Design of Equiripple FIR Filters With Constraint Using a Multiple Exchange Algorithm

Soo-Chang Pei and Peng-Hua Wang

***Abstract*—We propose a method of designing equiripple linear-phase FIR filters with linear constraint by carrying out the Remèz exchange algorithm. A novel technique is derived to convert a linearly constrained problem into an equivalent unconstrained one. We proposed a technique to modify the original desired frequency response so that the original linear constraint can be reduced to a simpler one (the null constraint) for the new target frequency response. The filter with null constraint can be designed without constraint by a transformation of the original basis functions. The transformation is represented by a basis for the null space of the constraint. In this brief, we show that the set of transformed basis also forms a Tchebycheff set. This fact indicates the proposed design is optimal in Tchebycheff sense. The optimal filter is deigned by Remèz method according to the new target frequency response in transformed basis. Design examples suggest that the proposed algorithm converges fast and stably.**

***Index Terms*—Equiripple FIR filter, linear constraint, multiple exchange algorithm.**

I. Introduction

Linear phase FIR filters with equiripple stopband and passband are important and have been investigated widely. The most important method of designing such filters is the Remèz exchange algorithm and its variants. Based on the alternation theorem, these filters are optimal in the Tchebycheff or minimax sense [7]. The Parks–McClellan algorithm makes use of the polynomial interpolation to calculate the frequency response [1], [2], [7], [8]. In [4] Shpak and Antioniou give a generalization of the Remèz algorithm to eliminate the transition band anomalies. Although the Parks–McClellan algorithm is powerful to design a wide class of FIR filters such as lowpass filters, differentiators, or Hilbert transformers, it is hard to design digital filters with constraint by this algorithm. In [3] Vaidyanathan presents a method to design digital filters with flat passbands and equiripple stopbands. This method carries out the Remèz algorithm based on a special filter structure that guarantees flat passband. In fact, the requirement of flat passband could be written as a set of linear equations of the filter coefficients and solved by our proposed method. Linear constraint occurs to the other filters. For example, the FIR notch filter is equivalent to an allpass filter with null constraint at the notch frequency. In [5] Er presents a least squared design of the notch filter with controlled null bandwidth. The bandwidth can be controlled by a set of linear equations established by posing the flat constraint at the notch frequency. In [6], Tseng and Pei design an equiripple FIR notch filter by a Remèz-like algorithm. Their proposed algorithm can be used for efficiently designing single-notch and multiple-notch filters. However, such filters can be designed with specifications of suitable target frequency response.

In this brief, we propose an algorithm to design linear phase FIR filters with constraint. The constraint is represented as a set of linear equations of the filter coefficients. The original constrained filter design problem is converted to a new problem without constraint. We carry out the Remèz exchange algorithm to solve the new problem. In Section II, we will formulate the filter design problem and put the four types of FIR filters in unified framework. The proposed algorithm are described and discussed in Section III. Examples are provided in Section IV to demonstrate the proposed method. Conclusions and remarks are given in Section V. Finally, in the Appendix we provide a proof to show that a linearly transformed Tchebycheff set still forms a Tchebycheff set.

Manuscript received August 1, 2000; revised August 17, 2001. This paper was recommended by Associate Editor W.-S. Lu.

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Publisher Item Identifier S 1057-7122(02)00294-5.

II. Problem Formulation

The frequency response of a causal Nth-order FIR filter is expressed by

$$H(e^{j\omega}) = \sum_{n=0}^{N} h_n e^{-jn\omega} \tag{1}$$

where the impulse responses h_n may be complex numbers or real ones. In this paper, we assume that the impulse responses are real. Since the frequency response $H(e^{j\omega})$ is a complex-value function, it can be represented as $H(e^{j\omega}) = e^{j\alpha(\omega)}A(\omega)$ where the amplitude response $A(\omega)$ is a real function of ω and $\alpha(\omega)$ is the corresponding phase response. If the impulse response sequence h_n exhibits symmetric properties about the index n, the phase response is a linear function of ω, and $H(e^{j\omega})$ is called a linear-phase filter. The impulse responses are symmetric if $h_n = \pm h_{N-n}$ for $0 \leq n \leq N$. Real coefficient linear phase filters can be classified into four types that depend on symmetry of h_n and on whether N is even or odd [7]–[8]. For each type, we can express the amplitude response as

$$A(\omega) = \sum_{m=1}^{M} a_m \phi_m(\omega) \tag{2}$$

where $\{\phi_m(\omega)\}$ is a set of appropriate trigonometric functions and M, determined by filter order N, is the number of independent coefficients in the filter. The value of M and the relationship among h_n, a_m, and $\phi_m(\omega)$ are summarized and presented in [6]. We may express the amplitude response in matrix form. Let the coefficient vector $\mathbf{a} = (a_1, a_2, \ldots, a_M)^t$ and the basis vector $\mathbf{\Phi}(\omega) = (\phi_1(\omega), \phi_2(\omega), \ldots, \phi_M(\omega))^t$ where the superscript $(\cdot)^t$ denotes the matrix transposition. The amplitude response in (2) is accordingly expressed as

$$A(\omega) = \mathbf{a}^t\mathbf{\Phi}(\omega) = \mathbf{\Phi}^t(\omega)\mathbf{a}. \tag{3}$$

This matrix form will facilitate the derivation and discussion of our proposed algorithm.

The filter design problem is to find a set of impulse response $\{h_n\}$ such that the frequency response $H(e^{j\omega})$ is approximated to a given frequency response. That is, to find the coefficients a_m in (2) such that the amplitude response $A(\omega)$ is close to a given amplitude response $D(\omega)$. If the performance of the filter has to be controlled precisely for obtaining good approximation within some frequency bands or at some frequency points, the coefficients a_m may be restricted by linear constraint. The linear constraint could be expressed in matrix form and written as

$$\mathbf{C}\mathbf{a} = \mathbf{b} \tag{4}$$

where the $K \times M$ constraint matrix $\mathbf{C}$ represents K linear equations of unknowns a_m. In the following section, a multiple change algorithm is used for solving the filter design problem.

III. Filter Design With Constraint

In this section, we will describe the proposed algorithm to solve the filter design problem with linear constraint on the filter coefficients.

Let $\tilde{\mathbf{a}} = \mathbf{a} - \mathbf{x}$ where $\mathbf{x}$ is a vector that satisfies $\mathbf{Cx} = \mathbf{b}$. A choice of $\mathbf{x}$ is $\mathbf{x} = \mathbf{C}^{+}\mathbf{b}$ which $\mathbf{C}^{+}$ is the pseudo inverse of $\mathbf{C}$ [9]. Let $D'(\omega) = D(\omega) - \mathbf{x}^t\mathbf{\Phi}(\omega)$. It is easy for the following error function to show that $E(\omega) = D(\omega) - \mathbf{a}^t\mathbf{\Phi}(\omega) = D'(\omega) - \tilde{\mathbf{a}}^t\mathbf{\Phi}(\omega)$. The constraint becomes $\mathbf{C}\tilde{\mathbf{a}} = \mathbf{Ca} - \mathbf{Cx} = \mathbf{0}$. That is, the original desired response $D(\omega)$ is equivalent to a new one of $D'(\omega)$ with a new constraint $\mathbf{C}\tilde{\mathbf{a}} = \mathbf{0}$.

The next key step is to transform the constrained problem to an unconstrained one. Let the $M \times R$ matrix $\mathbf{B}$ denote the null space of $\mathbf{C}$. Suppose $\tilde{\mathbf{a}} = \mathbf{Bw}$ where $\mathbf{w} = (w_1, w_2, \ldots, w_R)^t$ is a new set unknowns to be determined, the frequency response can be represented by $\tilde{\mathbf{a}}^t\mathbf{\Phi}(\omega) = \mathbf{w}^t\mathbf{B}^t\mathbf{\Phi}(\omega) = \mathbf{w}^t\mathbf{\Phi}'(\omega)$ where $\mathbf{\Phi}'(\omega) = \mathbf{B}^t\mathbf{\Phi}(\omega)$ represents a new set of bases. Based on this transformation, the filter design problem is equivalent to finding the optimal $\mathbf{w}$ using $\mathbf{\Phi}'(\omega)$. Moreover, since $\mathbf{B}$ represents a null space of $\mathbf{C}$, it is obvious to show that $\mathbf{C}\tilde{\mathbf{a}} = \mathbf{CBw} = \mathbf{0}$. Hence, we convert the original problem into an new one with desired frequency response $D'(\omega)$ but without any constraint on w_n.

The equiripple filter design problem for our new desired response $D'(\omega)$ is to find a set of frequencies ω_i, $i = 1, 2, \ldots, R+1$, and a set of coefficients w_j, $j = 1, 2, \ldots, R+1$ such that $\omega_1 < \omega_2 < \cdots < \omega_{R+1}$ and

$$W(\omega_i)(D'(\omega_i) - \mathbf{w}^t\mathbf{\Phi}'(\omega_i)) = (-1)^i\delta \tag{5}$$

for $i = 1, 2, \ldots, R+1$.

According to the former description, we propose a multiple exchange algorithm to design the equiripple linear-phase FIR filters as follows.

Step 1. Specify the filter order N, the type of filter, the desired frequency response $D(\omega)$, and the weighting function $W(\omega)$. Calculate M according to the filter type. Represent constraint by constraint matrix $\mathbf{C}$.

Step 2. Calculate $\mathbf{x}$ and $D'(\omega)$. Calculate the null space $\mathbf{B}$ of $\mathbf{C}$. Evaluate the new basis $\mathbf{\Phi}'(\omega)$.

Step 3. Obtain a set of initial extremal frequencies. One choice is to select the frequencies evenly on the passbands and stopbands.

Repeat

Step 4. Calculate the intermediate filter coefficients w_j and peak error δ by solving (5).

Step 5. Calculate the intermediate error function $E(\omega) = W(\omega)(D'(\omega) - \mathbf{w}^t\mathbf{\Phi}'(\omega))$ and search for the extremal frequencies ω_i.

Until some criterion is satisfied. We use the relative difference between the maximal error and the minimal error to test termination of the algorithm. The difference is represented by $Q = (\max|E(\omega)| - \min|E(\omega)|)/\max|E(\omega)$. Steps 4 and 5 are repeated until $Q < \varepsilon$. We choose $\varepsilon = 0.001$ in this paper.

Step 6. Calculate the coefficient vector $\mathbf{a}$ by $\mathbf{a} = \mathbf{Bw} + \mathbf{x}$. Finally, obtain the impulse response h_n.

Remark: It is well-known that the trigonometric basis $\phi_n(\omega)$ is a set of Tchebycheff basis for a suitable interval [10]. However, the transformed basis $\phi'_n(\omega)$ also forms Tchebycheff set. We provide a proof for this property in the Appendix.

IV. Design Examples

In this section, different kinds of FIR filters are to be designed by proposed method.

Example 1: In this example, we design 78th-order linear phase lowpass FIR filters with flat passband of $[0, 0.4\pi]$ and stopband of $[0.45\pi, \pi]$. Constraint of flatness on frequency response is represented by the equations of

$$\frac{d^k}{d\omega^k}A(\omega_m) = \frac{d^k}{d\omega^k}D(\omega_m) \tag{6}$$

for $k = 0, 1, \ldots, K-1$ at a given frequency point ω_m where K is called the degrees of flatness. Filters designed with such constraint will obtain good approximation around ω_m. The results of $K = 3$ and $\omega_m = 0$ is shown in Fig. 1. Fig. 2 shows the result of $K = 3$ and $\omega_m = 0.2\pi$.

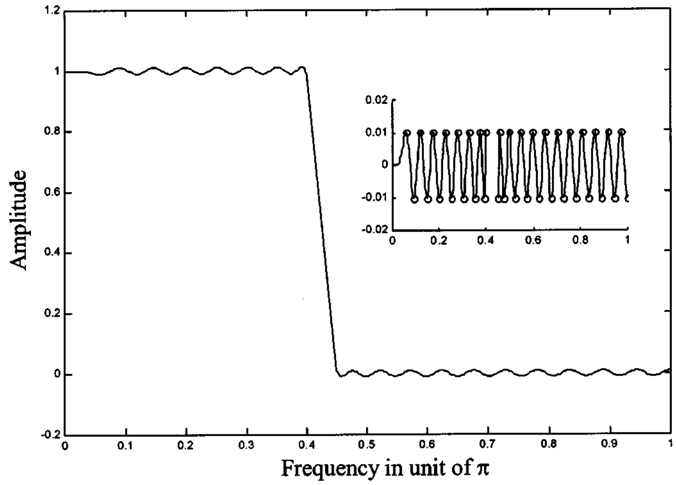

Fig. 1. The amplitude response and associated error function of a 78th-order FIR lowpass filter with passband edge of 0.4π and stopband edge of 0.45π. 3 degrees of flatness are set at $\omega = 0$. Convergence is achieved after 9 iterations with peak error of 0.010 226.

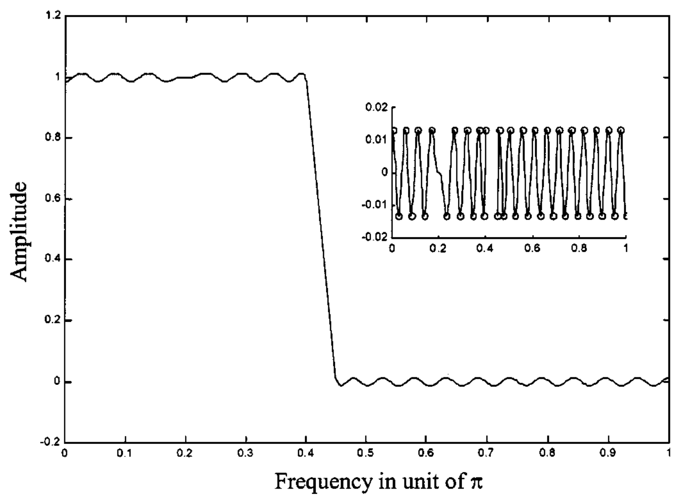

Fig. 2. The amplitude response and associated error function of a 78th-order FIR lowpass filter with passband edge of 0.4π and stopband edge of 0.45π. 3 degrees of flatness are set at $\omega = 0.2\pi$. Convergence is achieved after 7 iterations with peak error of 0.013 161.

Example 2: 78th-order linear phase digital differentiators with lowpass frequency responses will be designed in this example. The lowpass differentiators have ideal frequency responses $j\omega$ for lower frequency band and null frequency response on higher band. The lower band edge of 0.8π and higher edge of 0.85π are used for this example. 3 degrees of flatness constraint are set at $\omega_m = 0$ and $\omega_m = \pi/2$, respectively. Figs. 3 and 4 show the amplitude responses and associated error functions of these two filters.

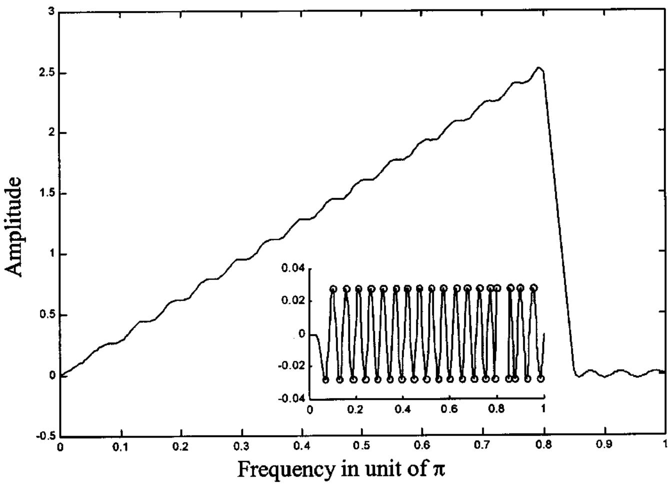

Fig. 3. The amplitude response and associated error function of a 78th-order FIR lowpass differentiator with passband edge of 0.8π and stopband edge of 0.85π. 3 degrees of flatness are set at $\omega = 0$. Convergence is achieved after 15 iterations with peak error of 0.028 057.

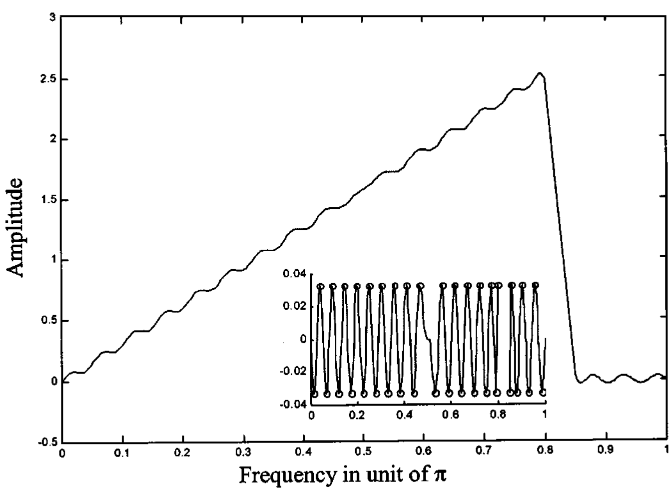

Fig. 4. The amplitude response and associated error function of a 78th-order FIR lowpass differentiator with passband edge of 0.8π and stopband edge of 0.85π. 3 degrees of flatness are set at $\omega = \pi/2$. Convergence is achieved after 20 iterations with peak error of 0.032 853.

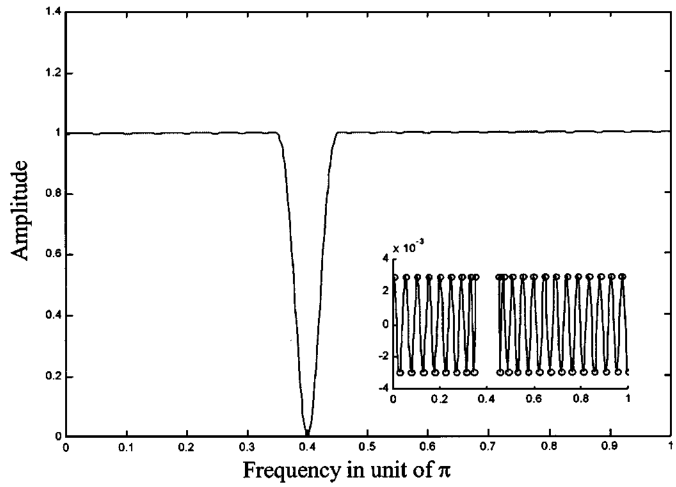

Fig. 5. The amplitude response and associated error function of an 84th-order FIR notch filter with notch frequency $\omega_n = 0.4\pi$. Convergence is achieved after 12 iterations with peak error of 0.002 944.

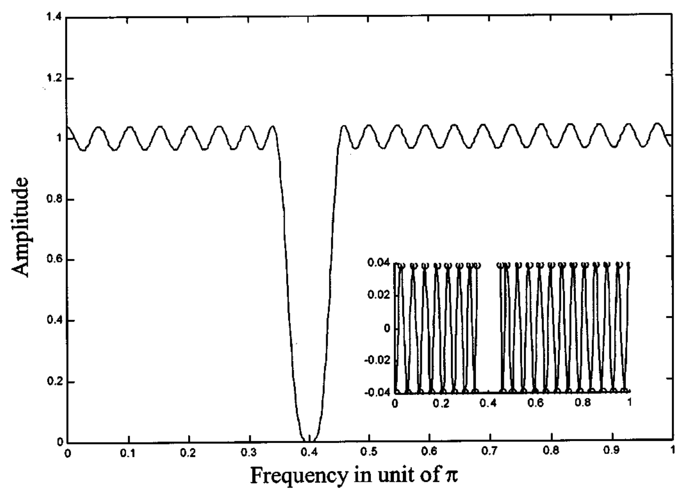

Fig. 6. The amplitude response and associated error function of an 84th-order FIR notch filter with notch frequency $\omega_n = 0.4\pi$. 3 degrees of flatness are set at the notch frequency. Convergence is achieved after 12 iterations with peak error of 0.038 875.

Example 3: FIR Notch filters with different null bandwidths will be designed in this example. The ideal frequency response $D_n(\omega)$ of a notch filter is expressed by

$$D_n(\omega) = \begin{cases} 1, & \omega \neq \omega_n \\ 0, & \omega = \omega_n. \end{cases}$$

The frequencyω_n is called the notch frequency. Hence, design of notch filter is equivalent to the problem of designing a filter with the desired frequency response $D(\omega) = 1$ and the null constraint on ω_n. 84th-order linear phase FIR notch filters are designed with $\omega_n = 0.4\pi$ and different degrees of flatness on the notch frequency. Figs. 5 and 6 show the amplitude responses and associated error function of 1 and 3 degrees of flatness, respectively.

V. Conclusions

In this brief, we propose a method to design equiripple linear-phase FIR filters with linear constraint by carrying out the Remèz exchanges algorithm. A novel technique is derived to convert a linearly constrained problem to an equivalent unconstrained one. The key step is to modify the original desired frequency response such that the constraint of $\mathbf{Ca} = \mathbf{b}$ is reduced to a null constraint of $\mathbf{C}\tilde{\mathbf{a}} = \mathbf{0}$ for the new target frequency response. Then the filter constrained by such constraint can be designed without any constraint by a set of bases obtained by transforming the original basis by the null space of $\mathbf{C}$. In the appendix, we show that this set of transformed bases forms a Tchebycheff set. This fact indicates that our design is optimal in Tchebycheff sense. Design examples suggest that the proposed algorithm converges quickly and stably.

Appendix

Let $\mathbf{\Phi}(\omega) = (\phi_1(\omega), \phi_2(\omega), \ldots, \phi_M(\omega))^t$ and $\mathbf{\Phi}'(\omega) = (\phi'_1(\omega), \phi'_2(\omega), \ldots, \phi'_R(\omega))^t$ satisfies $\mathbf{\Phi}'(\omega) = \mathbf{B}^t\mathbf{\Phi}(\omega)$ where $\mathbf{B}$ is a matrix of $M \times R$, $R \leq M$.

Property: If the set $\{\phi_n(\omega)\}$ forms a Tchebycheff set in an interval X and the rank of $\mathbf{B}$ is equal to R, $\{\phi'_n(\omega)\}$ also forms a Tchebycheff set in X.

Proof: According to [10], $\{\phi'_n(\omega)\}$ forms a Tchebycheff set if and only if the following matrix is nonsingular for every set $\{\omega_i | i = 1, 2, \ldots R\}$ of distinct points in X

$$\mathbf{P} = \begin{pmatrix} \phi'_1(\omega_1) & \phi'_1(\omega_2) & \cdots & \phi'_1(\omega_R) \\ \phi'_2(\omega_1) & \phi'_2(\omega_2) & \cdots & \phi'_2(\omega_R) \\ \vdots & \vdots & \ddots & \vdots \\ \phi'_R(\omega_1) & \phi'_R(\omega_2) & \cdots & \phi'_R(\omega_R) \end{pmatrix}. \qquad (7)$$

It is obvious to show that $\mathbf{P} = \mathbf{B}^t\mathbf{Q}$ where

$$\mathbf{Q} = \begin{pmatrix} \phi_1(\omega_1) & \phi_1(\omega_2) & \cdots & \phi_1(\omega_R) \\ \phi_2(\omega_1) & \phi_2(\omega_2) & \cdots & \phi_2(\omega_R) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_M(\omega_1) & \phi_M(\omega_2) & \cdots & \phi_M(\omega_R) \end{pmatrix}. \qquad (8)$$

Because $\{\phi_n(\omega)\}$ forms a Tchebycheff set, the rank of $\mathbf{Q}$ is equal to R. Hence both the ranks of $\mathbf{P}$ and of $\mathbf{B}$ are equal to R and accordingly $\mathbf{P} = \mathbf{B}^t\mathbf{Q}$ is nonsingular.

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Discretized Quadratic Optimal Control for Continuous-Time Two-Dimensional Systems

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***Abstract*—In this brief, a discretized quadratic optimal control for continuous-time two-dimensional (2-D) system is newly proposed. It introduces a new state vector (a new virtual control input) to directly convert the original continuous-time 2-D quadratic cost function into a decoupled discretized form. As a result, a new virtual discrete-time 2-D model with the new virtual control input is constructed for indirectly finding the desired discretized quadratic optimal regulator for the continuous-time 2-D system. The recently developed dynamic programming in discrete-time 1-D descriptor form is utilized to determine the desired discretized quadratic optimal regulator. This method provides a novel approach for discretized quadratic optimal control of continuous-time 2-D systems. An illustrative example is presented to demonstrate the effectiveness of the proposed procedure.**

***Index Terms*—Digital design, optimal control, Roesser's model, two-dimensional systems.**

I. INTRODUCTION

The majority of distributed parameter systems, such as smart materials, smart structure, heat flow, transmission lines, gas absorption, etc., are formulated by continuous-time two-dimensional (2-D) framework. However, little has been accomplished in the development of optimal analog regulators for continuous-time 2-D systems. Motivated by the applications in digital picture processing, seismic data processing, X-ray image processing, etc., the continuous-time 2-D system is often converted into an equivalent discrete-time 2-D system via some approximation methods, such as the first difference method [1] and the high-order discretization method [2] with the assumptions that the sampling time and the sampling distance are sufficiently small. Then, by using the approximate discrete-time 2-D model together with a discrete-time performance index suited to discrete-time 2-D systems, many digital linear quadratic regulators (LQRs) are developed for optimal digital control of discrete-time 2-D system [1], [3]–[7].

In this paper, we utilize the well-developed discrete-time 2-D model, recently proposed by the authors [5] together with a continuous-time performance index suited to continuous-time 2-D system to design a discretized quadratic optimal regulator for the continuous-time 2-D system. The well-developed discrete-time 2-D model [5] has capability of allowing the use of relatively long sampling time and long sampling distance. A new state vector is introduced in this paper to eliminate the cross terms arisen in discretizing the pre-selected continuous performance index. As a result, a new virtual discrete-time 2-D model suited to the development of digital LQRs for discrete-time 2-D systems can be established. The recently developed dynamic programming in discrete-time 1-D descriptor form [6] is then applied to the new virtual discrete-time 2-D model for indirectly determining the desired digital LQR for the continuous-time 2-D system.

Manuscript received April 3, 2001; revised August 15 2001, September 6 2001, and September 8 2001. This work was supported in part by the National Science Council of Republic of China under Contract NSC-89-2213-E-006-189 and in part by the Army Research Office under Grant DAAG-55-98-1-0198. This paper was recommended by Associate Editor G. Chen.

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Publisher Item Identifier S 1057-7122(02)00295-7.