CLOSED-FORM DESIGN OF GENERALIZED MAXIMALLY FLAT LOW-PASS FIR FILTERS USING GENERATING FUNCTIONS

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ABSTRACT

In this paper, we propose a closed-form design of the generalized maximally flat low-pass FIR filters. By representing the transfer function as the Bernstein polynomial form, a simple generating function of the weighting coefficients is derived. We show that the linear phase maximally flat FIR filters and the nonlinear phase maximally flat half-band FIR filters are both special cases of the proposed filter. Closed-form expressions of both linear and nonlinear phase maximally flat FIR filters with prescribed degrees of flatness and delay can be easily obtained by the proposed method.

1. INTRODUCTION

The problem of designing and implementing the maximally flat low-pass FIR filters has been studied for a long time. In [1], a closed-form expression of the maximally flat linear phase low-pass FIR filter was proposed. In [2], another closed-form representation of the filter was derived in another domain rather the z-domain based on a conformal mapping. The two closed-forms in [1] and [2] were identical although the structures of filters were different. In [3], the authors proposed a method of designing the maximally flat FIR filters with arbitrary cutoff frequency. The method was basically an improvement of the transformation in [2]. In [4], an efficient implementation for the maximally flat FIR filter was proposed using a property of the weighting coefficients of the filter structure in [1]. There are few and small coefficients required in this implementation. In [5], the authors showed that the maximally flat FIR filters in [1, 2] could be also derived using the Bernstein polynomial. This provides an insight to the design of filters.

A method of designing the nonlinear phase maximally flat low-pass FIR filters was studied in [7]. The degrees of flatness on the magnitude response and the group delay could be adjusted independently. Nonlinear equations were involved in this method and these equations were solved numerically using the Gröbner bases. A closed-form design of the nonlinear phase half-band maximally flat FIR filters was recently proposed in [6] using the Chebyshev polynomial. Since the half-band filter is a special case of the low-pass filter, this may be regarded as a generalization of the linear phase filter cases.

In this paper, we propose a very simple design of the generalized maximally flat low-pass FIR filters. By representing the transfer function as the Bernstein polynomial form, we derive simple generating function of the weighting coefficients. It is shown that the linear phase maximally flat FIR filters in [1] and the nonlinear phase maximally flat half-band FIR filters in [6] are both the special cases of proposed design. However, the closed-forms of both linear and nonlinear phase maximally flat FIR filter with prescribed degrees of flatness and delay can be obtained by the proposed method.

2. PROBLEM FORMULATION

An Nth order FIR filter is characterized by its transfer function

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n} \tag{1}$$

where $h(n)$ is the impulse response of the filter. The frequency response of the filter is expressed by

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n) e^{-jn\omega} \tag{2}$$

which is a complex-valued function. The filter design problem is to find the impulse response such that the frequency response $H(e^{j\omega})$ is approximated to a desired frequency response $H_d(\omega)$. Since we want to design a low-pass filter with arbitrary delay, the desired frequency response can be written as

$$H_d(\omega) = \begin{cases} e^{-j\tau\omega}, & \text{for } 0 \le \omega \le \omega_p, \\ 0, & \text{for } \omega_s \le \omega \le \pi \end{cases} \tag{3}$$

where τ, ω_p and ω_s denote the desired delay, the passband edge, and the stopband edge, respectively. The designed FIR filter is said to be maximally flat if

$$\left.\frac{d^u}{d\omega^u} H(e^{j\omega})\right|_{\omega=0} = \left.\frac{d^u}{d\omega^u} H_d(e^{j\omega})\right|_{\omega=0} \tag{4}$$

for $u = 0, 1, \ldots, P-1$, and,

$$\left.\frac{d^v}{d\omega^v} H(e^{j\omega})\right|_{\omega=\pi} = \left.\frac{d^v}{d\omega^v} H_d(e^{j\omega})\right|_{\omega=\pi} \tag{5}$$

for $v = 0, 1, \ldots, Q-1$; where P and Q represent the degrees of flatness at $\omega = 0$ and $\omega = \pi$, respectively. Since there are $N+1$ unknowns to be determined by the $P+Q$ equations in Eqs. (4) and (5), we obtain the following relation between P, Q and N

$$N = P + Q - 1. \tag{6}$$

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In order to solve the impulse response of the maximally flat low-pass filter, we substitute the frequency response $H(e^{j\omega})$ in Eq. (2) and the desired frequency response $H_d(e^{j\omega})$ in Eq. (3) for those in Eqs. (4) and (5). These equations can be simplified and expressed as

$$\sum_{n=0}^{N} h(n)n^u = \tau^u, \quad \text{for } u = 0, 1, \ldots, P-1, \tag{7}$$

and

$$\sum_{n=0}^{N} h(n)n^v(-1)^n = 0, \quad \text{for } v = 0, 1, \ldots, Q-1. \tag{8}$$

We can express the $P+Q$ equations of Eqs. (7) and (8) in matrix form as follow

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{9}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 3 & \cdots & N \\ 0 & 1 & 4 & 9 & \cdots & N^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^{P-1} & 3^{P-1} & \cdots & N^{P-1} \\ 1 & -1 & 1 & -1 & \cdots & (-1)^N \\ 0 & -1 & 2 & -3 & \cdots & N(-1)^N \\ 0 & -1 & 4 & -9 & \cdots & N^2(-1)^N \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 2^{Q-1} & -3^{Q-1} & \cdots & N^{Q-1}(-1)^N \end{bmatrix},$$

$$\mathbf{x} = [h(0), h(1), h(2), \ldots, h(N)]^t,$$

and

$$\mathbf{b} = [1, \tau, \tau^2, \ldots, \tau^{P-1}, \overbrace{0, 0, \ldots, 0}^{Q \text{ zeros}}]^t.$$

The superscript ^{t} denotes the matrix transpose.

Although the impulse response of the maximally flat FIR filter can be obtained by solving Eq. (9), there are some drawbacks to solve these equations directly. First, $\mathbf{A}$ is ill-conditioned when N is large. That is, the numerical solution may be not accurate if Eq. (9) is solved by numerical software. Secondly, even if the numerical solution is solved accurately, it cannot reveal explicitly the properties of the impulse response or the structure of the filter. We may obtain deeper properties or structure of the filter if we know the closed-form expression of impulse response. However, it is difficult to calculate the closed-form solution directly from Eqs. (7) and (8), or equivalent, Eq. (9). Therefore, to obtain the closed-form solution, we will design the filter by another approach rather than solving the equations directly.

3. CLOSED-FORM DESIGN BY GENERATING FUNCTION

In order to obtain the closed-form expression of the transfer function, we use the Bernstein form of the transfer function expressed by

$$H(z) = \sum_{m=0}^{N} c(m) \left(\frac{1-z^{-1}}{2}\right)^m \left(\frac{1+z^{-1}}{2}\right)^{N-m}. \tag{10}$$

Since $H(z)$ has Q zeros at $z=-1$, i.e., $\omega = \pi$, we can set

$$c(P) = c(P+1) = \cdots = c(N) = 0. \tag{11}$$

Accordingly, the transfer function is written by

$$H(z) = \sum_{m=0}^{P-1} c(m) \left(\frac{1-z^{-1}}{2}\right)^m \left(\frac{1+z^{-1}}{2}\right)^{N-m}. \tag{12}$$

The fundamental difference between Eqs. (1) and (12) is that the latter has desired degree of flatness at $\omega = \pi$ structurally. Now, the problem of designing the FIR low-pass filter is to find the coefficients $c(0)$, $c(1)$, ..., $c(P-1)$ such that the following approximation is achieved

$$H(z) \approx z^{-\tau} \tag{13}$$

where $z^{-\tau}$ is the desired transfer function in the passband.

To solve $c(0)$, $c(1)$, ..., $c(P-1)$, we make the following variable change

$$z^{-1} = \frac{1-t}{1+t}. \tag{14}$$

Substituting Eqs. (12) and (14) for Eq. (13), we obtain the following approximation

$$\sum_{m=0}^{P-1} c(m) \left(\frac{t}{1+t}\right)^m \left(\frac{1}{1+t}\right)^{N-m} \approx \left(\frac{1-t}{1+t}\right)^\tau$$

or

$$\sum_{m=0}^{P-1} c(m)t^m \approx (1-t)^\tau (1+t)^{N-\tau}. \tag{15}$$

Therefore, the coefficients $c(0), c(1), \ldots, c(P-1)$ is the first P coefficients in $(1-t)^\tau(1+t)^{N-\tau}$ if we expand it into power series. We may regard $(1-t)^\tau(1+t)^{N-\tau}$ as the generating function of the coefficients.

Property 1. *The coefficient generating function of the Bernstein-type transfer function of Nth order maximally flat low-pass FIR filter with desired delay τ is*

$$G(t) = (1-t)^\tau (1+t)^{N-\tau}. \tag{16}$$

If the desired delay τ is an integer, the generating function expressed by Eq. (16) has only finite terms, and the coefficients $c(0)$, $c(1)$, ..., $c(P-1)$ has the following simple closed-form expression

$$c(m) = \sum_{i=0}^{m} (-1)^i \binom{\tau}{i} \binom{N-\tau}{m-i} \tag{17}$$

for $m = 0, 1, \ldots, P-1$ where $\binom{k}{\ell}$ is the binomial coefficient. However, if the desired delay τ is not an integer, the power series expansion of the generating contains infinite terms. Based on the well-known power series

$$(1+t)^q = \sum_{k=0}^{\infty} \frac{(q-k+1)_k}{k!} t^k,$$

the coefficients $c(0)$, $c(1)$, ..., $c(P-1)$ could be expressed by

$$c(m) = \sum_{i=0}^{m}(-1)^i \frac{(\tau-i+1)_i}{i!}\frac{(N-\tau-m+i+1)_{m-i}}{(m-i)!} \tag{18}$$

for $m = 0, 1, \ldots, P-1$ where $(a)_k$ is defined by

$$(a)_k = a(a+1)\cdots(a+k-1)$$

for $k \geq 1$ and $(a)_0 = 1$.

It is interesting that the generating function expressed by Eq. (16) is independent from the degree of flatness P or Q. That is, for a fixed filter order N, the coefficients of the Bernstein-type transfer function are identical even if the degrees of flatness are different. What controls the degree of flatness is the structure of the filter and the number of the coefficient rather than the values of the coefficients.

4. RELATIONS TO SOME WELL-KNOWN MAXIMALLY FLAT FILTERS

In this section, we will investigate the relationship between the proposed low-pass FIR filter and two well-known maximally flat FIR filters. We will show that these two filters are special cases of the proposed filters.

4.1. RELATION TO LINEAR-PHASE FIR FILTERS

In [1], the author derived the closed-form expression of the maximally flat linear-phase low-pass filter. In [2], another closed-form expression of the filter was derived based on a conformal mapping which is identical to change the proposed variable. The filter derived in [2] was equivalent to the one in [1]. We will show that the linear-phase filter is a special case of the proposed filter.

For the linear-phase FIR filter, the desired delay is a half of the filter order. However, to obtain an integer delay, N has to be an even number. Substituting $N = 2\tau$ for Eq. (16), we obtain the generating function of the maximally flat linear phase FIR low-pass filter as follows

$$G_{lp}(t) = (1-t^2)^\tau \tag{19}$$

That is, the coefficients in the Bernstein-type transfer function are

$$c(m) = \begin{cases} (-1)^{m/2}\binom{\tau}{m/2}, & \text{for even } m \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

for $m = 0, 1, \ldots, P-1$. Substituting Eq. (20), $P-1 = 2\tau - 2K$ nd $N = 2\tau$ for Eq. (12), we obtain the following transfer function

$$H_{lp}(z) = \sum_{m'=0}^{\tau-K}(-1)^{m'}\binom{\tau}{m'}\left(\frac{1-z^{-1}}{2}\right)^{2m'}\left(\frac{1+z^{-1}}{2}\right)^{2\tau-2m'} \tag{21}$$

where $P-1 = 2\tau - 2K$, i.e., $Q = 2K$, is the degree of flatness at $\omega = \pi$ specified in [1, 2]. Since the transfer function $H_{lp}(z)$ in Eq. (21) is identical to the one given in [2], we conclude that the well-known maximally flat low-pass FIR filter is a special case of the proposed filter.

4.2. RELATION TO GENERALIZED HALF-BAND FIR FILTERS

In [6], an expression of the generalized half-band maximally flat FIR filters was derived based on relating the frequency response of the filter to the Chebyshev polynomial. An closed-form representation of the Bernstein-type transfer function was also obtained. We will show that the result in [6] is also a special case of the proposed filter. For the specifications in [6], we have

$$N = 2M, \quad \tau = M + d, \quad \text{and } Q = M + 1.$$

Substituting these specifications for Eq. (16), we obtain the following generating function for the maximally flat half-band FIR filters

$$G_{hb}(t) = (1-t)^{M+d}(1+t)^{M-d}. \tag{22}$$

In [6], the Bernstein form of the transfer function is defined by

$$H(z) = \sum_{i=0}^{N} b(i)\binom{N}{i}\left(\frac{1+z^{-1}}{2}\right)^i\left(\frac{1-z^{-1}}{2}\right)^{N-i}$$

where the coefficients $b(i)$ and our coefficients $c(m)$ is related by

$$\binom{N}{i}b(i) = c(N-i).$$

It is easy to show that

$$c(N-i) = (-1)^{M+d}c(i)$$

if $c(i)$ is the coefficients in Eq. (22). Therefore,

$$\binom{N}{i}b(i) = (-1)^{M+d}\sum_{j=0}^{i}(-1)^{i-j}\binom{M-d}{j}\binom{M+d}{i-j}.$$

which is identical to the coefficients given in [6]. Consequently, the generalized maximally flat half-band FIR filter is a special case of the proposed filter.

5. DESIGN EXAMPLES

In this section, there are several design examples using the proposed design method.

Example 1. In this example, the maximally flat 10th order half-band FIR filters are to be designed. The degrees of flatness are $P = 6$ and $Q = 5$ and the desired delay is $\tau = 5$. The generating function is $G_{hb}(t) = (1-t^2)^5$ and its first P coefficients are {1, 0, -5, 0, 10, 0}. Therefore, the Bernstein-type transfer function is $H(z) = \left(\frac{1+z^{-1}}{2}\right)^{10} - 5\left(\frac{1-z^{-1}}{2}\right)^2\left(\frac{1+z^{-1}}{2}\right)^8 + 10\left(\frac{1-z^{-1}}{2}\right)^4\left(\frac{1+z^{-1}}{2}\right)^6$. After expanding and combining the above function into a polynomial in z^{-1}, we obtain the impulse response of {3/512, -25/512, 75/256, 1/2, 75/256, -25/512, 3/512}.

Example 2. In this example, the maximally flat 20th order low-pass FIR filters are to be designed. The degrees of flatness are $P = 10$ and $Q = 11$, respectively. The desired delays are $\tau = 9, 9.5, 10, 10.5, 11$, respectively. Fig. 1(a) and (b) show the magnitude responses and their group delays, respectively. Note that two amplitude curves of $\tau = 9.5$ and $\tau = 10.5$ are exactly the same. The straight line for $\tau = 9.5$, 10 and 10.5 in Fig. 1(b) indicates that the associated frequency response is of linear phase. Fig. 1(c) shows the zero plots of the corresponding transfer function.

Example 3. We will design the maximally flat 20th order low-pass FIR filters. The degrees of flatness are $P = 7$ and $Q = 14$, respectively. The desired delays are $\tau = 9, 9.5, 10, 10.5, 11$, respectively. Fig. 2(a) shows the magnitude responses and Fig. 2(b) shows the their group delays. Note that two amplitude curves of $\tau = 9.5$ and $\tau = 10.5$ are the same. The straight line for $\tau = 10$ in Fig. 2(b) indicates that the associated frequency response is of linear phase. Fig. 2(c) shows the zero plots of the corresponding transfer function.

6. CONCLUSIONS

In this paper, we proposed a closed-form design of the generalized maximally flat low-pass FIR filters. A simple generating function of the weighting coefficients was derived if the transfer function was expressed by the Bernstein polynomial form. The generating function is a function of the filter order and the desired delay. If the desired delay is an integer, the generating function is a simple polynomial. This means that the coefficients can be evaluated easily. We showed that some maximally flat low-pass FIR filters are special cases of proposed filter. Closed-form expressions of both linear and nonlinear phase maximally flat FIR filters with prescribed degrees of flatness and delay are obtained by the proposed method. Several examples were provided to demonstrate the effectiveness for the proposed design.

7. REFERENCES

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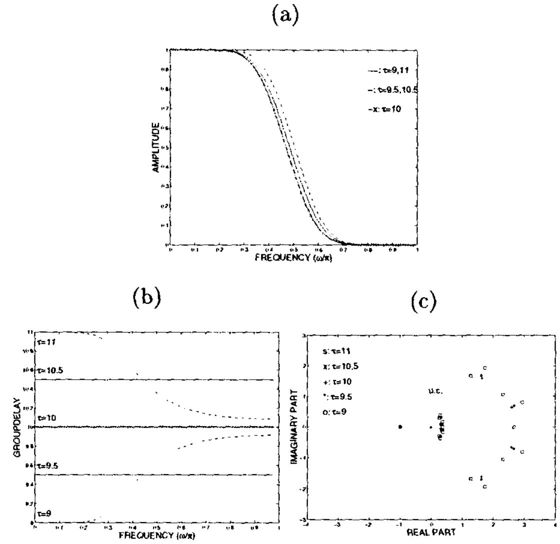

Figure 1: Design of the maximally flat 10th order low-pass FIR filters The degrees of flatness are $P = 10$ and $Q = 11$. The desired delays are $\tau = 9, 9.5, 10, 10.5, 11$, respectively. (a) Magnitude responses, (b) group delays and (c) zero plot. 'u.c.' stands for the unit circle.

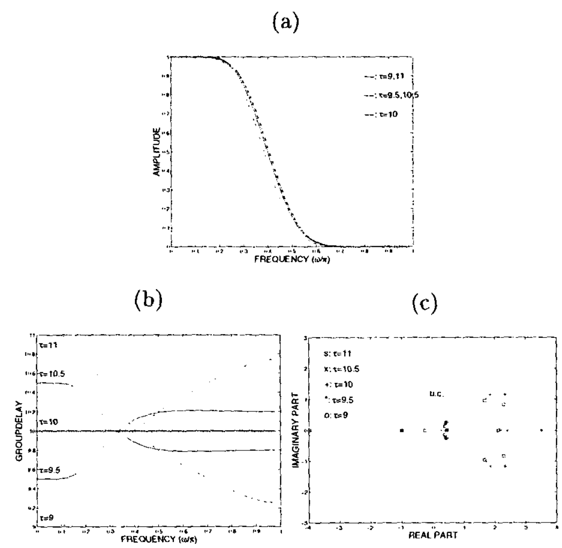

Figure 2: Design of the maximally flat 10th order low-pass FIR filters The degrees of flatness are $P = 7$ and $Q = 14$. The desired delays are $\tau = 9, 9.5, 10, 10.5, 11$, respectively. (a) Magnitude responses, (b) group delays and (c) zero plot.