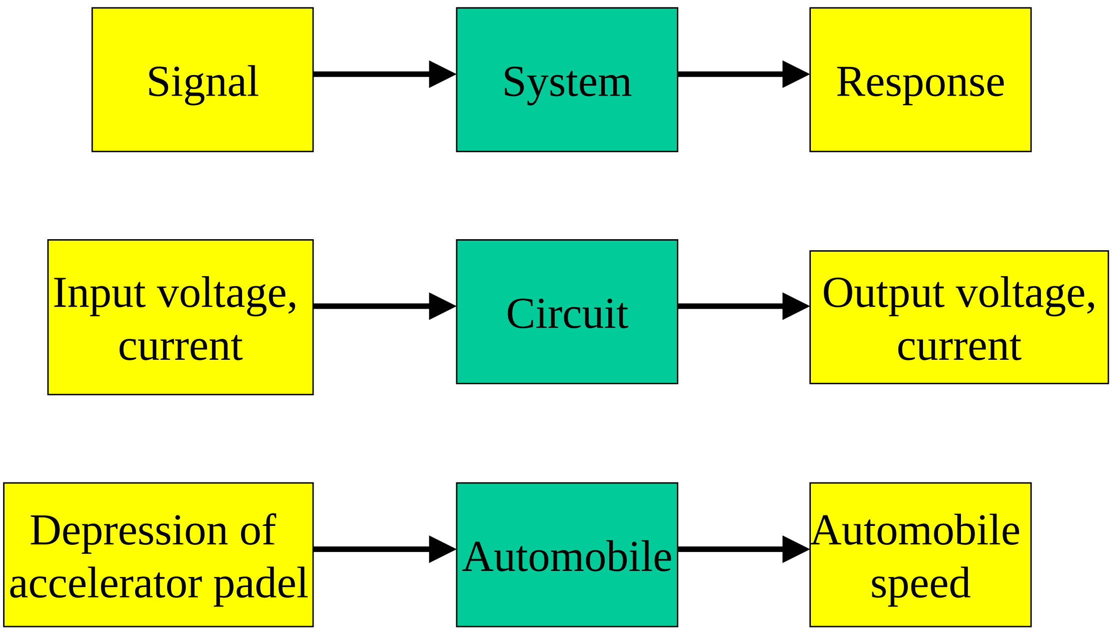
# Signal & Systems

Hsin-chia Lu

https://ceiba.ntu.edu.tw/941s_and_s_vlsi



# Forward



Signal: characterization, enhancement, noise suppression
examples: image restoration, speech enhancement

System: characterization, synthesis
examples: chemical processing plants

Analysis tool: Fourier analysis, Fourier series, Fourier transform
Signal type:
continuous: voltage in a circuit, temperature,…
discrete: closing stock market average, …



- Quantization of continuous signals into discrete signals.
- DSP ( digital signal process) is possible due to advance in computer and digital signal processing power.
  - Audio (CD, MP3) , Video (VCD, DVD, DVB) , …

- Continuous and discrete formulations are presented is parallel in this book. They are similar but not identical.

# What is a signal?

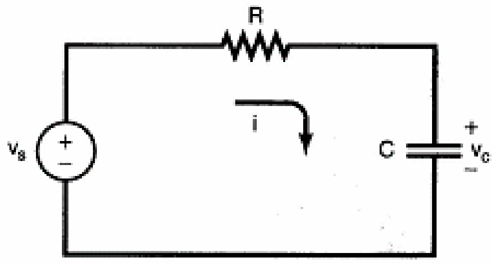



圖 1.1　具有電源 $v_s$ 和電容電壓 $v_c$ 的簡單 *RC* 電路

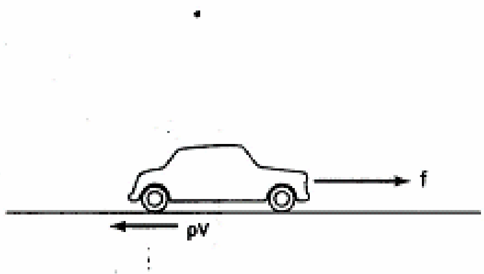



圖 1.2　汽車引擎產生的應用力 $f$ 和與汽車速度 $v$ 成正比的摩擦力 $\rho v$ 的響應



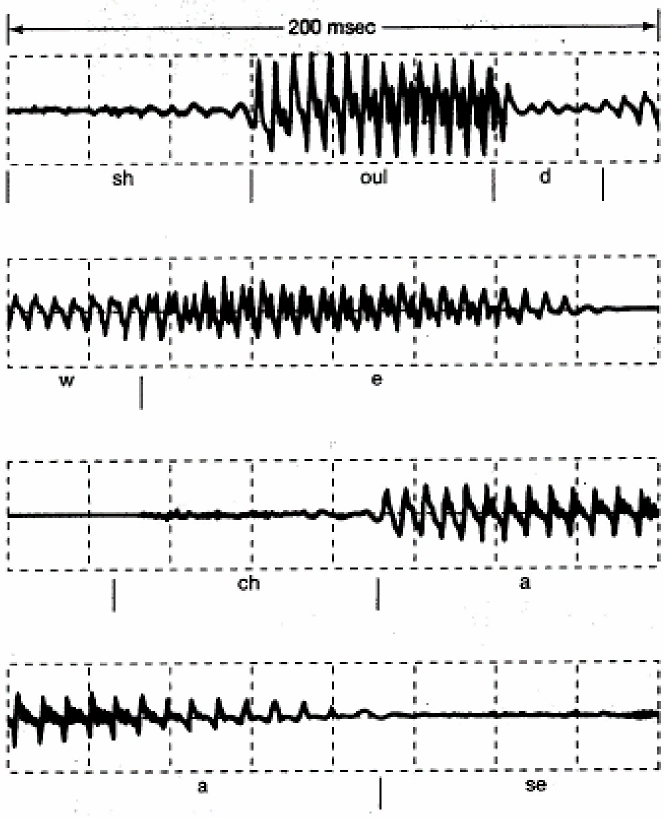
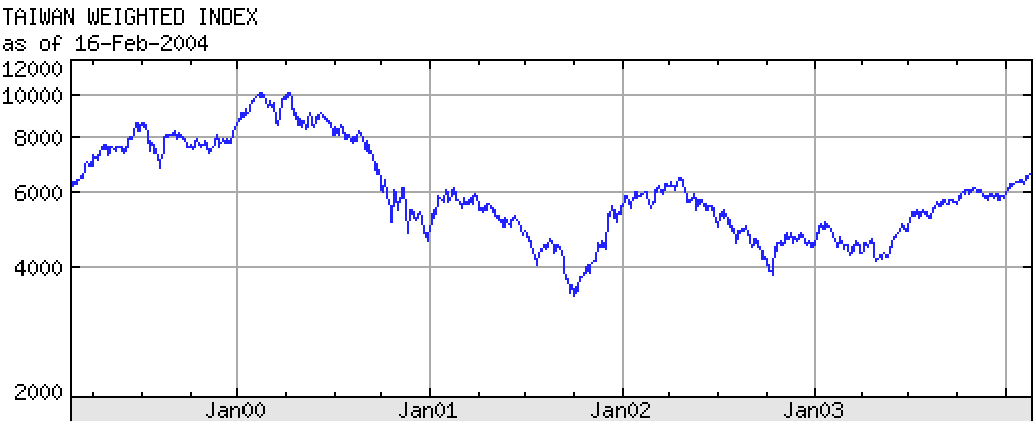



圖 1.3　某個語音信號的波形。圖的第一行對應到 "should" 這個字，第二行的字是 "we"，最後兩行的字是 "chase"。（在每個字中，我們也大約標示出每個連續聲音的開始和結束。）

# Other Signal Types: photos



# Other Signal Types



# Continuous Time vs. Discrete Time

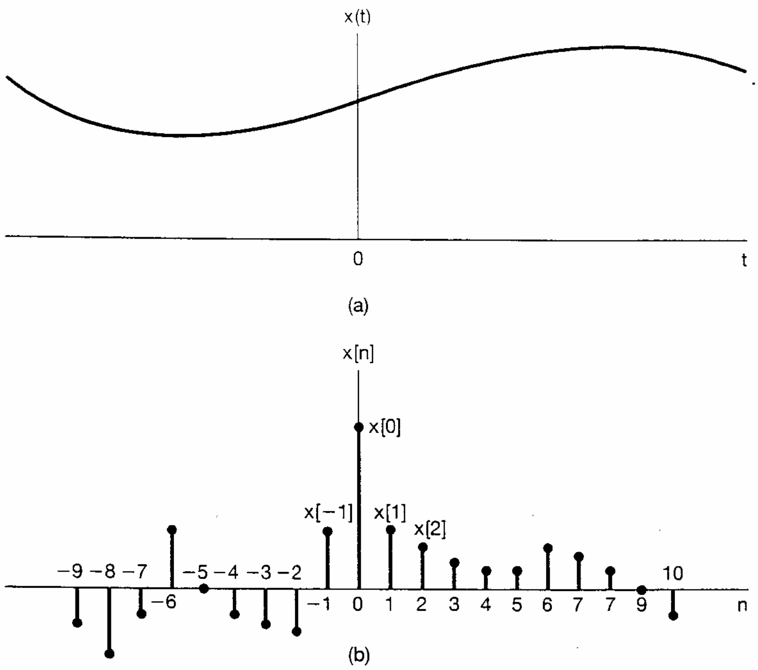



**Figure 1.7** Graphical representations of (a) continuous-time and (b) discrete-time signals.



# Signal Energy & Power

- For a resistor with voltage $v(t)$ and current of $i(t)$, the power is $$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

with total energy of $$\int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

and power of $$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt.$$

- Our definition is

  Energy: $$\int_{t_1}^{t_2} |x(t)|^2 dt, \qquad \sum_{n=n_1}^{n_2} |x[n]|^2$$

# Signal Energy & Power (cont.)

- For $-\infty < t < +\infty$, the energy is

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

- Power is

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$



| Class of signals | $E_\infty$ | $P_\infty$ | Examples |
|---|---|---|---|
| First | Finite | Zero | Finite duration signals |
| Second | Infinite | Finite | X[n]=4 |
| Third | Infinite | Infinite | X(t)=t |

# Transformations of Signal (Time shift)

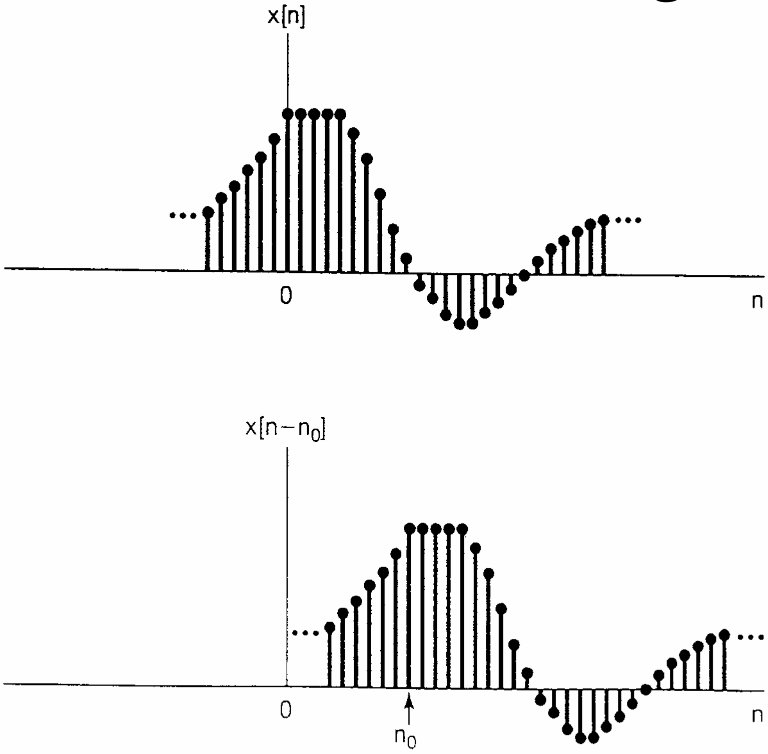



**Figure 1.8** Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n - n_0]$ is a delayed verson of $x[n]$ (i.e., each point in $x[n]$ occurs later in $x[n - n_0]$).

# Transformations of Signal (Time shift)

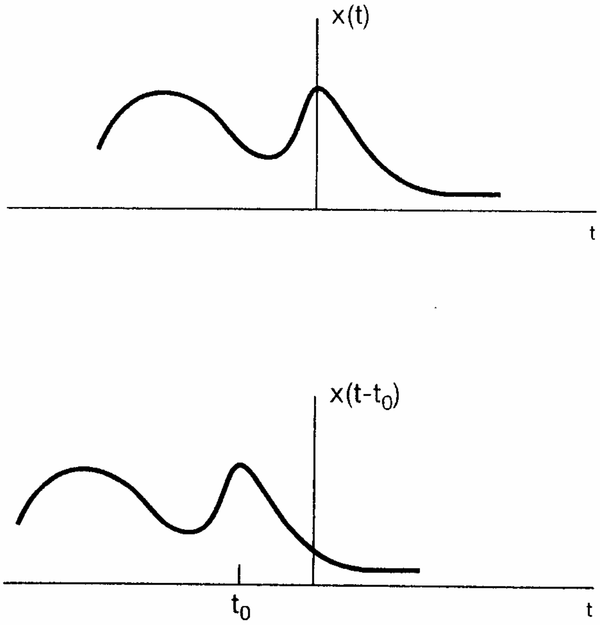



**Figure 1.9** Continuous-time signals related by a time shift. In this figure $t_0 < 0$, so that $x(t - t_0)$ is an advanced version of $x(t)$ (i.e., each point in $x(t)$ occurs at an earlier time in $x(t - t_0)$).

Transformations of Signal (Reflection)

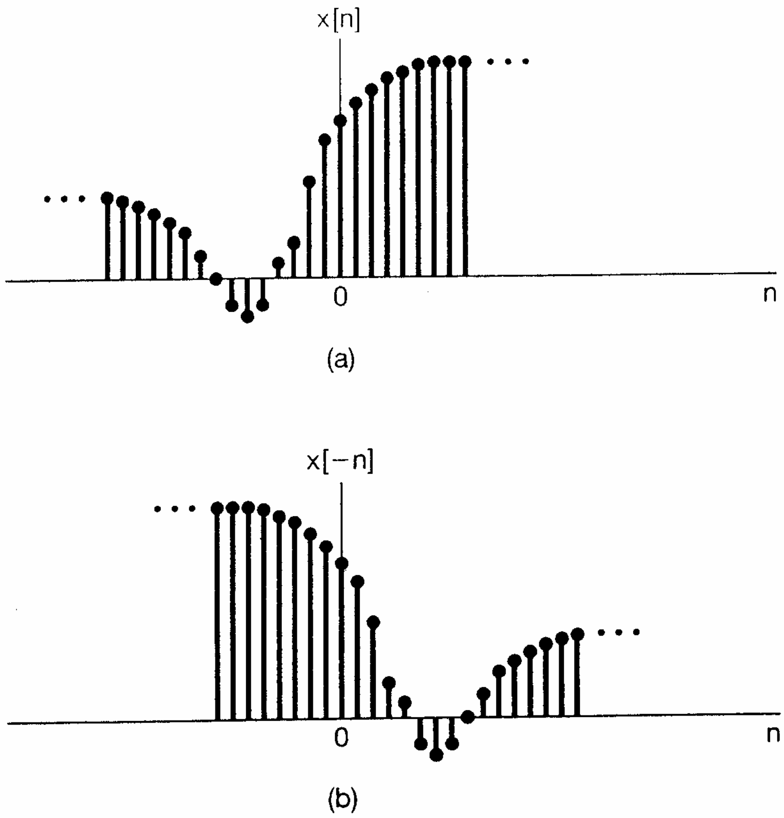



**Figure 1.10** (a) A discrete-time signal $x[n]$; (b) its reflection $x[-n]$ about $n = 0$.

Transformations of Signal (Reflection)

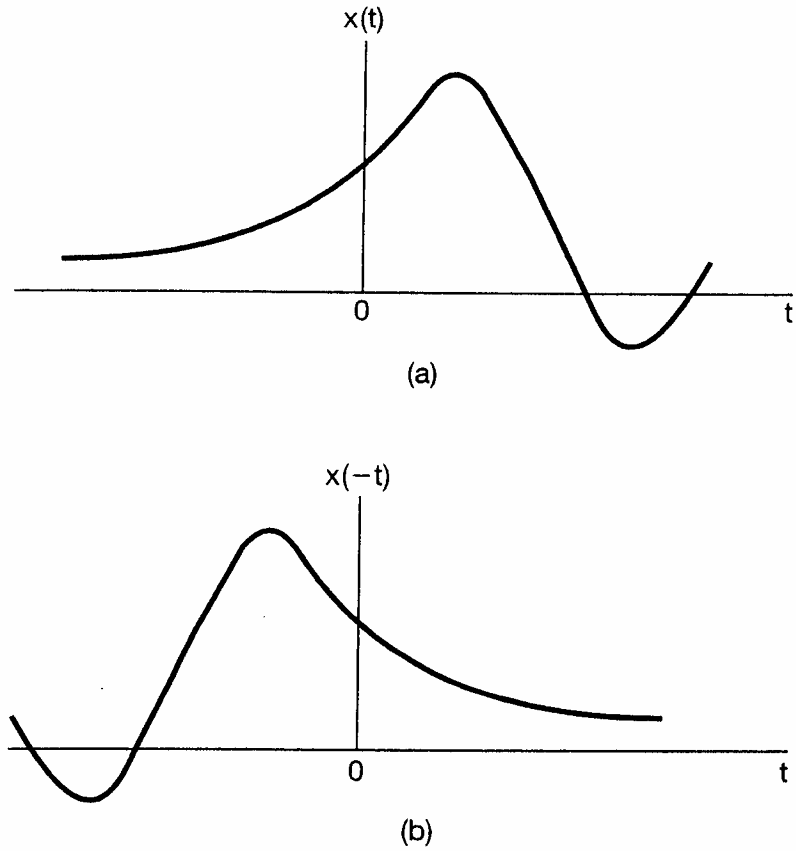



**Figure 1.11** (a) A continuous-time signal $x(t)$; (b) its reflection $x(-t)$ about $t = 0$.

# Transformations of Signal (Scaling)

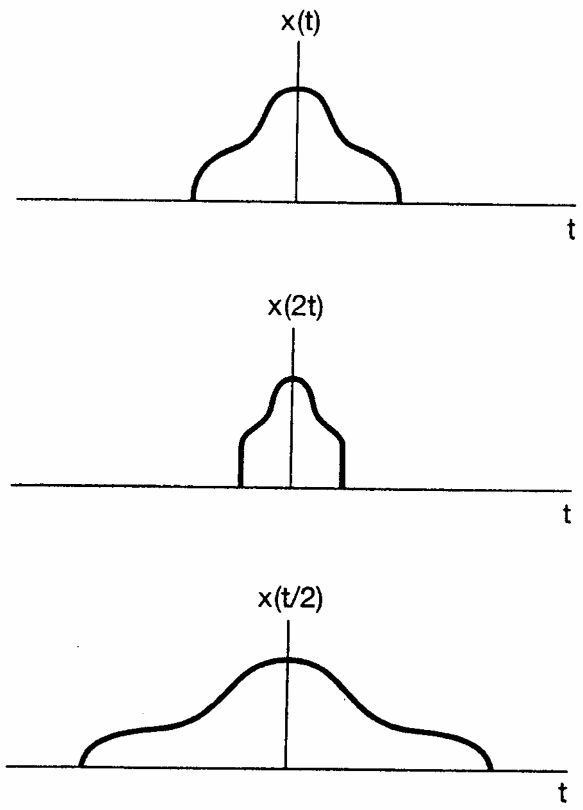



**Figure 1.12** Continuous-time signals related by time scaling.

# Transformations of Signal (Examples)

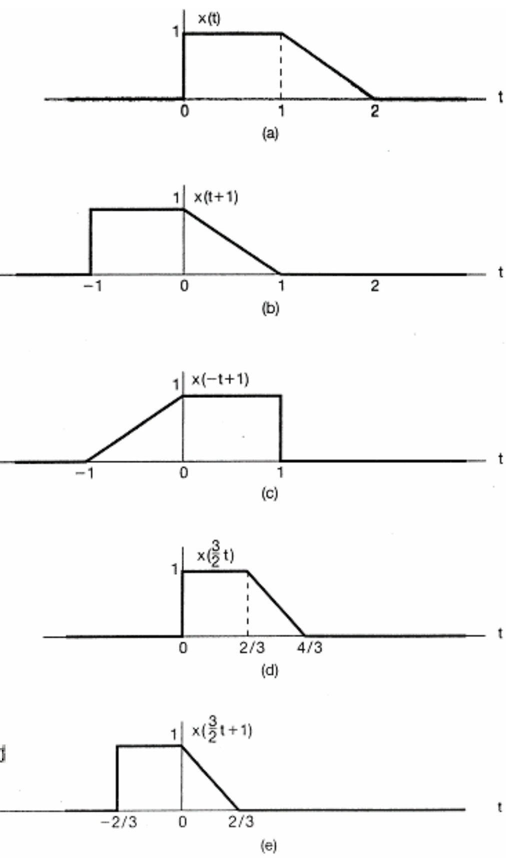



**Figure 1.13** (a) The continuous-time signal $x(t)$ used in Examples 1.1–1.3 to illustrate transformations of the independent variable; (b) the time-shifted signal $x(t+1)$; (c) the signal $x(-t+1)$ obtained by a time shift and a time reversal; (d) the time-scaled signal $x(\frac{3}{2}t)$; and (e) the signal $x(\frac{3}{2}t+1)$ obtained by time-shifting and scaling.

# Periodic Signals

$$x(t) = x(t + T)$$

$x(t)$ is *periodic with period* $T$.

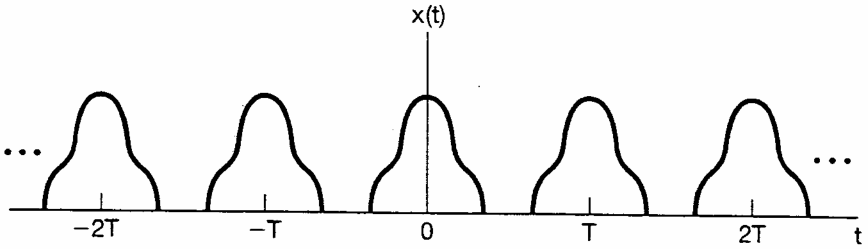



**Figure 1.14** A continuous-time periodic signal.



# Periodic Signals

$$x[n] = x[n + N]$$

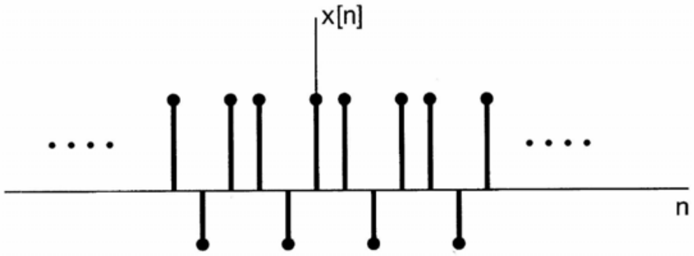
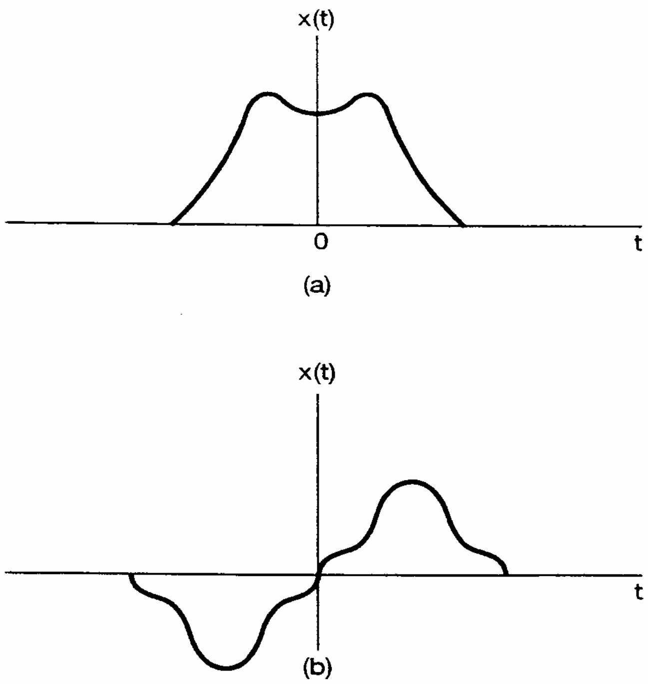



**Figure 1.15** A discrete-time periodic signal with fundamental period $N_0 = 3$.

# Even and Odd Signals




$$x(-t) = x(t);$$

$$x[-n] = x[n].$$

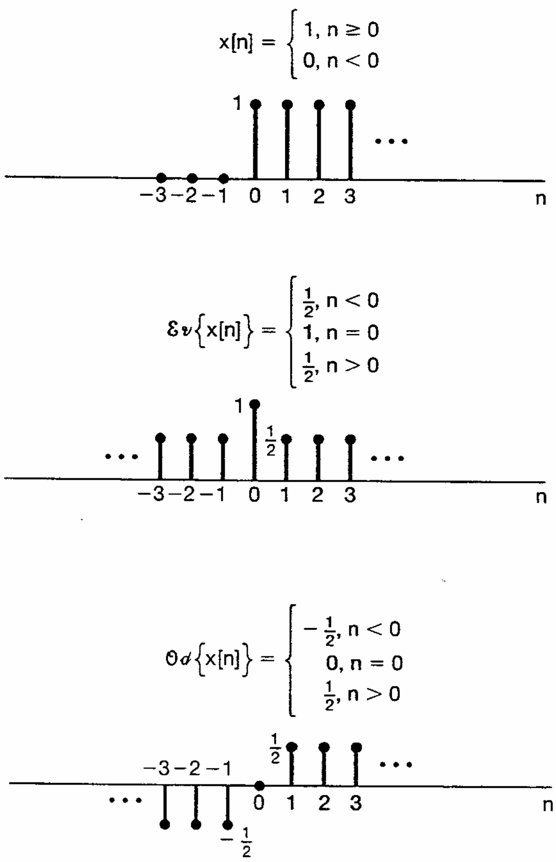
$$x(-t) = -x(t),$$

$$x[-n] = -x[n].$$

**Figure 1.17** (a) An even continuous-time signal; (b) an odd continuous-time signal.



# Even and Odd Decomposition




$$\mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

$$\mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

**Figure 1.18** Example of the even-odd decomposition of a discrete-time signal.

# Real Exponential Signals

$x(t) = Ce^{at}$

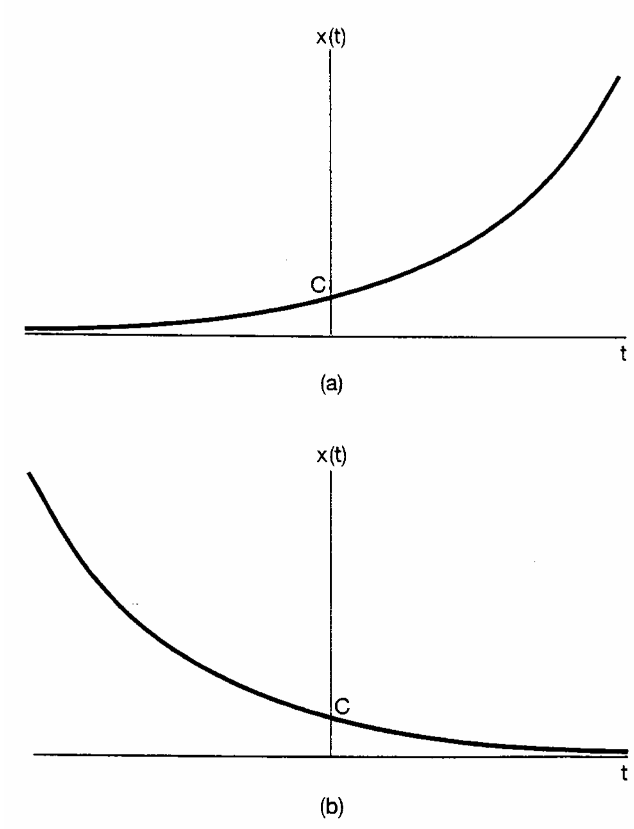



**Figure 1.19** Continuous-time real exponential $x(t) = Ce^{at}$: (a) $a > 0$; (b) $a < 0$.



# Periodic Sinusoidal Signals (Imaginary Exponential)

$$x(t) = e^{j\omega_0 t}$$

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T} = e^{j\omega_0 t} \quad \text{if} \quad e^{j\omega_0 T} = 1$$

$$T_0 = \frac{2\pi}{|\omega_0|}$$

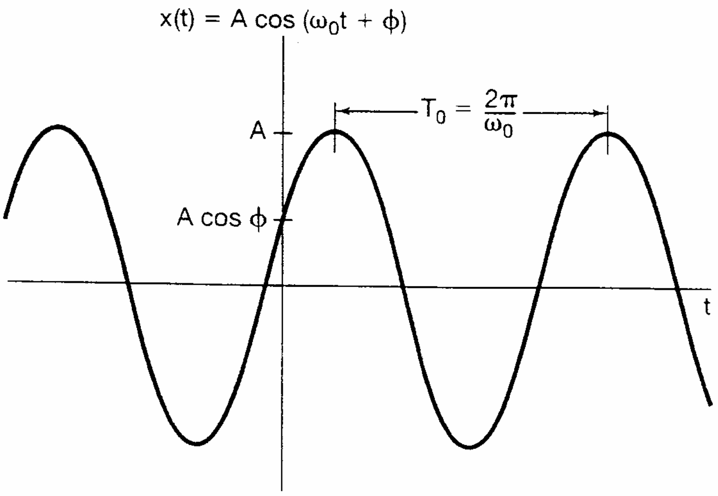



**Figure 1.20** Continuous-time sinusoidal signal.

Relationship between Frequency and period

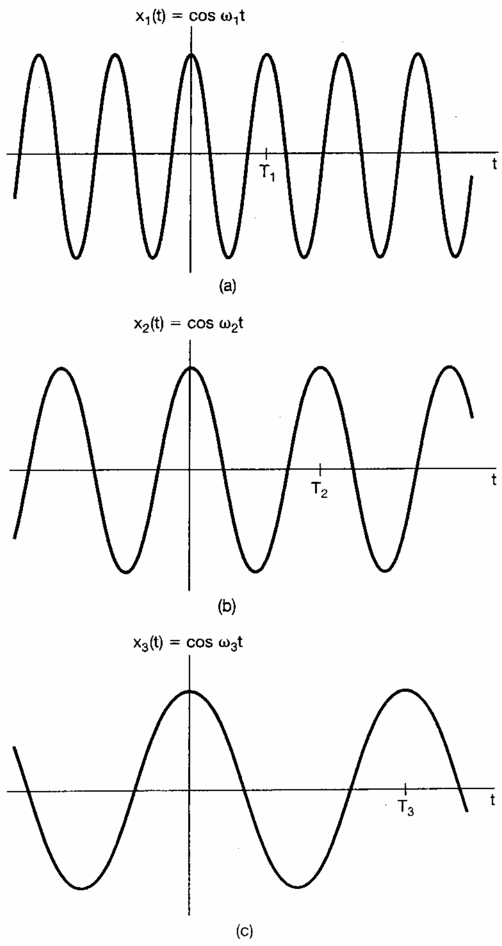



**Figure 1.21** Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.



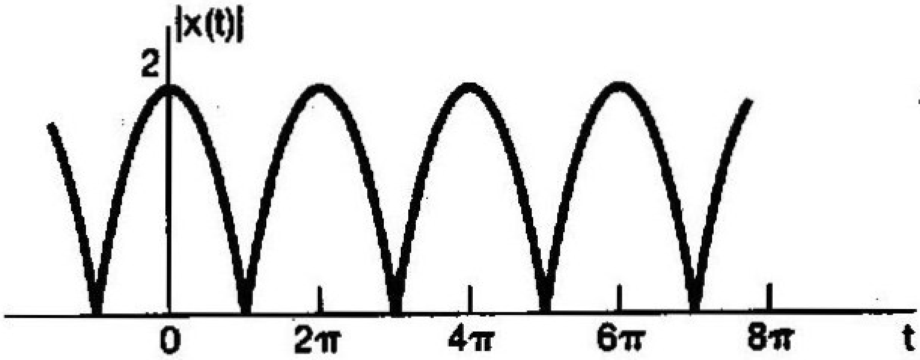



**圖 1.22　例題 1.5 全波整流的弦波信號**

# Periodic Sinusoidal Signals (cont.)

- Euler's relation

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

- We obtain

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$$A\cos(\omega_0 t + \phi) = A\,\mathcal{R}e\{e^{j(\omega_0 t+\phi)}\}$$

$$A\sin(\omega_0 t + \phi) = A\,\mathcal{I}m\{e^{j(\omega_0 t+\phi)}\}$$

- Energy per period

$$E_{\text{period}} = \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt = \int_0^{T_0} 1 \cdot dt = T_0$$

$$P_{\text{period}} = \frac{1}{T_0}E_{\text{period}} = 1$$

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|e^{j\omega_0 t}\right|^2 dt = 1$$



# Harmonically Related Complex Exponentials

- All complex exponentials with period of $T_0$

$$e^{j\omega T_0} = 1$$

or

$$\omega T_0 = 2\pi k, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

- Define

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega = k\omega_0$$

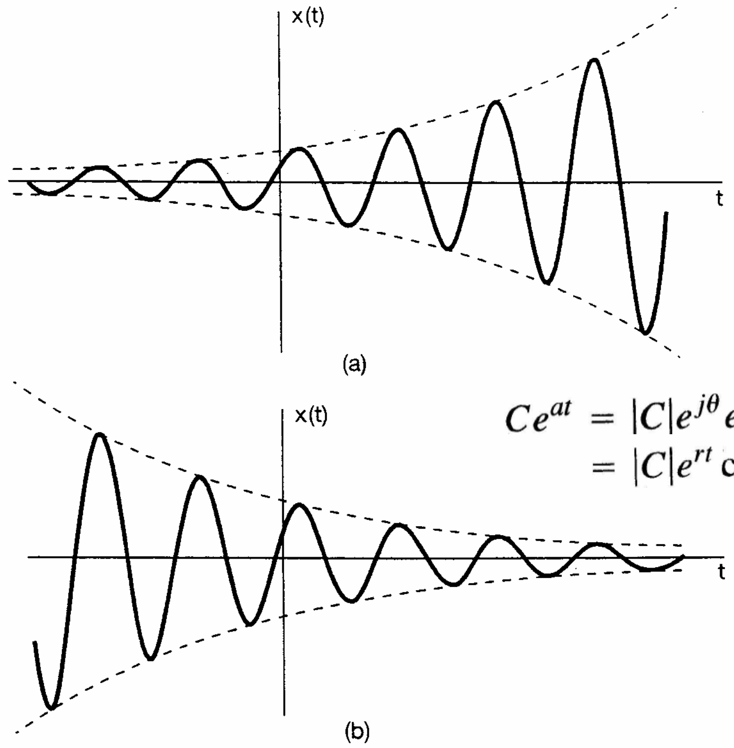
- Sets of

$$\phi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

with period of

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

# General Complex Exponential Signals




$x(t) = Ce^{at}$

$C = |C|e^{j\theta}$

$a = r + j\omega_0$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$
$$= |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

**Figure 1.23** (a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.



# Discrete-Time Complex Exponential

- Complex exponential signal or sequence

$x[n] = Ca^n$

$x[n] = Ce^{\beta n}$ $\quad a = e^{\beta}$

- Real complex exponential:

$C$ and $a$ are real

- Sinusoidal Signal

$x[n] = e^{j\omega_0 n}$ or $x[n] = A\cos(\omega_0 n + \phi)$

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

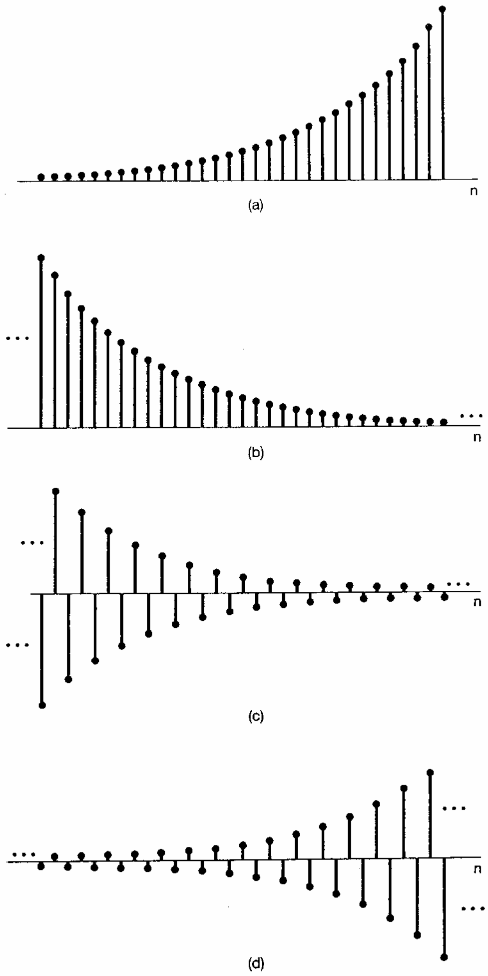



**Figure 1.24** The real exponential signal $x[n] = C\alpha^n$:
(a) $\alpha > 1$; (b) $0 < \alpha < 1$;
(c) $-1 < \alpha < 0$; (d) $\alpha < -1$.



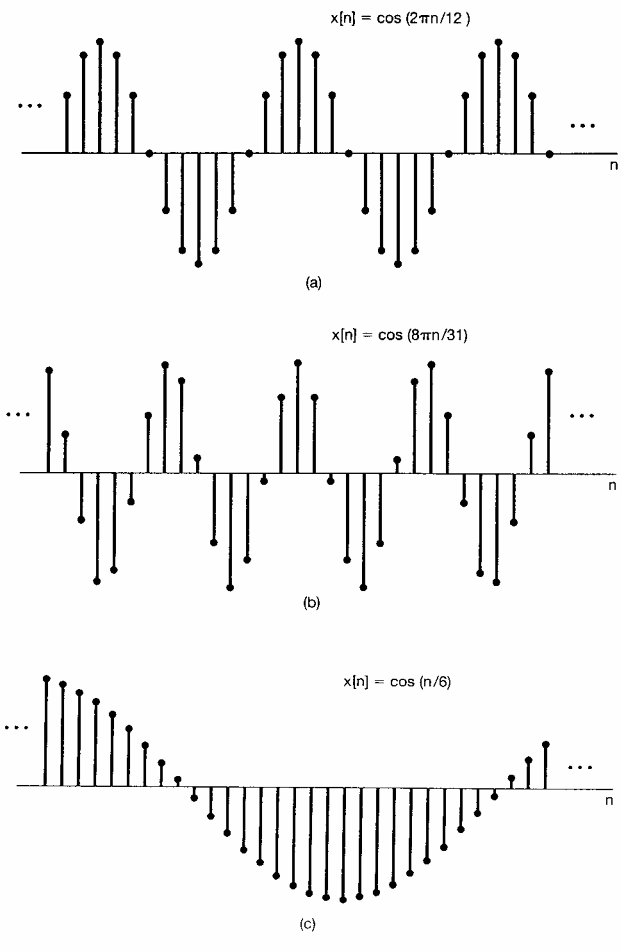



**Figure 1.25** Discrete-time sinusoidal signals.

## General Complex Exponential Signals

$x(t) = Ce^{at}$ $\quad C = |C|e^{j\theta}$ $\quad \alpha = |\alpha|e^{j\omega_0}$

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

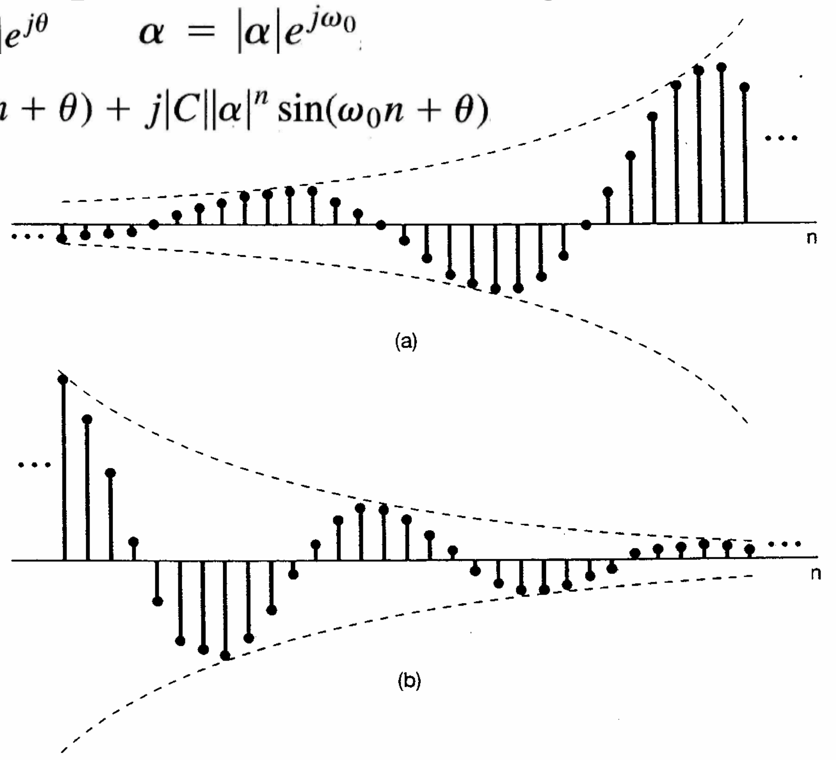



**Figure 1.26** (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

## Periodicity of Discrete-Time Complex Exponentials

- Continuous time

  $x(t) = e^{j\omega_0 t}$

  - Rate of oscillation increases with $\omega_0$
  - Periodic for any value of $\omega_0$
- Discrete-time

  $$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

  Meaningful in the interval $0 \leq \omega_0 < 2\pi$ or the interval $-\pi \leq \omega_0 < \pi$
- For $\omega_0 = \pi$, the highest oscillation is

  $$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

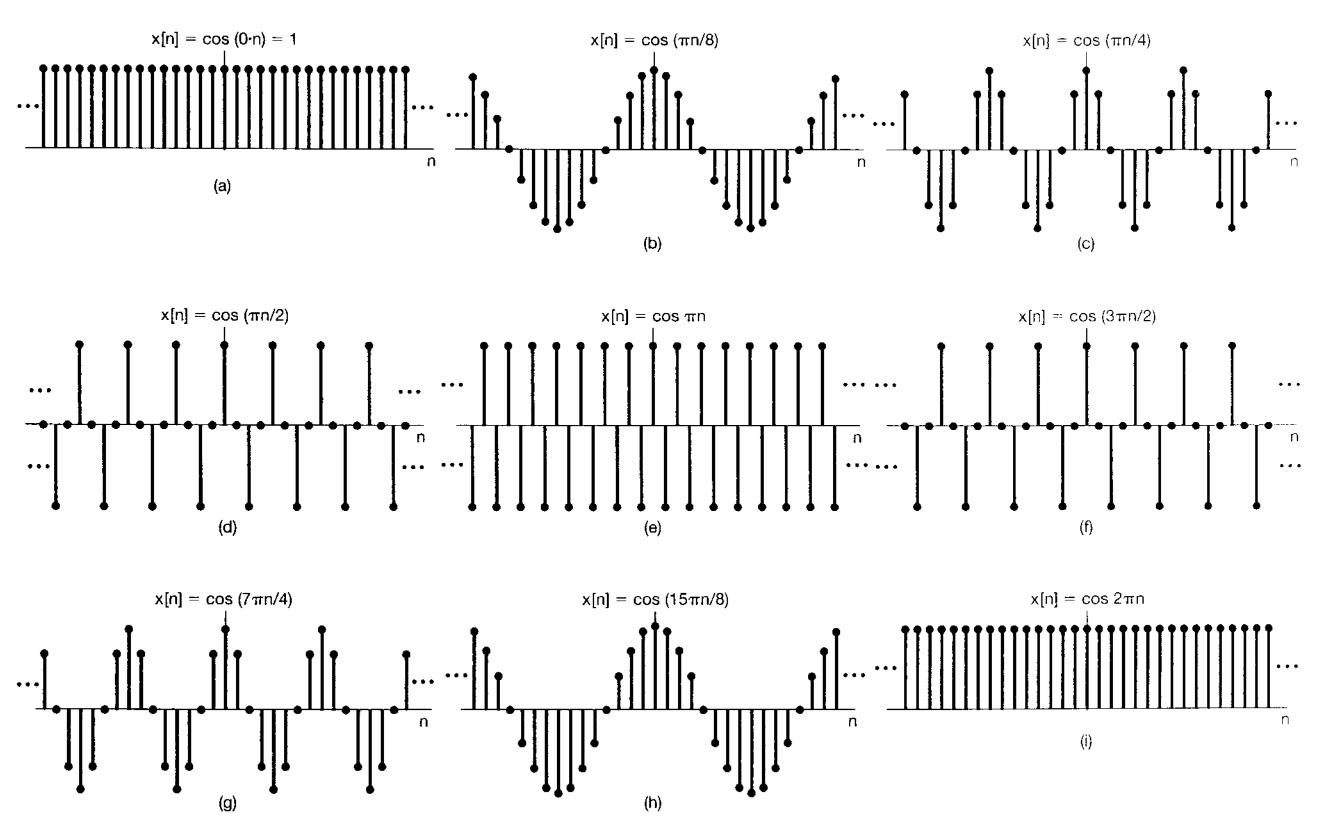



**Figure 1.27** Discrete-time sinusoidal sequences for several different frequencies.



# Periodicity of Complex Exponentials (cont.)

- For a period of $N$

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

we need

$$e^{j\omega_0 N} = 1$$

or

$$\omega_0 N = 2\pi m \qquad \frac{\omega_0}{2\pi} = \frac{m}{N}$$

- Fundamental period

$$N = m\left(\frac{2\pi}{\omega_0}\right)$$

**TABLE 1.1** Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

| $e^{j\omega_0 t}$ | $e^{j\omega_0 n}$ |
|---|---|
| Distinct signals for distinct values of $\omega_0$ | Identical signals for values of $\omega_0$ separated by multiples of $2\pi$ |
| Periodic for any choice of $\omega_0$ | Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and $m$. |
| Fundamental frequency $\omega_0$ | Fundamental frequency* $\omega_0/m$ |
| Fundamental period<br>$\omega_0 = 0$: undefined<br>$\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$ | Fundamental period*<br>$\omega_0 = 0$: undefined<br>$\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$ |

*Assumes that $m$ and $N$ do not have any factors in common.



# Harmonically Related Periodic Exponentials (A common period of $N$)

- Frequencies are multiple of $2\pi/N$

$$\phi_k[n] = e^{jk(2\pi/N)n}, \qquad k = 0, \pm 1, \ldots.$$

- $N$ distinct periodic exponentials

$$\phi_0[n] = 1,\ \phi_1[n] = e^{j2\pi n/N},\ \phi_2[n] = e^{j4\pi n/N}, \ldots, \phi_{N-1}[n] = e^{j2\pi(N-1)n/N}$$

because

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n} e^{j2\pi n} = \phi_k[n]$$

# Unit Impulse Function

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

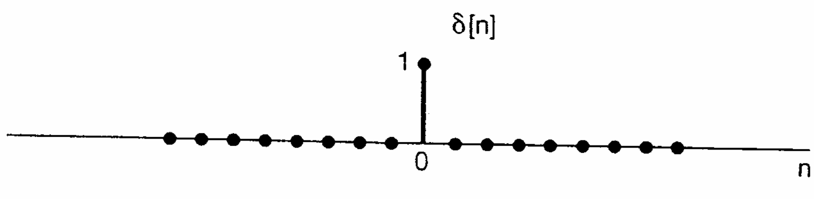



**Figure 1.28** Discrete-time unit impulse (sample).



# Unit Step Function

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

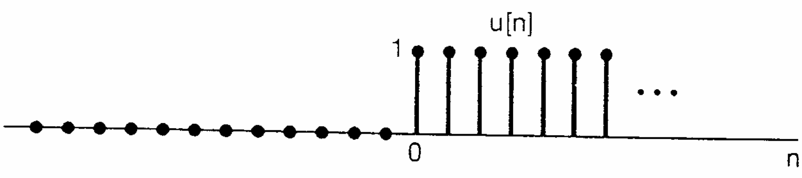



**Figure 1.29** Discrete-time unit step sequence.

$$\delta[n] = u[n] - u[n-1]$$

# Unit Step Function (cont.)

$$u[n] = \sum_{m=-\infty}^{+\infty} \delta[m]$$

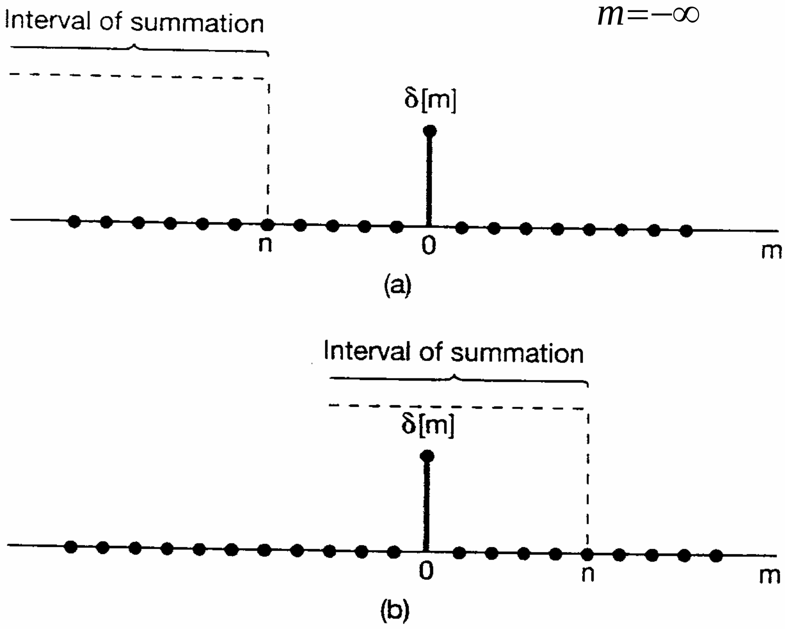
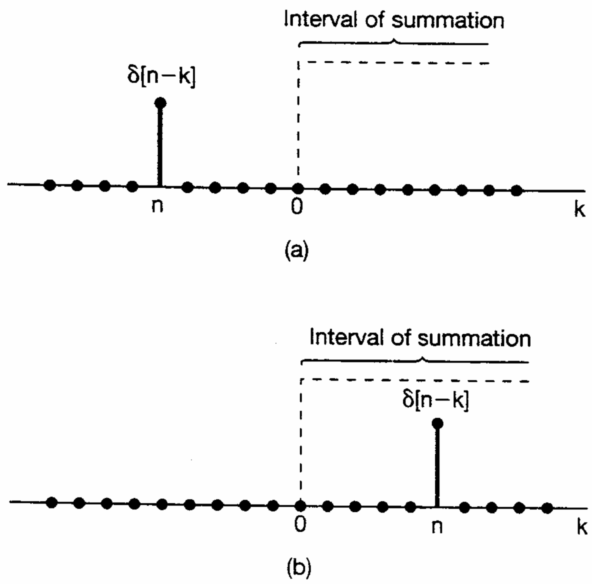



**Figure 1.30** Running sum of eq. (1.66): (a) $n < 0$; (b) $n > 0$.

# Unit Step Function (cont.)




*m* to *k* = *n*-*m*

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

**Figure 1.31** Relationship given in eq. (1.67): (a) $n < 0$; (b) $n > 0$.

Some properties:

$$x[n]\delta[n] = x[0]\delta[n]$$

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

# Continuous-Time Unit Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Discontinuous at $t = 0$

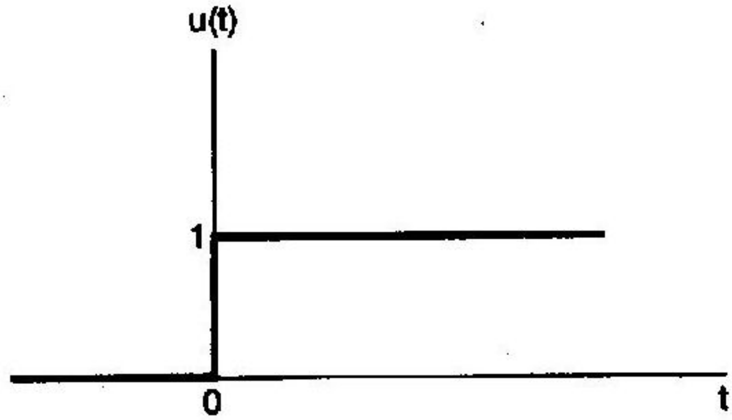



圖 1.32 連續時間單位步階函數



# Unit Impulse Function

$$\delta(t) = \frac{du(t)}{dt}$$

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

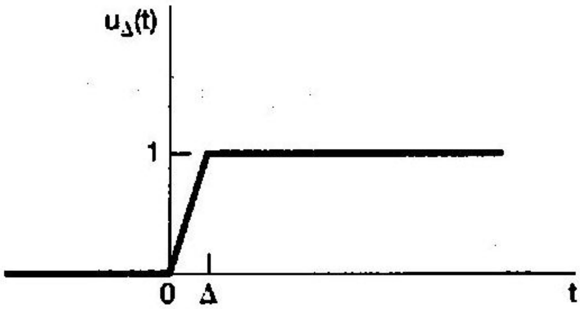



圖 1.33 單位步級 $u_\Delta(t)$ 的連續近似

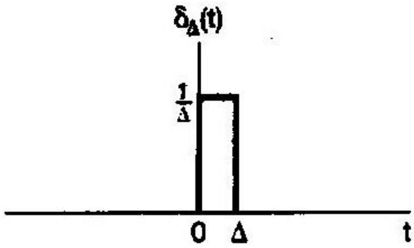



圖 1.34 $u_\Delta(t)$ 的導數

# Relation between unit step and impulse functions

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

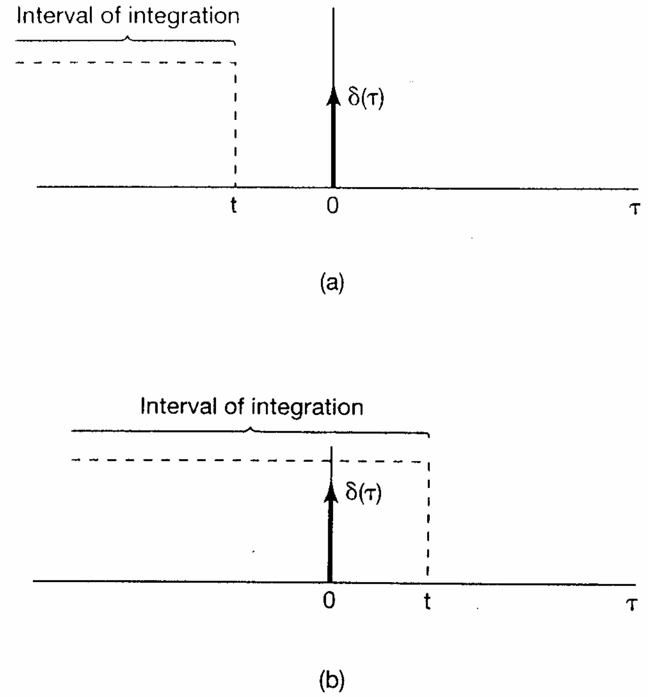



**Figure 1.37** Running integral given in eq. (1.71): (a) $t < 0$; (b) $t > 0$.

# Relation between unit step and impulse functions

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$= \int_{\infty}^{0} \delta(t-\sigma)d(-\sigma)$$

$$u(t) = \int_{0}^{\infty} \delta(\tau-\sigma)d\sigma$$

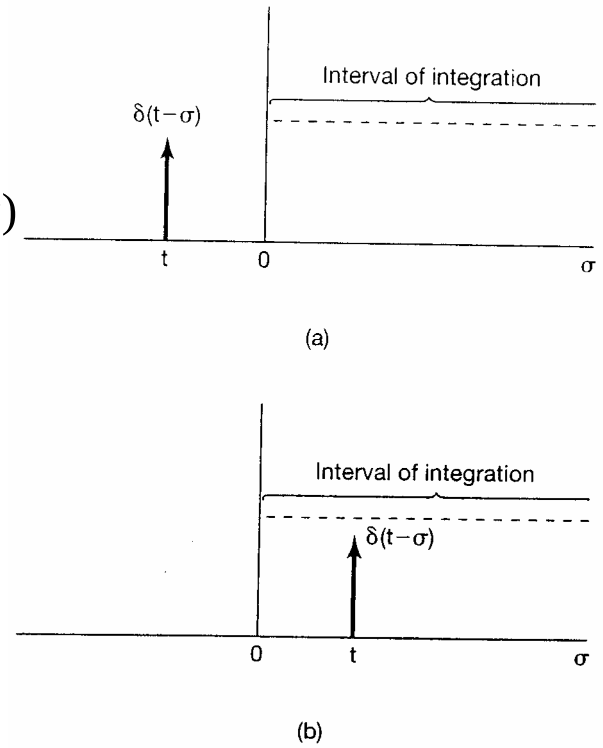



**Figure 1.38** Relationship given in eq. (1.75): (a) $t < 0$; (b) $t > 0$.

# Scaled Unit Impulse Function

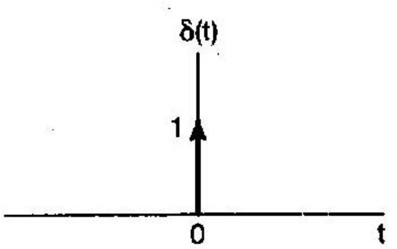



圖 1.35 連續時間單位脈衝

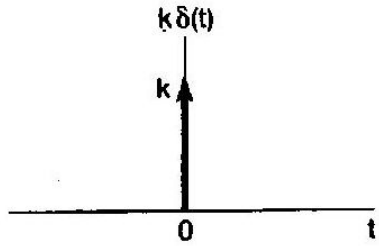



圖 1.36 脈衝強度 $k$ 的脈衝

$$ku(t) = \int_{-\infty}^{t} k\delta(\tau)d\tau$$



# Some Properties of Impulse Functions

$$x(t)\delta_{\Delta}(t) \approx x(0)\delta_{\Delta}(t)$$

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

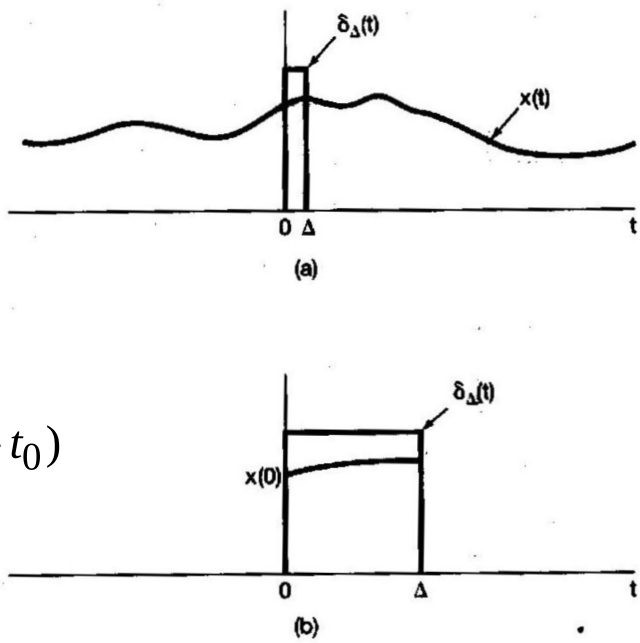



圖 1.39 $x(t)\delta_{\Delta}(t)$ 的乘積: (a)兩個相乘函數的圖; (b)乘積非零部分的放大

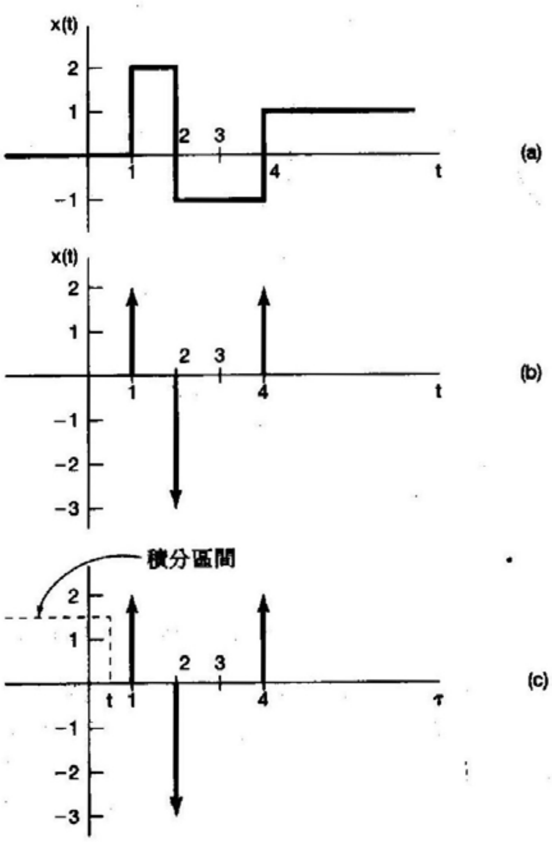
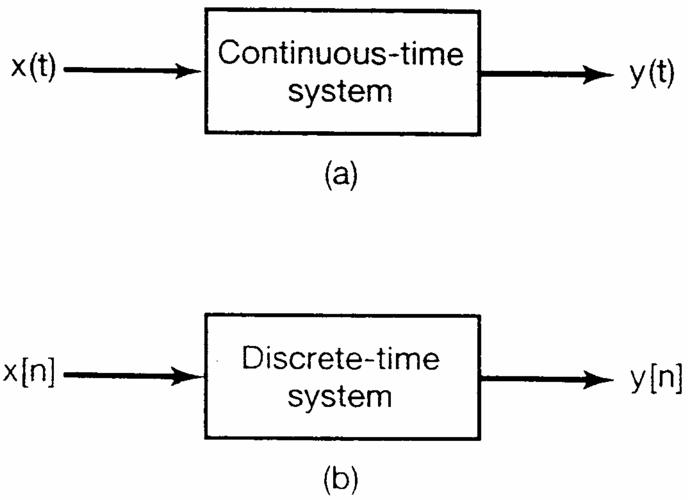



圖 1.40 (a)範例 1.7 的非連續信號；(b) $x(t)$ 的導函數 $\dot{x}(t)$；(c)利用 $\dot{x}(t)$ 在 $t$ 介於 0 和 1 之間的連續積分以恢復 $x(t)$ 的描述



# Continuous- and discrete-time systems

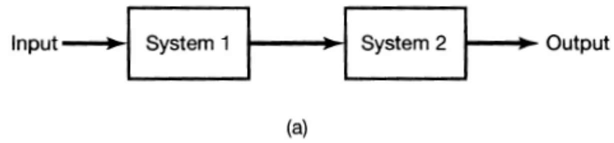



$$x(t) \rightarrow y(t)$$

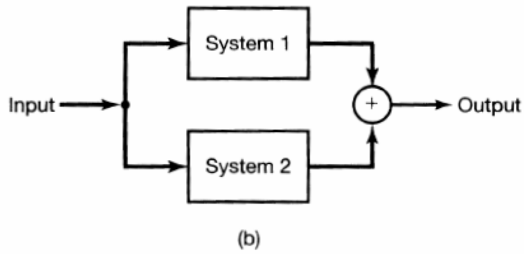
$$x[n] \rightarrow y[n]$$

**Figure 1.41** (a) Continuous-time system; (b) discrete-time system.

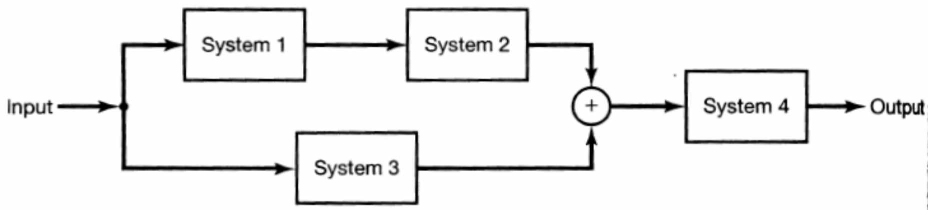
# Interconnection of Systems




Series (cascade)




parallel




Series-parallel



# Feedback Connections

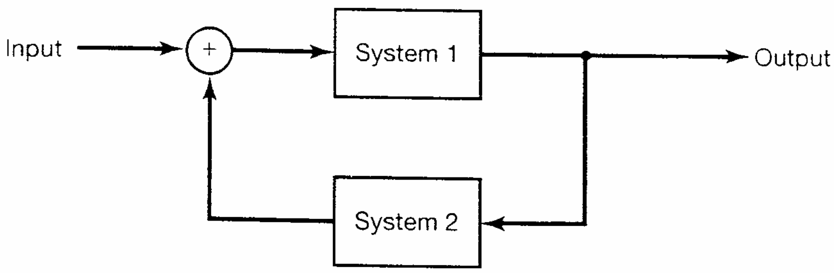
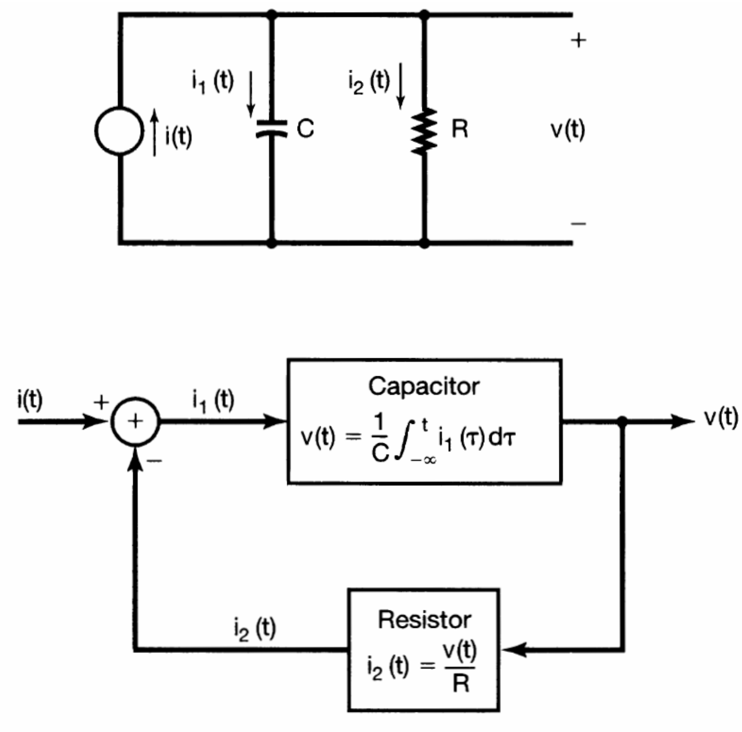



**Figure 1.43** Feedback interconnection.

# Example of Feedback Systems






# Systems without Memory

- Memoryless, depends on input at the same time

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = Rx(t)$$

- Identity system

$$y(t) = x(t) \qquad y[n] = x[n]$$

# Systems with Memory

- Examples

$$y[n] = \sum_{k=-\infty}^{n} x[k], \qquad y[n] = x[n-1]$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

- The system must remember or store something

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$

$$y[n] = y[n-1] + x[n]$$



# Invertibility and Inverse Systems

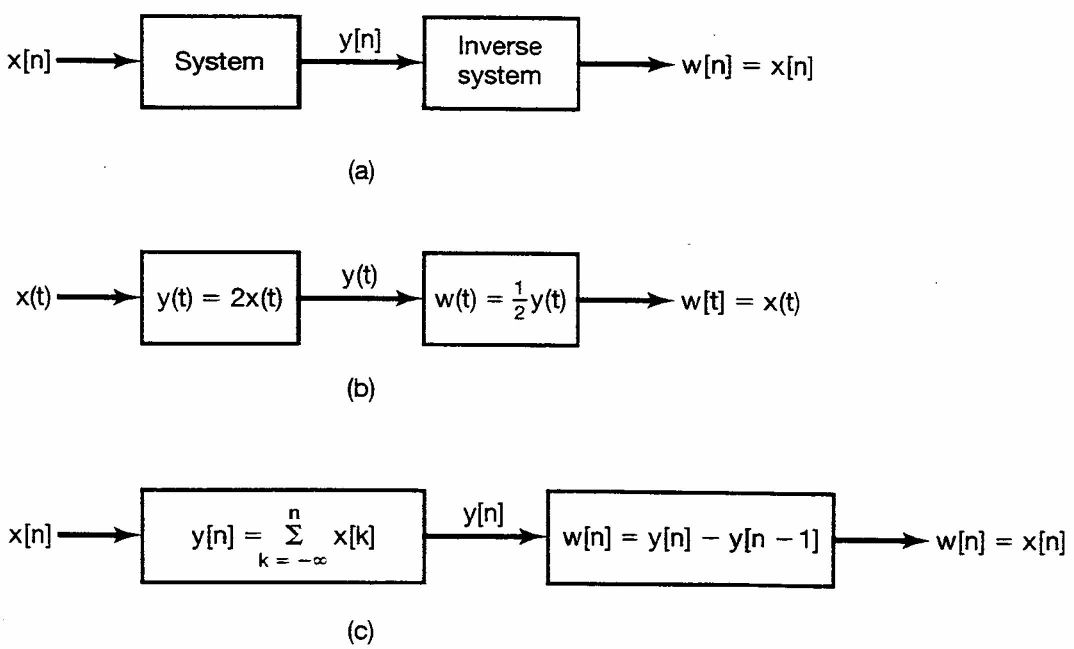



**Figure 1.45** Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

# Noninvertible Systems

- Examples

$$y[n] = 0$$

$$y(t) = x^2(t)$$

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = |x(t)|$$



# Causality

- A system is causal if the output at any time depends on values of the input at present and past times.
- Causal system is *nonanticipative*
- Examples of causal system

$$y[n] = \sum_{k=-\infty}^{n} x[k], \qquad y[n] = x[n-1]$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

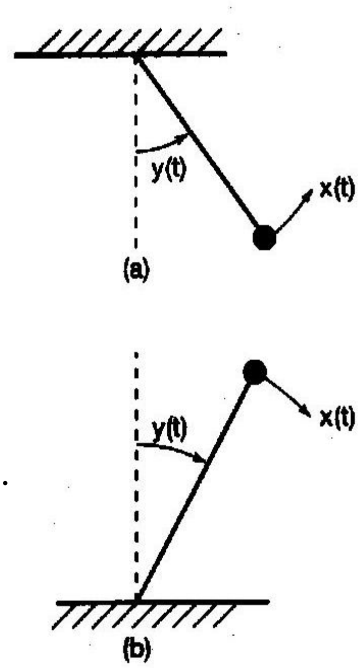
- Examples of noncausal system

$$y[n] = x[n] - x[n+1] \qquad y(t) = x(t+1)$$

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k].$$

# System Stability

- Stable Systems
  - Small perturbation does not give large divergence.
- Examples of unstable systems
  - Inverted pendulum
  - Population growth??
  - Bank account with interest??
  - Accumulator
- Examples of stable systems
  - Normal pendulum
  - Population growth with limited resources.
  - RC circuits
  - Practical integrator




1.

# Time Invariance Systems

- The characteristic of the system are fixed over time.
- Mathematically,

$$x(t) \rightarrow y(t)$$

$$x(t-t_0) \rightarrow y(t-t_0)$$

$$x[n] \rightarrow y[n]$$

$$x[n-n_0] \rightarrow y[n-n_0]$$

- Example: Time-invariance $y(t) = \sin[x(t)]$

  Time-variance $y[n] = nx[n]$

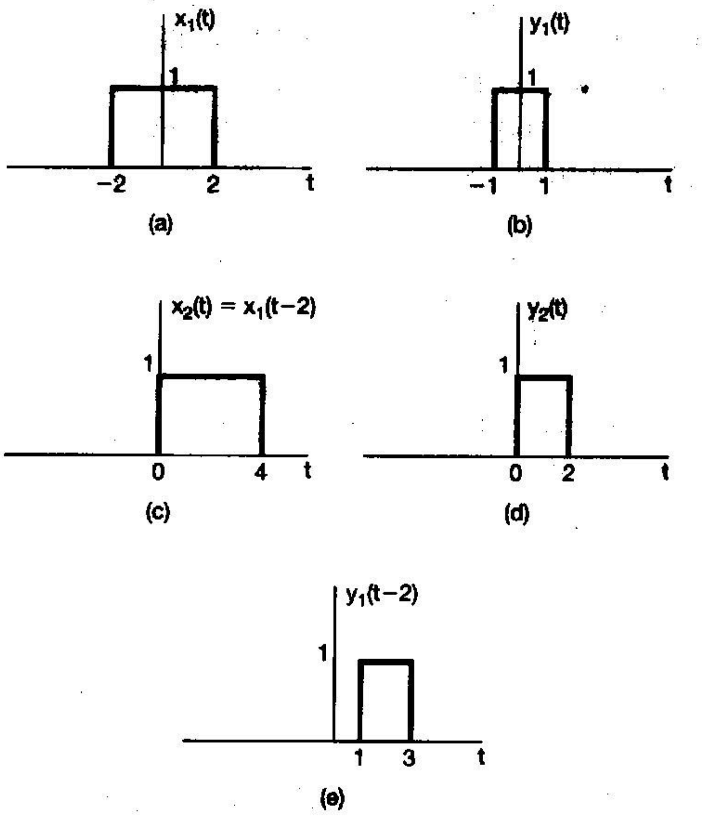



圖 1.47 (a)對例題 1.16 系統的輸入 $x_1(t)$；(b) $x_1(t)$ 的相對應輸出 $y_1(t)$；(c)移位的輸入 $x_2(t) = x_1(t-2)$；(d) $x_2(t)$ 相對應的輸出 $y_2(t)$；(e)移位信號 $y_1(t-2)$。注意：在 $y_2(t) \neq y_2(t-2)$ 時證明系統是時變的。

# System Linearity

- Linear system

if $x_1(t) \to y_1(t) \qquad x_2(t) \to y_2(t)$

then $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$

$$a x_1(t) \to a y_1(t)$$

Called additive and homogeneity properties

Combined conditions: Superposition properties

$$a x_1(t) + b x_2(t) \to a y_1(t) + b y_2(t)$$

$$a x_1[n] + b x_2[t] \to a y_1[t] + b y_2[t]$$

# Examples

- Linear systems

$$y(t) = tx(t)$$

$$y[n] = 2x[n] + 3$$

- Nonlinear systems

$$y(t) = x^2(t)$$

$$y[n] = \text{Re}\{x[n]\}$$



# Incrementally Linear System

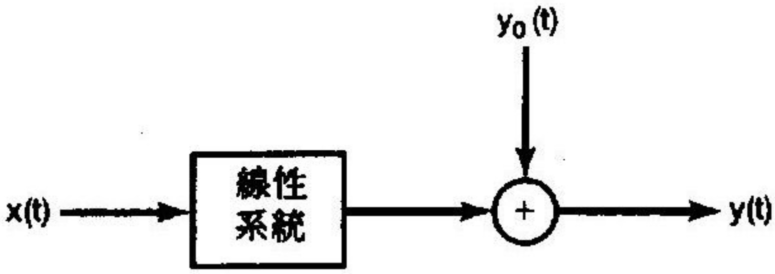



圖 1.48 增量線性系統的結構，其中 $y_0[n]$ 是系統的零輸入響應

Homework #1, due: Oct. 12
O: 1.21, 22, 37,54
B: 1.3(a),(b), 1.7(a),(b),(c)