Robust Synthesis in ℓ_1 via μ Approach and *D-K* iteration

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ABSTRACT -

In this paper, we propose the linear fractional transformation (*LFT*) method to synthesize controllers that achieve globally robust stability, robust performance, and satisfactory worst-case performance, within any prescribed tolerance, against structured norm-bounded time-varying and/or nonlinear uncertainty, respectively. The uncertainty is characterized by any prescribed tolerance, against structured norm-bounded time-varying and/or nonlinear behaviors. The output feedback synthesis problem when the signal norm is proposed to get the ℓ_∞ norm and the perturbations are structured time-varying and/or nonlinear systems with an induced ∞-norm bound. A global optimal solution to the robust synthesis problem is obtained. A synthesizing ℓ_1 controller for a simplified space-shuttle model is taken from the μ-Analysis and *D-K* Iteration method to compare each other.

1. Introduction

Accurate system models are rarely available for the design of effective controllers. Therefore, one must develop robust controllers that is insensitive to various sources of uncertainties. Study of robust system design has begun since early 1980s. In [1], [12], the performance preserving controller reduction method is discussed. However, the bound of performance degradation due to controller reduction has yet not been developed. The coprime factor controller reduction problem was investigated in [4]. The problem of synthesis of controllers that minimize the structured singular value (SSV) function remains largely unsolved. On the other hand, controllers designed using *D-K* iterations method [5]-[8] may achieve locally optimal solutions. When perturbations are characterized by an induced ℓ_∞ norm bound, a *D-K* type iteration procedure is first available [9]. However, as in the case of the SSV, this iteration scheme does not guarantee that a global minimum is achieved. It appears to be an inherently non-convex problem in either norm, and is difficult to solve in general [10]. It is shown that the solution involves only resulting from relaxation of the original non-convex problem [10].

A globally optimal solution to the robust synthesis in ℓ_1 is treated in [6]-[8], where an algorithm is proposed to find an approximation of the globally optimal solution to the constantly scaled ℓ_1 norm control problem. The controllers achieve globally optimal robust performance, against structured norm-bounded time-varying and/or nonlinear uncertainty was developed. However, the controller with satisfactorily optimal worst-case perturbation is not considered. Thus, the problem of synthesis of controllers achieving globally robust stability, optimal robust performance, and satisfactorily worst-case performance is proposed to solve the constantly scaled ℓ_1 synthesis problem via μ and the modified D-K iteration method.

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2. Preliminaries

Problem Formulation and μ Approach

In this paper, we review the robust stability, robust performance, and worst-case performance synthesis problems, respectively. A controller *K* is sought that achieves robust stability, robust performance, and satisfactory worst-case performance in the presence uncertainty Δ. Then, we will restrict the study to Linear Fractional Transformation (*LFT*) description of control problem from the domain of ℓ_∞ norm to characterize allowable perturbations and exogenous disturbances. Model uncertainty is reflected by the uncertainty block Δ which is restricted to lie in the following set of admissible perturbations:

$$\underline{\Delta} := \left\{ \Delta = diag(\Delta_1, \cdots, \Delta_n) : \Delta_i : \ell_\infty \to \ell_\infty \text{ is causal and } \|\Delta_i\| := \sup_{u \neq 0} \frac{\|\Delta_i u\|_\infty}{\|u\|_\infty} \leq 1 \right\}.$$

The perturbation may be therefore nonlinear or time-varying.

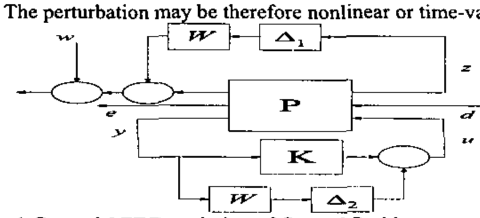

Fig.1 General *LFT* Description of Control Problem.

The system in Fig. 1 is said to be robustly stable if it is ℓ_∞-stable for all admissible perturbations, i.e. for all $\Delta \in \underline{\Delta}$. ℓ_1 is the Banach space of right sided absolutely summable real sequences with the norm given by

$$\|x\|_1 := \sum_{k=0}^{\infty} |x(k)|, \text{ or, } \|\Phi\|_1 = |D_{ij}| + \sum_{k=0}^{\infty} |C_i A^k B_j|.$$

Problem Statement: Find a linear finite-dimensional controller *K* such that:

1) the system achieves robust stability,
2) the system achieves robust performance, and
3) using μ to analyze worst-case performance.

The SSV (μ) approach can be used to solve above problem ([6]-[8]). It has three major functions: 1) to analyze the robustness of the system to the structured uncertainty that enters in the feedback path; 2) to analyze robust performance; 3) to analyze worst-case

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performance for whole system. The following robustness theorems will be the basis for the proposed synthesis method.

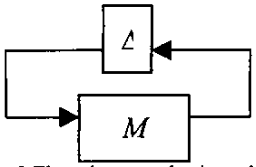

Fig.2 The robust synthesis problem.

Theorem 1 ([6]-[8]): The system in Fig.2 achieves robust stability if and only if (*iff*) the following condition holds: $\inf_{D\in \underline{D}} \left\| D^{-1}\Phi D \right\|_1 < 1$ where $D := \{diag(d_1,\cdots,d_n) : d_i > 0\}$.

The dependence of Φ on the controller K can be captured through the Youla parameter Q which provides a parameterization of all possible Φ's that can be obtained with a stabilizing controller. Hence, the robust synthesis problem can be stated as follows:

$$\inf_{D\in \underline{D}} \inf_{Q\in \ell_1} \left\| D^{-1}\Phi(Q)D \right\|_1 := \gamma^*, \qquad (1)$$

where $\Phi(Q) = H - U*Q*V$. For each fixed $D = \mathrm{diag}(d_1,\ldots d_n)$, this problem is a standard ℓ_1 norm-minimization problem. By introducing a regularizing bound Q of the form: $\|Q\|_1 \le \alpha$, we define

$$\gamma_{opt} := \inf_{\substack{subject\ to\\ \|Q\|_1 \le \alpha \\ \Phi = H-U*Q*V \\ D\in D_{feas}}} \left\| D^{-1}\Phi_\varepsilon D \right\|_1 . \qquad (2)$$

Clearly, if the optimization problem (1) has a solution Q_{opt}, then for any $\alpha \ge \|Q_{opt}\|_1$ the two problems (1) and (2) are equivalent. The objective function as well as the constraints is linear. A solution based on Finitely Many Variable (*FMV*) and Finitely Many Equations (*FME*) methods can be obtained ([6]-[8]).

1) **Robust stability**

- Let β_u and β_l denote the peak of the upper/lower bounds of μ.

  a) For all perturbation matrices $\Delta \in \underline{\Delta}$ satisfying

  $$\max_{\omega} \bar{\sigma}[\Delta(j\omega)] < \frac{1}{\beta_u}, \qquad (3a)$$

  the perturbed system is stable.

  b) There is particular perturbation matrix

  $$\Delta \in \underline{\Delta} \text{ satisfying } \max_{\omega} \bar{\sigma}[\Delta(j\omega)] = \frac{1}{\beta_l}, \qquad (3b)$$

  that causes instability. Hence, the gap between upper and lower bounds translates into gaps between the conclusions guaranteed robust stability and not robustly stable.

2) **Robust Performance**

- Specifically, we assume that good performance is equivalent to

$$\left\| T_{zw}(K,\Delta) \right\|_\infty < 1 \quad \forall \Delta \in \underline{\Delta};\ \text{or}$$
$$\left\| T_{zw} \right\|_\infty := \max_{\omega\in\Re} \bar{\sigma}(T_{zw}(j\omega)) \le 1, \qquad (4)$$

where T_{zw} is the map from w to z, some weighted, closed-loop transfer function, matrix and the norm used above is the induced ℓ_∞ operator norm.

3) **Worst-case Perturbation**

- If a system has robust performance to uncertainty, it is useful to get the worst-case perturbation of a given size. For instance, using perturbation of a particular structure Δ, and restricting to those of size $\le \alpha$, what is the worst performance (as measured in $\|\cdot\|_1$ norm) and what is the worst performance that causes the largest degradation of performance?

The above problems have been solved from the *LTF* ([5] - [8]) and therefore, are summarized as below.

i) Well Posedness and Performance for constant *LFTs*

Let M be a complex matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and suppose there are two defined block structures Δ_1 and Δ_2, which are compatible in size with M_{11} and M_{22} respectively. Define a third structure Δ as

$$\Delta = \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \underline{\Delta}_1 ; \Delta_2 \in \underline{\Delta}_2 \right\} \qquad (5)$$

For $\Delta_2 \in \underline{\Delta}_2$, consider the following loop equations,

$$\begin{aligned} e &= M_{11}d + M_{12}w \\ z &= M_{21}d + M_{22}w \\ w &= \Delta_2 z. \end{aligned} \qquad (6)$$

The set of equations is called *well posedness* if for any vector d, there exists unique vectors w, z and e satisfying the loop equations. When the inverse of $I-M_{22}\Delta_2$ exists, the vectors e and d must satisfy $e = F_L(M,\Delta_2)d$, where

$$F_L(M,\Delta_2) = M_{11} + M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}. \quad (7)$$

$F_L(M, \Delta_2)$ is a linear fractional transformation on M by Δ_2, and in feedback diagram.

ii) Main Loop Theorems

Theorem 2: ([2], Them[4.2]) The *LTF* $F_L(M, \Delta_2)$ is well posed for all $\Delta_2 \in B_2$, where

$$B_i = \{\Delta_i \in \underline{\Delta}_i : \bar{\sigma}(\Delta_i) \le 1\}, \ \textit{iff}\ \mu_2(M_{22}) < 1. \quad (8)$$

3. Control Design via μ Synthesis and Redesign Controller Using *D-K* Iteration

In order to apply the general structure singular value theory to control system design, the control problem has been recast into the *LFT* setting as in Fig. 3, redrawn from Fig. 1. The set of system to be controlled is described by the *LFT*

$$\{F_U(P,\Delta_{pert}) : \Delta_{pert} \in \Delta_{pert} \max_{\omega} \bar{\sigma}[\Delta_{pert}(jw)] \le 1\}. \ (9)$$

The goal of μ synthesis is to minimize over all stabilizing controller K, the peak value of $\mu_\Delta(\cdot)$ of the closed-loop transfer function $F_L(P, K)$. More formally,

$$\min_{\substack{K\\ stabilizing}} \max_{w} \mu_\Delta(F_L(P,K)(jw)). \qquad (10)$$

For tractability of the μ synthesis problem of above section, it is necessary to replace $\mu_\Delta(\cdot)$ with the upper bound. We saw that for a constant matrix M and an uncertainty structure Δ, an upper bound for $\mu_\Delta(M)$ is an optimally scaled maximum singular value,

$$\mu_\Delta(M) \le \inf_{D\in D_\Delta} \bar{\sigma}(DMD^{1}). \tag{11}$$

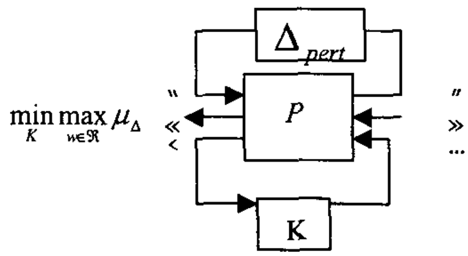

Fig3. μ Synthesis for system with *LFT* Description.

Using the upper bound, the optimization in (10) is reformulated as

$$\min_{\substack{K \\ \text{stabilizing}}} \max_{w} \min_{D_w \in D_\Delta} \bar{\sigma}[D_w F_L(P,K)(jw)D_w^{-1}]. \tag{12}$$

Remember, the D minimization is simply an approximation to $\mu[F_L(P,K)(jw)]$. D_w is chosen from the set of scalings D_Δ, independently at every w. Hence, we have

$$\min_{\substack{K \\ \text{stabilizing}}} \min_{D, D_w \in D_\Delta} \max_{w} \bar{\sigma}[D_w F_L(P,K)(jw)D_w^{-1}].$$

By D, $D_w \in D_\Delta$, we mean a frequency dependent function D that satisfies $D_w \in D_\Delta$ for each w. Suppose U is a complex matrix with the same structure as D, but satisfying $U^*U = UU^* = I$. Then,

$$\bar{\sigma}[(UD)(M(UD)^{-1}] = \bar{\sigma}[UDMD^{-1}U^*] = \bar{\sigma}(DMD^{-1})$$

So, replacing D by UD does not affect the upper bound. Thus, the new optimization is

$$\min_{\substack{K \\ \text{stabilizing}}} \min_{\substack{\hat{D}(s)\in D_\Delta \\ \text{stable min. phase}}} \left\| \hat{D} F_L(P,K) \hat{D}^{-1} \right\|_1. \tag{13}$$

This optimization is currently solved by an iterative approach, referred to as the modified *D-K* iteration. A block diagram-depicting scheme is shown in the following.

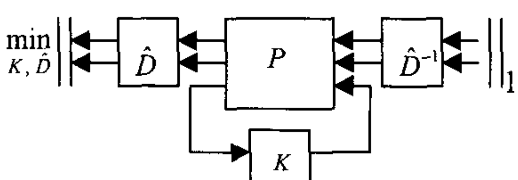

Fig. 4 New Optimization Scheme for μ synthesis.

The scaling set D_Δ is easily seen to be

$$D_\Delta = \{diag[d_1 I, d_2 I, \cdots, d_N I] : d_i > 0\}. \tag{14}$$

The element of D_Δ, which is defined in Eqn. (14) to be real and positive, can be allowed to take any nonzero complex values and not change the value of upper bound, $\inf_{D\in D_\Delta} \bar{\sigma}(DMD^{-1})$. Using this freedom in the phase of each entry of D, we can restrict the frequency dependent scaling matrix D_w of Eqn. (14) to be a real-rational, stable, minimum-phase transfer function, $d(s)$. Now, the optimization is shown in equation (15).

The performance of the closed loop system will be evaluated using the output sensitivity transfer function $(I+GK)^{-1}$.

$$\min_{K,\hat{d}} \left\| \begin{bmatrix} \hat{d}_1 I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \hat{d}_N I & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} F_L(P,K) \begin{bmatrix} \hat{d}_1 I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \hat{d}_N I & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}^{-1} \right\|_1. \tag{15}$$

Good performance will be characterized in terms of a weighted ℓ_1 norm on this transfer function. Given a 2x2 stable, rational transfer matrix W_p. We say that:

1) Nominal performance. It is achieved if the performance objective is satisfied for the nominal plant model, G_{nom}.

$$\text{Nominal Performance} \Leftrightarrow \left\| W_P (I + G_{nom} K)^{-1} \right\|_1 < 1. \tag{16}$$

2) Robust Stability. The closed-loop system achieves robust stability if the closed loop system is internally stable for all of possible plant models $G \in \mathcal{G}$.

$$\text{Robust Stability} \Leftrightarrow \left\| W_{del} K G_{nom} (I + K G_{nom})^{-1} \right\|_1 < 1 \tag{17}$$

3) Robust Performance. The closed-loop system achieves robust performance if the closed-loop system is internally stable for all $G \in \mathcal{G}$, and, in addition to that, the performance objective,

$$\text{Robust Performance} \Leftrightarrow \left\| W_P (I + GK)^{-1} \right\|_1 < 1 \tag{18}$$

4) Worst-Case Performance. The worst-case performance can be stated as following. More precisely, given $\alpha > 0$, the worst-case performance associated with structured perturbations of size a is defined as

$$W(M,\Delta,a) := \max_{\substack{\Delta\in\Delta \\ \max_\omega \bar{\sigma}[\Delta(j\omega)] \le \alpha}} \left\| F_U(M,\Delta) \right\|. \tag{19}$$

4. Simulation Results

Design Example: Consider a simplified model of space shuttle

[12]. The objective is to design (if possible) a controller that satisfies the design objectives of (16)-(18), and the worst-case norm (19) of the mapping from *w* to *z* less than one, respectively. The analysis procedure involves several steps: 1) Build uncertain model of plant, 2) Define performance specifications and uncertainty bounds; 3) Construct open-loop interconnection; 4) Close feedback loop with controller; 5) Perform a variety of real and complex μ and ℓ_1 analysis tests on the closed-loop system, and explore the impact of uncertain model (real vs. complex) on the robust stability and robust performance requirements; 6) Construct worst-case perturbations, and see their effect on the closed-loop system in the frequency and time domain. The state-space aircraft and actuator models for acnom, actuator, and weightings are created by the M-file mk_acnom, mk_act, mk_wts, and mk_olic; respectively ([11], [12]). The three (*k_h*, *k_mu*, and *k_x*) controllers, are designed to optimize the norms of H_∞ and ℓ_1, to perform modified *D-K* iteration, and to tradeoff between two controllers, respectively.

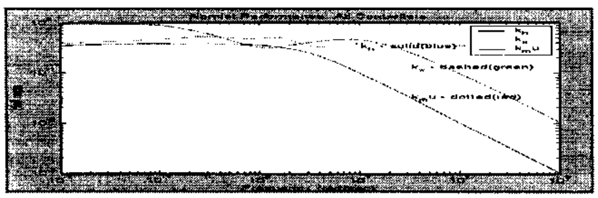

Fig. 5. Nominal Performance of all controllers

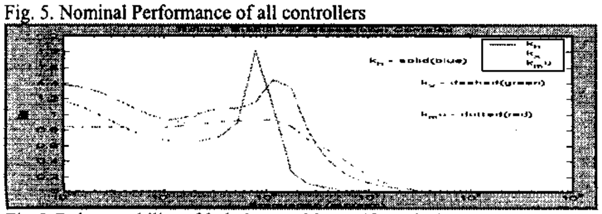

Fig.6. Robust stability of *k_h*, *k_x*, and *k_mu* (Complex).

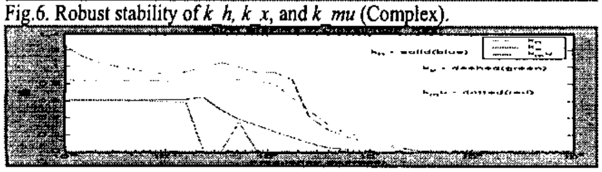

Fig. 7. Robust stability of *k_h*, *k_x*, and *k_mu* (Real).

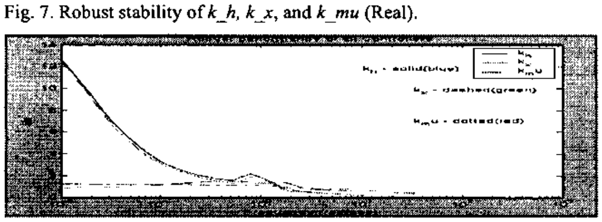

Fig. 8. Robust Performance m plots of *k_h*, *k_x*, and *k_mu*.

5. Conclusion

In this paper we propose a method to synthesize controllers that achieve globally optimal robust stability, robust performance, and satisfactory worst-case performance within any prescribed tolerance, against structured norm-bounded time-varying and/or nonlinear uncertainty, respectively.

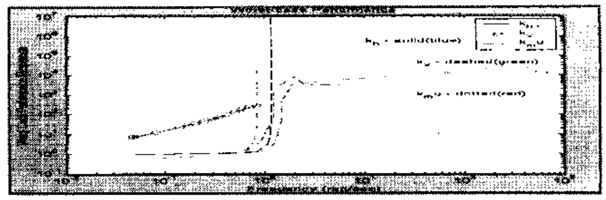

Fig. 9. Performance Degradation of closed-loop.

The output feedback synthesis problem when the signal norm is proposed to get the ℓ_∞ norm and the perturbations are structured time-varying and/or nonlinear systems with an induced ∞-norm bound. A synthesizing ℓ_1 controller for a simplified space-shuttle model is taken from the μ-Analysis and *D-K* Iteration method.

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