# Practically Realizable Digital Transmission Significantly Below the Nyquist Bandwidth

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**Abstract**

In this paper we propose a practically realizable digital signaling scheme that requires a bandwidth significantly less than the Nyquist limit, and at the same time achieves the same asymptotic performance as memoryless PAM. A three-step iterative procedure for constructing such schemes is presented, and examples that occupy down to about 60 percent of the Nyquist bandwidth are demonstrated using this simple procedure. We also discuss a practical receiver structure using adaptive breath-first trellis search algorithm.

## 1 Introduction

It has long been believed that to transmit a symbol per $T$ seconds by pulse amplitude modulation (PAM) without significant performance degradation, one needs a signal bandwidth wider or equal to the Nyquist bandwidth $1/2T$. The origin of this belief lies in the pioneering work of Nyquist [1] on intersymbol-interference (ISI) free transmission . The Nyquist bandwidth is achieved only when the unrealizable ideal rectangular filter is used. For realizable filters such as the raised-cosine family, an excess bandwidth of 15 to 100 percent is required. In 1964, Lender [2] introduced correlative-level coding, or partial response signaling (PRS) to confine the signal bandwidth completely within the Nyquist bandwidth. This signaling technique is later found [3] to have the same asymptotic performance as ideal memoryless signaling.

In optimum receiver principles [4] and detection theory [5], when one comes to bandlimited channels, the work of Nyquist usually is also adopted, even though both theories give no indication that the Nyquist bandwidth is the ultimate limit. Mazo [6] was the first to consider the signaling techniques below the Nyquist bandwidth essentially without any performance degradation, and there has been some new interest in this problem [7, 8, 9] in recent years. However, in all these previous works only the unrealizable ideal rectangular filters with binary input were considered. In this paper, on the other hand, we will investigate multi-level signaling below the Nyquist bandwidth using realizable filters. In Section 2 the necessary background is first briefly summarized, and the general approach to design realizable filters for sinaling below the Nyquist bandwidth is presented in Section 3. Some design examples are then given in Section 4, and practical receiver structures are discussed in Section 5. Section 6 discusses the channel capacity of PAM and QAM, and the conclusion is finally given in Section 7.

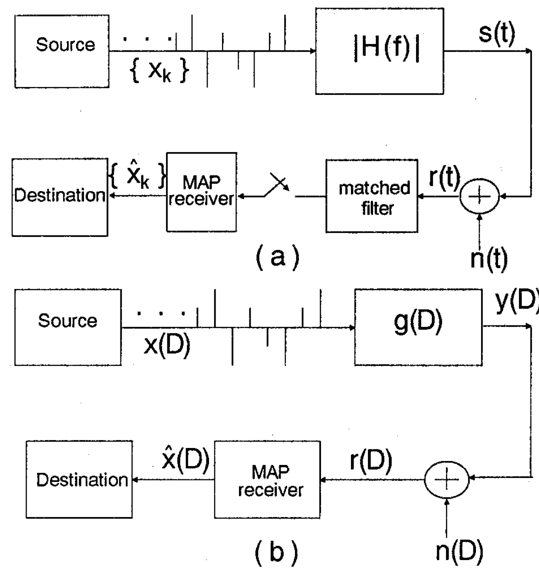


Figure 1: (a) Block diagram of a PAM transmission system, (b) Equivalent discrete model of the system.

## 2 Background

The block diagram of a baseband PAM transmission system is depicted in Figure 1(a). Let $\{x_k\}$ be a sequence of interger symbols with finite length $N$ such that $0 \leq x_k \leq m-1$, $0 \leq k \leq N-1$. The impulse response and transfer function of the transmitting filter are $h(t)$ and $H(f)$, respectively. Then the transmitted signal $s(t)$ is

$$s(t) = \sum_{k=0}^{N-1} x_k h(t-kT) \tag{1}$$

where $T$ is the symbol duration, and the received signal $r(t)$ is the additive-noise corrupted version of $s(t)$

$$r(t) = s(t) + n(t) \tag{2}$$

where $n(t)$ is assumed to be a white Gaussian noise with two-sided power spectral density (PSD) $N_0/2$. If $H(f)$ is bandlimited, $h(t)$ will have infinite response. For convenience in discussion, we normalize the energy of the filter to unity, i. e.,

$$\int_{-\infty}^{\infty} h^2(t)\,dt = \int_{-\infty}^{\infty} |H(f)|^2\,df = 1. \tag{3}$$

  

Let $D$ denotes the delay operator such that $D^k h(t) = h(t-kT)$, then the input sequence can be expressed as $x(D) = \sum_{k=0}^{N-1} x_k D^k$, and the transmitted signal expressed as

$$s(t) = x(D)h(t). \quad (4)$$

Since $s(t)$ is simply a linear combination of $h(t)$ and its time-shifted versions $h(t-kT)$, by Gram-schmidt procedure [4] an orthonormal basis of dimension not greater than $N$ can be found to expand $s(t)$, and at the receiver side a matched filter followed by a symbol-rate sampler [4, 5] can be used for maximum a posteriori (MAP) detection to obtain the received symbols $\hat{x}_k$ and sequence $\hat{x}(D)$ with minimum error probability. The corresponding discrete model of the system is depicted in Figure 1(b), where $g(D)$ is the equivalent discrete transfer function of the system and $r(D) = x(D)g(D) + n(D)$ is the received sequence. For equally probable input symbols, a MAP receiver becomes a maximum-likelihood (ML) receiver, and the event error probability and the symbol error probability are respectively given by

$$Pr(e) \geq Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) \quad (5)$$

and

$$P_s \geq w_H(d_{min})Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) \quad (6)$$

where

$$d_{min}^2 = \min_{x(D)\neq x'(D)} \int_{-\infty}^{\infty} |s(t) - s'(t)|^2\, dt \quad (7)$$

is the minimum squared Euclidean distance (ED), and $w_H(d_{min})$ is the Hamming distance between the sequences with squared ED $d_{min}^2$. The linear property of PAM enables us to rewrite Equation (7) as

$$d_{min}^2 = \min_{e(D)} \int_{-\infty}^{\infty} |e(D)h(t)|^2\, dt \quad (8)$$

where

$$e(D) = \sum_{k=0}^{N-1} e_k D^k = x(D) - x'(D),\ |e_k| \leq m-1 \quad (9)$$

is an error sequence. The signal to noise ratio (SNR) is defined as

$$\text{SNR} = \frac{\sigma_x^2 \int_{-\infty}^{\infty} h^2(t)\, dt}{N_0} = \frac{\sigma_x^2}{N_0} \quad (10)$$

where $\sigma_x^2$ is the variance of the input alphabets $x_k$.

It is clear from Equation (5) that a PAM scheme achieves the best performance when $d_{min}^2$ is maximized. For all $h(t)$, it's trivial to show that $d_{min}^2 \leq \int_{-\infty}^{\infty} h^2(t)\, dt$, hence to achieve the best performance we must have

$$d_{min}^2 = \int_{-\infty}^{\infty} h^2(t)\, dt = \int_{-\infty}^{\infty} |H(f)|^2\, df = 1. \quad (11)$$

In this paper we are interested in bandlimited PAM schemes that satisfy Equation (11), with realizable $H(f)$ sharply bandlimited below the Nyquist limit.

The simplest examples of bandlimited PAM schemes that satisfy Equation (11) use the unrealizable ideal rectangular filters and the realizable raised-cosine family filters, in which $h(t)$ is orthonormal to its shifted versions $h(t-kT), k = 1, 2, \cdots, N-1$. Another example is the class of $1 \pm D^n$ PRS schemes [3]. All these PAM schemes require a bandwidth equal to or wider than the Nyquist bandwidth $1/2T$. The first attempt to devise PAM schemes that require less bandwidth than $1/2T$ is the faster-than-Nyquist signaling [6] proposed by Mazo, which consists of a binary input sequence and an ideal filter of bandwidth $\rho/2T, \rho \leq 1$. Mazo [9] showed that for $0.802\cdots \leq \rho \leq 1$, the minimum distance does not drop below 1.

For a passband PAM system, basically everything is the same as the baseband PAM system described above except for the presence of a modulated carrier, therefore the baseband PAM model presented above can be directly used as an baseband-equivalent representation. The results for PAM are basically applicable for QAM.

## 3 General Approach

The faster-than-Nyquist signaling is attractive except that the ideal rectangular filter used is not realizable and that only binary input is considered. In this section we will consider multi-level input and realizable $H(f)$, i.e., $H(f)$ that has gradual roll-off at its band-edge, which is lower than the Nyquist frequency, and a three-step iterative procedure will be provided to construct such filters. Before we present the three-step iterative procedure, some preparatory material has to be provided first.

Given an arbitrary realizable $H(f)$ such that $|H(f)| = 0, |f| > \rho/2T, \rho \leq 1$, Equation (11) in general does not hold. Thus we have to first find $d_{min}^2$ and the corresponding error sequences based on Equation (8). The minimum distance problem for low-complexity finite-impulse-response channels has been solved [10], yet for infinite-impulse-response channels discussed here, this problem is still open in general. However, in some particular situations, this problem in fact has very simple solution, as was discussed below.

Let $e_F(f)$ be the Fourier transform of an error sequence $e(D)$ with length $L+1$, i.e.,

$$e_F(f) = e(D)|_{D=e^{j2\pi Tf}} \quad (12)$$

where

$$e(D) = \sum_{k=n}^{n+L} e_k D^k,\ e_n, e_{n+L} \neq 0,\ n, L \geq 0. \quad (13)$$

Using Parseval theorem, Equation (8) can be rewritten as

$$\begin{aligned} d_{min}^2 &= \min_{e(D)} \int_{-\rho/2T}^{\rho/2T} e_F(f) e_F^*(f) H(f) H^*(f)\, df \\ &= \min_{e(D)} \left[\sum_{k=n}^{n+L} e_k^2 + \sum_{k=n}^{n+L-1} \sum_{i=1}^{L+n-k} e_k e_{k+i}(R_i + R_{-i})\right] \end{aligned} \quad (14)$$

where

$$R_i = \int_{-\rho/2T}^{\rho/2T} |H(f)|^2 e^{j2\pi iTf}\, df = \int_{-\infty}^{\infty} h(t)h(t-iT)\, dt = R_{-i} \quad (15)$$

is the $i$th autocorrelation coefficient of $h(t)$. It can then be easily found from Equation (14) that if $R_1 > 0$ and $R_1 \gg |R_i|$, $i = 2, 3, \cdots, L$, the error sequences that produce $d_{min}^2$ will inevitably belong to the set of alternating sequences $\pm D^n(1 - D + D^2 - \cdots)$. Hence we have the following proposition:

**Proposition 1** *For a bandlimited filter $H(f), |H(f)| = 0, |f| \geq \rho/2T, \rho \leq 1$, if $R_1 \gg |R_i| \geq 0$, $i = 2, 3, \cdots, L$, then the minimum ED can be found by simply searching through the set $\{e(D)|e_{n+i} = -e_{n+i+1} = \pm 1,\ 0 \leq i \leq L,\ n, L \geq 0\}$.*

This proposition is very useful in the three-step construction procedure presented below to find realizable filters below the Nyquist bandwidth. When the above assumption fails, which is often the case as one tries to push the bandwidth to the bottom, no solution is avaliable at the present time.

However, as long as the minimum distance $d^2_{min}$ and the corresponding error sequences are found in some way, we have either $d^2_{min} = 1$, or $d^2_{min} < 1$. In the former case, this is exactly the desired solution, whereas in the latter case we can, under a weak condition, find the desired solution by making some modifications on $H(f)$ based on the following theorem, which is originally attributed to Forney [3], but is modified here for the present purposes.

**Theorem 1** *Let $D^n p(D)$ be one of the error sequences which produce $d^2_{min} < 1$ for a bandlimited filter $H(f)$, $|H(f)| = 0, |f| > \rho/2T, \rho \le 1$, and let $p_F(f)$ be the Fourier transform of $p(D)$. Define the modified error sequence $\tilde{e}(D)$ as $\tilde{e}(D) = e(D)p(D)$ for an arbitrary error sequence $e(D)$ as defined in Equation (13). If $\min_{\tilde{e}(D)} \int_{-\infty}^{\infty} |\tilde{e}(D)h(t)|^2\,dt = \min_{e(D)} \int_{-\infty}^{\infty} |e(D)p(D)h(t)|^2\,dt = d^2_{min}$, the filter $\tilde{H}(f) = (1/d_{min})p_F(f)H(f)$ satisfies Equation (11), and occupies no more bandwidth than $H(f)$.*

Now we are ready for the three-step iterative construction procedure:

**step 1:** Given an arbitrary linear phase bandlimited filter with transfer function $H(f)$ such that $|H(f)| = 0, |f| > \rho/2T$, $\rho \le 1$. Go to step 2.

**step 2:** For the chosen $H(f)$ find $d^2_{min}$ and one of the error sequences $D^n p(D)$ that produce $d^2_{min}$. Go to step 3.

**step 3:** If $d^2_{min} = 1$, then $H(f)$ is exactly the desired solution, thus terminate the procedure. If $d^2_{min} < 1$, construct $\tilde{H}(f) = (1/d_{min})p_F(f)H(f)$, where $p_F(f)H(f)$ is the Fourier transform of $p(D)$. Assign this $\tilde{H}(f)$ to $H(f)$, and go to step 2.

## 4 Examples

We will consider a class of filters as the initial choices of step 1 in the construction procedure in this section.

**Step 1:** The class of the chosen transfer function $H(f)$ is the frequency scaled version of the duobinary filter with $|H(f)| = \sqrt{2T/\rho}\cos(\pi T f/\rho), |f| \le (\rho/2T), \rho \le 1$ which has gradual roll-off at its band-edge, and is thus realizable.

**Step 2:** For $\rho$ reduced from 1.00 down to about 0.65, Proposition 1 can be applied with no difficulty. For example, we list the autocorrelation coefficients $R_i$ of $H(f)$ for $\rho$ =1.00,0.95,0.90,0.85, 0.80,0.75,0.70, and 0.65 in Table 1(a). It can be found that $R_1 > 2.5|R_2| > 11|R_i| \ge 0$, $i = 3, 4, \cdots$, which can be regarded to comply with the criterion of Proposition 1: $R_1 \gg |R_i| \ge 0, i = 2, 3, \cdots$. The minimum distances for these cases and the error sequences that produce these minimum distances are found numerically by searching through the set of alternating sequences $\pm D^n(1 - D + D^2 - \cdots)$, and are listed in Table 1(b). The minimum distances are all smaller than unity except when $\rho = 1.00$, we can then go to step 3 at this point. Extensive computer search has also been performed to verify all the data in Table 1(b), and no disagreement is located so that we are convinced Proposition 1 applies very well in this example.

**Step 3:** For all the $H(f)$ with $d^2_{min} < 1$, we can invoke Theorem 1 to construct $\tilde{H}(f)$ to improve the minimum distance. For example, for $\rho = 0.65$ the minimum distance $d^2_{min}$ is 0.4889 and the corresponding error sequence is $\pm D^n(1 - D)$, thus the power efficient $\tilde{H}(f)$ is $\sqrt{1/0.4889}(1 - e^{j2\pi Tf})H(f)$. Go to step 2.

**Step 2:** The filter $\tilde{H}(f)$ given above is verified to have minimum distance equal to unity. We have thus found the solution, and the procedure is terminated. The transmitter implementations of this filter both with direct synthesis, or by cascading a transversal filter are illustrated in Figure 2.

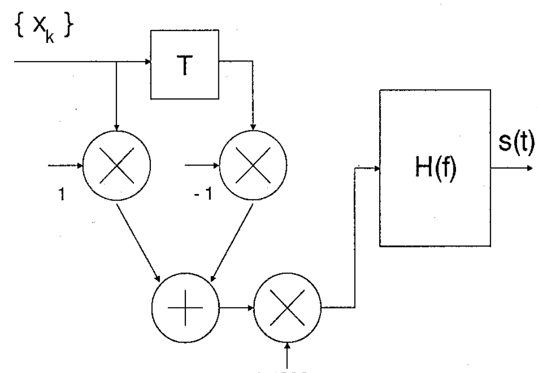


Figure 2: Implementation of the class of example filters with $\rho = 0.65$ using a transversal filter in cascade with $H(f)$.

## 5 Practical Receiver Structure

The optimum receiver structure for the signaling scheme discussed here can be implemented by the whitened matched filter (WMF) [3]. The end-to-end discrete impulse response $g(D)$ of the system is causal and the samples of additive noise sequence $n(D)$ at the output are Gaussian and uncorrelated. The system is then described by the discrete model as depicted in Figure 1(b):

$$r(D) = x(D)g(D) + n(D) \tag{16}$$

Theoretically a Viterbi decoder can be used to process the WMF output $r(D)$ for the purpose of ML detection. In practice, two problems have to be taken into account. First, the bandlimited filters considered in this paper will have infinite impulse response such that memory truncation is necessary for receiver implementation. A possible approach for memory truncation is to truncate all $R_i, |i| \ge \nu$ if $|R_i| < \epsilon$, $\forall\, |i| \ge \nu$, where $\epsilon$ can be set to a very small value, say $10^{-4}$. Take the class of filters with $\rho = 0.65$ presented in Section 4 as an example. Refering to Figure 2, let $f(D)$ be the discrete transfer function model of the initial choice of $H(f)$, then the overall discrete transfer function of the system is $g(D) = 1.4302(1 - D)f(D)$. Following the above memory truncation approach we obtain a discrete transfer function $f(D)$ of length 21 for the initially chosen filter, whose coefficients are shown in Table 2(a). The overall transfer function obtained is then $g(D) = 1.4302(1 - D)f(D)$ whose coefficients are given in Table 2(b).

In the second problem, after the truncation is performed, the memory length $\nu$ of the system can still be very long such that even the parallel implementation of Viterbi algorithm [11] is inadequate for a practical receiver, since the complexity of a Viterbi decoder grows exponentially with the channel memory length. Alternative trellis search algorithms are thus necessary.

Table 1: The class of example filters, (a) autocorrelation coefficients, (b) $d^2_{min}$ and the corresponding error sequences $p(D)$.

| $\rho$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $\cdots$ |
|---|---|---|---|---|---|---|
| 1.00 | 0.5000 | 0 | 0 | 0 | 0 | $\cdots$ |
| 0.95 | 0.5370 | 0.0198 | 0.0071 | 0.0037 | −0.0022 | $\cdots$ |
| 0.90 | 0.5752 | 0.0464 | −0.0152 | 0.0070 | −0.0037 | $\cdots$ |
| 0.85 | 0.6127 | 0.0801 | −0.0224 | 0.0084 | −0.0031 | $\cdots$ |
| 0.80 | 0.6467 | 0.1235 | −0.0277 | 0.0063 | 0.0012 | $\cdots$ |
| 0.75 | 0.6860 | 0.1698 | −0.0246 | 0.0000 | 0.0046 | $\cdots$ |
| 0.70 | 0.7213 | 0.2252 | −0.0137 | −0.0098 | 0.0081 | $\cdots$ |
| 0.65 | 0.7556 | 0.2871 | 0.0091 | −0.0202 | 0.0072 | $\cdots$ |

(a)

| $\rho$ | $d^2_{min}$ | $p(D)$ |
|---|---|---|
| 1.00 | 1 | $\pm(1-D+\cdots)$ |
| 0.95 | 0.8274 | $\pm(1-D+\cdots-D^{13}+D^{14})$ |
| 0.90 | 0.7435 | $\pm(1-D+\cdots-D^5+D^6)$ |
| 0.85 | 0.6861 | $\pm(1-D+D^2-D^3+D^4)$ |
| 0.80 | 0.6402 | $\pm(1-D+D^2-D^3)$ |
| 0.75 | 0.5957 | $\pm(1-D+D^2)$ |
| 0.70 | 0.5573 | $\pm(1-D)$ |
| 0.65 | 0.4889 | $\pm(1-D)$ |

(b)

Table 2: Coefficients for the discrete model of the class of example filters with $\rho = 0.65$: (a) the originally chosen filter, (b) the final filter.

| | | | | | |
|---|---|---|---|---|---|
| $f_0$ | 1.5217E − 1 | $f_8$ | −3.1409E − 2 | $f_{16}$ | 5.7681E − 3 |
| $f_1$ | 5.3170E − 1 | $f_9$ | 4.7458E − 3 | $f_{17}$ | −4.0297E − 3 |
| $f_2$ | 7.2402E − 1 | $f_{10}$ | 1.5472E − 2 | $f_{18}$ | 1.5158E − 4 |
| $f_3$ | 3.9518E − 1 | $f_{11}$ | −1.2690E − 2 | $f_{19}$ | 2.0035E − 3 |
| $f_4$ | −4.1042E − 2 | $f_{12}$ | −1.4100E − 3 | $f_{20}$ | −1.7199E − 3 |
| $f_5$ | −9.0729E − 2 | $f_{13}$ | 9.4345E − 3 | $f_{21}$ | 5.9172E − 4 |
| $f_6$ | 4.1723E − 2 | $f_{14}$ | −6.1055E − 3 | | |
| $f_7$ | 2.2500E − 2 | $f_{15}$ | −1.7356E − 3 | | |

(a)

| | | | | | |
|---|---|---|---|---|---|
| $g_0$ | 2.1763E − 1 | $g_8$ | −7.7099E − 2 | $g_{16}$ | 1.0732E − 2 |
| $g_1$ | 5.4279E − 1 | $g_9$ | 5.1708E − 2 | $g_{17}$ | −1.4013E − 2 |
| $g_2$ | 2.7506E − 1 | $g_{10}$ | 1.5341E − 2 | $g_{18}$ | 5.9800E − 3 |
| $g_3$ | −4.7030E − 1 | $g_{11}$ | −4.0278E − 2 | $g_{19}$ | 2.6485E − 3 |
| $g_4$ | −6.2388E − 1 | $g_{12}$ | 1.6133E − 2 | $g_{20}$ | −5.3250E − 3 |
| $g_5$ | −7.1061E − 2 | $g_{13}$ | 1.5510E − 2 | $g_{21}$ | 3.3060E − 3 |
| $g_6$ | 1.8943E − 1 | $g_{14}$ | −2.2225E − 2 | $g_{22}$ | −8.4627E − 4 |
| $g_7$ | −2.7491E − 2 | $g_{15}$ | 6.2498E − 3 | | |

(b)

There are many other trellis search algorithms that can be used, in which the category of breath-first trellis-search algorithm features a stable computational load, inherent parallelism, synchronous symbol release rule, and attractive performance [12] such that it is widely used in both source coding and channel coding. The Viterbi algorithm belongs to this category, but requires too much computation. Another attractive algorithm of this category is the $(M,L)$ algorithm, which is originally used in source coding, but is gaining more attention for use in channel coding and sequence detection.

The $(M,L)$ algorithm reduced the computation required dramatically as compared to the Viterbi algorithm, but it also tends to reject the best path when the noise is severe because of its fixed number of $M$ survivors. Computer simulation show that this fixed number of survivors is the major shortcoming of the $(M,L)$ algorithm when used in sequence detection or channel coding. One remedy is to allow $M$ to vary adaptively according to the channel condition. Computer simulation show that the adaptive approach can eliminate the shortcoming of the $(M,L)$ algorithm as well as other breath-first trellis search algorithms [13, 14].

## 6 Information Theory Aspect

In this section we will discuss the significance of this signaling scheme from information theory point of view. In the simplest performance analysis of PAM and QAM, one usually model the system as a discrete-time memoryless channel with additive white Gaussian noise. The performance limit of this channel is given by its channel capacity $C$ in bits,

$$C = \frac{1}{2}\log_2(1 + \mathrm{SNR_d}) \qquad (17)$$

where $\mathrm{SNR_d}$ is the discrete-time signal-to-noise ratio at the channel output. At symbol error rate $10^{-5}$ memoryless PAM and QAM fall below this capacity by about 9 dB. This 9 dB gap can be further narrowed down by coded modulation [15]. Note that this discrete-time memoryless model for PAM and QAM did not take into account the spectral behavior of the signaling scheme, which is the major concern of the discussions here in this paper. A more natural capacity formula for the discussions here is then the channel capacity for bandlimited waveform channel [16] in bits per second,

$$C = W\log_2(1 + \frac{S}{WN_0}) \qquad (18)$$

where $W$ is the signal bandwidth, $S$ is the signal power at the channel output and $N_0/2$ is the two-sided PSD of the white Gaussian noise. This capacity formula is valid whether the channel has memory or not. Define $C_s = CT$ as the channel capacity per symbol duration where $T$ is the symbol duration, $S_s = ST$ as the average energy per symbol, and let $\rho = 2WT$ be the ratio of $W$ to the Nyquist bandwidth $1/2T$, Equation (18) can be rewritten as

$$C_s = \frac{\rho}{2}\log_2(1 + \frac{\mathrm{SNR_d}}{\rho}) \qquad (19)$$

where $\mathrm{SNR_d} = S_s/(N_0/2)$ for PAM and $S_s/N_0$ for QAM, is the discrete-time equivalent signal to noise ratio per symol. For $\rho = 1$, Equation (19) is identical to Equation (17). This is the case for ideal unrealizable memoryless PAM and QAM where unrealizable ideal rectangular filter with exactly the Nyquist bandwidth is assumed. However, for practically realizable memoryless signaling scheme using realizable filter such as the raised-cosine family, the signal bandwidth is wider than the Nyquist bandwidth and $\rho > 1$ such that Equation (19) is larger than Equation (17). This means that the conventional calculation of channel capacity based on Equation (17) is somewhat too conservative if the entire signal bandwidth is taken into account. The channel capacity given by Equation (19) for $\rho = 1.00$, 1.15 and 2.00 are plotted in Figure 3. Also plotted in Figure 3 are the performance of memoryless 2, 4, 8 and 16 PAM at symbol error

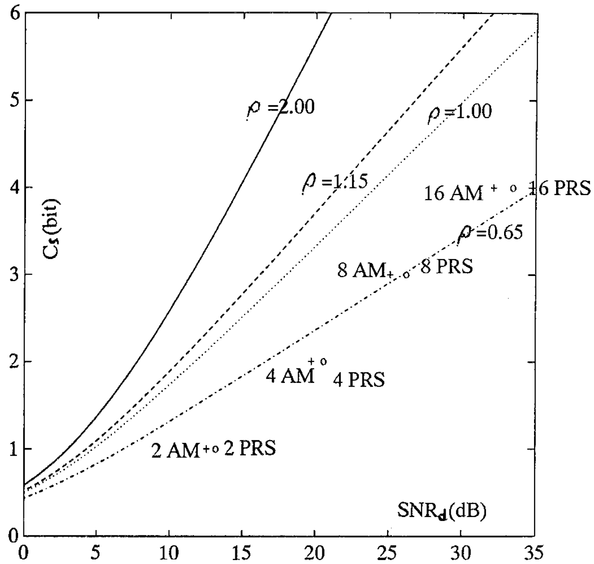


Figure 3: Capacity of waveform channels: $\rho = 2.00, 1.15$ and 1.00 with memoryless PAM, and $\rho = 1.00$ with $1 + D$ PRS, and $\rho = 0.65$ .

rate $10^{-5}$, which fall below the capacity for $1.15 < \rho < 2.00$ by more than 9 dB, instead of the conventional belief of only 9 dB.

There are also realizable digital signaling schemes that occupy exactly the Nyquist bandwidth, e.g., the partial response signalings with a null at the Nyquist frequency such as the $1+D$. Some of them can actually achieve the same asymptotic performance as memoryless PAM or QAM, yet at practical SNR a modest penalty is usually inevitable. The calculated capacity will be identical to either Equation (17) or Equation (19) with $\rho = 1.00$, and is plotted in Figure 3. Also plotted in Figure 3 are the performance of 2, 4, 8 and 16 $1 + D$ partial response schemes (the "o" mark) at symbol error rate $10^{-5}$, which also fall below the capacity curve by more than 9 dB.

To summarize the above discussions of digital transmission at and above the Nyquist bandwidth, we note that the channel capacity is a function of the bandwidth, and the practical uncoded transmission schemes are unable to attain it by a gap of more than 9 dB. This gap can be further narrowed down by coded modulation. We may wonder if the situation is similar for the case of signaling below the Nyquist bandwidth, even though the available channel capacity is now less than that of discrete-time memoryless channel, as can be found by evaluating Equation (19) with $\rho < 1.00$. That is, here we are looking for signaling schemes that occupy less than the Nyquist bandwidth, and at the same time attain the same asymptotic performance as memoryless PAM and QAM and $1+D$ PRS. If such signaling schemes exist, they are expected to be below the channel capacity by approximately 9 dB, or probably slightly more than 9 dB as the above analysis indicates. Also coded modulation should be useful to narrow down this gap. The answer is certain at least for digital transmission with a bandwidth down to 65 percent of the Nyquist limit. It has already been shown in the previous section that the signaling scheme proposed in this paper can occupy less than the Nyquist bandwidth, and at the same time attain the same asymptotic performance as memoryless PAM and QAM and $1 + D$ PRS. Furthermore, as this signaling scheme can be modeled by an IIR transfer function or approximated by an FIR transfer function, a coded modulation scheme known as Vector coding [17] can be used to obtain significant coding gain essentially without any bandwidth expansion. We plot the channel capacity of the bandlimited Gaussian channel with $\rho = 0.65$ in Figure 3 for comparison.

## 7 Conclusion

Realizable digital signaling schemes that achieve the best asymptotic performance and occupy less than the Nyquist bandwidth has been demonstrated. This indicates that the Nyquist bandwidth indeed is not the ultimate limit of efficient bandwidth utilization if advanced transmission and detection techniques are employed.

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