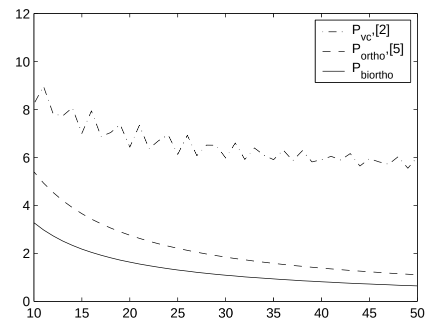

圖 4: Performance comparison of $P_{biortho}$, $P_{ortho,[3]}$ and $P_{vc,[2]}$.

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