Stability Analysis of Systems with Commensurate Time Delays[1]

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Abstract: This paper presents a computationally tractable method to check if a linear system with commensurate time delays is stable independent of delay. The method needs only finite computations, and is stronger than previous results which are only sufficient conditions. We also discuss the robust delay-independent stability of such systems when there are uncertain parameters.

Introduction

Recently, stability analysis of linear systems with commensurate time delays in the system dynamics has received considerable attentions [1-5]. This is because time delay is encountered in many control problems, especially the process control systems. Brierley *et al.* [1] first provides a necessary and sufficient condition for a linear system with commensurate time delays to check if the system is asymptotically stable independent of the delay time. Since the condition has to be checked for an interval of a parameter, several sufficient conditions [2-5] are derived to ensure the delay independent asymptotic stability. Although these conditions are easy to use by just calculating some matrix measures and norms of certain matrices, they are often conservative. In this paper, we give a method which involves only eigenvalue calculations of some matrices to verify the necessary and sufficient condition. If the system contains uncertain parameters in the system matrices, we also construct a hypersphere region in the parameter space to assure the robust delay independent asymptotic stability.

Main Results

Consider the following system with commensurate time delays

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t-ih),\ t \geq 0, \tag{1}$$

where $A_i \in \Re^{n\times n}$ is the ith system matrix, and h is the delay duration. Define

$$A(z) \equiv \sum_{i=0}^{m} A_i z^i. \tag{2}$$

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It is shown in [1] that (1) is asymptotically stable independent of delay if and only if $A(z)$ is a stable matrix, i.e., with all eigenvalues in the open left half plane, for all $|z| = 1$. To propose an algorithm that does not need to check the condition on the entire unit circle $|z| = 1$, we need the following

Theorem 1 *$A(z)$ is stable for all $|z| = 1$ if and only if*

(a) A_0 and $A(1)$ are stable matrices;

(b) for $\bar{A}_i \equiv A_i \otimes I_n + I_n \otimes A_i$, $i = 0, \cdots, m$,

$$\det\left[I_{n^2} + \sum_{i=1}^{m} z^i \bar{A}_0^{-1}\bar{A}_i\right] \neq 0,\ \forall\ |z| = 1, \tag{3}$$

where $\otimes$ stands for the Kronecker product.

Proof: Since $A(z)$ should be stable $\forall\ |z| = 1$, and the system should be stable for $h \to \infty$, condition (a) can be seen to be a necessary condition. By using the stability of $A(1)$ and the continuity of the eigenvalues of $A(z)$ with respect to z, it can be seen that $A(z)$ is stable $\forall\ |z| = 1$ if and only if $A(z)$ has no $j\omega$-axis eigenvalues $\forall\ |z| = 1$. The property of Kronecker sum reveals that it is equivalent to that $A(z) \otimes I_n + I_n \otimes A(z)$ is nonsingular $\forall\ |z| = 1$. This can be checked by

$$\det\left[\bar{A}_0 + \sum_{i=1}^{m} z^i \bar{A}_i\right] \neq 0,\ \forall\ |z| = 1. \tag{4}$$

Since A_0 is stable, $\bar{A}_0$ is nonsingular. Therefore (4) implies and is implied by condition (b) of the Theorem. Thus, $A(z)$ is stable $\forall\ |z| = 1$ if and only if the conditions of Theorem 1 hold.

By using a property of determinant [7, p.181], (3) can be rewritten as

$$\det[I_{mn^2} + zF_0] \neq 0,\ \forall\ |z| = 1, \tag{5}$$

where

$$F_0 = \begin{bmatrix} \bar{A}_0^{-1}\bar{A}_1 & \bar{A}_0^{-1}\bar{A}_2 & \cdots & \cdots & \bar{A}_0^{-1}\bar{A}_m \\ -I_{n^2} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & -I_{n^2} & 0 \end{bmatrix}. \tag{6}$$

It is easy to verify that condition (b) of Theorem 1 is equivalent to that the matrix F_0 has no eigenvalue with unity absolute value. Thus the problem of determining the eigenvalues of $A(z)$, $\forall\, |z| = 1$ is converted into that for two $n \times n$ and an $mn^2 \times mn^2$ real matrices. This result can also be easily extended to assure the to assure the delay independent asymptotic stability with a decay rate a. It is achieved by substituting A_0 with $A_0 + aI_n$ and reconsidering the problem using the above method.

Next, consider the case in which the system dynamics of the delay part contain structured uncertain parameters. Suppose the system is asymptotically stable independent of delay when there are no uncertain parameters, and the ith system matrix, $i = 1, \ldots, m$, has the form

$$A_i + \sum_{j=1}^{p} k_j E_{ij},$$

where k_j and E_{ij} are respectively the jth uncertain parameter and the ijth structural matrix. Then (5) and $A(1)$ become

$$\det[I_{mn^2} + zF] \neq 0, \ \forall\, |z| = 1, \tag{7}$$

$$A(1) \equiv A_a = A_{a0} + \sum_{j=1}^{m} k_j A_{aj}, \tag{8}$$

where

$$\begin{aligned}
F &= F_0 + \sum_{j=1}^{p} k_j F_j, \\
F_j &= \begin{bmatrix} \bar{A}_0^{-1}\bar{E}_{1j} & \bar{A}_0^{-1}\bar{E}_{2j} & \cdots & \cdots & \bar{A}_0^{-1}\bar{E}_{mj} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \\
\bar{E}_{ij} &= E_{ij} \otimes I_n + I_n \otimes E_{ij}, \\
A_{a0} &= \sum_{i=0}^{m} A_i, \\
A_{aj} &= \sum_{i=1}^{m} E_{ij}.
\end{aligned}$$

According to Theorem 1, the robust delay-independent asymptotic stability can then be obtained if and only if A_a is stable and F has no eigenvalues with unity absolute values in the presence of uncertain parameters. By using the continuity of eigenvalues with respect to uncertain parameters k_j's, the condition is equivalent to preserving the nonsingularity of the following two matrices for each possible uncertain parameter vector $k = [k_1 \cdots k_p]^{\mathsf{T}}$:

$$\tilde{F} = I_{m^2n^4} - F \otimes F, \tag{9}$$

$$\bar{A}_a = A_a \otimes I_n + I_n \otimes A_a. \tag{10}$$

To apply the above results, if we want to test whether a hyperrectangle region in the parameter space of k is a robust delay-independent asymptotic stability region, we may use the method of Tesi and Vicino [6], as the problem considered here has the same form as that in [6]. If we want to construct a robust delay-independent asymptotic stability region in the parameter space, we may note that

$$\begin{aligned}
\tilde{F} &= I - F_0 \otimes F_0 - \sum_{j=1}^{p} k_j (F_j \otimes F_0 + F_0 \otimes F_j) \\
&\quad - \sum_{j,l=1}^{p} k_j k_l F_j \otimes F_l \\
&= \tilde{F}_0 \{ I - \sum_{j=1}^{p} k_j \tilde{F}_j - \sum_{j,l=1}^{p} k_j k_l \tilde{F}_{jl} \},
\end{aligned} \tag{11}$$

$$\begin{aligned}
\bar{A}_a &= \bar{A}_{a0} + \sum_{j=1}^{p} k_j \bar{A}_{aj}, \\
&= \bar{A}_{a0} \{ I_{n^2} + (k \otimes I_{n^2}) \tilde{A}_{ae} \},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\tilde{F}_0 &\equiv I_{m^2n^4} - F_0 \otimes F_0, \\
\tilde{F}_j &= \tilde{F}_0^{-1}(F_j \otimes F_0 + F_0 \otimes F_j), \\
\tilde{F}_{jl} &= \frac{1}{2}\tilde{F}_0^{-1}(F_j \otimes F_l + F_l \otimes F_j), \\
\bar{A}_{a0} &= A_{a0} \otimes I_n + I_n \otimes A_{a0}, \\
\bar{A}_{aj} &= A_{aj} \otimes I_n + I_n \otimes A_{aj}, \\
\tilde{A}_{aj} &= \bar{A}_{a0}^{-1}\bar{A}_{aj}, \\
\tilde{A}_{ae} &= \begin{bmatrix} \tilde{A}_{a1} \\ \vdots \\ \tilde{A}_{ap} \end{bmatrix}.
\end{aligned}$$

From (11) and (12), we have the following theorem.

Theorem 2 *The robust delay-independent asymptotic stability of (1) can be guaranteed if*

$$k^{\mathsf{T}} k < \min\{ \|\tilde{A}_{ae}\|_s^{-2}, \tau^2 \}, \tag{13}$$

where $\tau^2 = \max\{\tau_1^2, \tau_2^2(\alpha, r)\}$, $\|\cdot\|_s$ denotes the maximal singular value of the argument matrix, α and r are positive numbers chosen to make $\tau_2(\alpha, r) > 0$, and

$$\begin{aligned}
\tau_1 &= \frac{-\|\tilde{F}_e\|_s + \sqrt{\|\tilde{F}_e\|_s^2 + 4\|\tilde{F}_{ee}\|_s}}{2\|\tilde{F}_{ee}\|_s}, \\
\tau_2(\alpha, r) &= \frac{\alpha^{2+r} + 1}{\|\Phi\|_s} - \alpha^2, \\
\tilde{F}_e &= \begin{bmatrix} \tilde{F}_1 & \cdots & \tilde{F}_p \end{bmatrix}, \\
\tilde{F}_{ee} &= \begin{bmatrix} \tilde{F}_{11} & \cdots & \tilde{F}_{1p} \\ \vdots & \ddots & \vdots \\ \tilde{F}_{p1} & \cdots & \tilde{F}_{pp} \end{bmatrix}, \\
\Phi &\equiv \begin{bmatrix} \tilde{F}_{11} & \cdots & \tilde{F}_{1p} & \frac{\tilde{F}_1}{2\alpha} \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{F}_{p1} & \cdots & \tilde{F}_{pp} & \frac{\tilde{F}_p}{2\alpha} \\ \frac{\tilde{F}_1}{2\alpha} & \cdots & \frac{\tilde{F}_p}{2\alpha} & \alpha^r I_{m^2n^4} \end{bmatrix}.
\end{aligned}$$

Proof: Since the nominal system is asymptotically stable independent of delay when there are no uncertain parameters, $\tilde{F}$ and $\bar{A}_{a0}$ are nonsingular. The nonsingularity of (11) and (12) can then be preserved for each possible uncertain parameter vector k if and only if

$$I_{m^2n^4} - \sum_{j=1}^{p} k_j \tilde{F}_j - \sum_{j,l=1}^{p} k_j k_l \tilde{F}_{jl}, \text{ and} \tag{14}$$

$$I_{n^2} + (k \otimes I_{n^2}) \tilde{A}_{ae} \tag{15}$$

are nonsingular for each possible uncertain parameter vector k. It is seen that (14) can also be expressed as

$$I_{m^2n^4} - \tilde{F}_e(k \otimes I_{mn^2}) - (k \otimes I_{m^2n^4})^{\mathsf{T}} \tilde{F}_{ee}(k \otimes I_{m^2n^4}), \text{ or}$$

$$(\alpha^{2+r}+1)I_{m^2n^4} - ([k^\top \alpha] \otimes I_{m^2n^4})\Phi(\begin{bmatrix} k \\ \alpha \end{bmatrix} \otimes I_{m^2n^4}).$$

Thus (14) is nonsingular if

$$\|\tilde{F}_{ee}\|_s \|k\|_2^2 + \|\tilde{F}_e\|_s \|k\|_2 - 1 < 0, \text{ or} \quad (16)$$

$$\|\Phi\|_s (k^\top k + \alpha^2) < \alpha^{2+r} + 1. \quad (17)$$

From (15), (16) and (17), it can be easily seen that the delay independent asymptotic stability is guaranteed if the condition of Theorem 2 holds.

An example

Consider the following linear time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h)$$

where

$$A_0 = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.5 \end{bmatrix} + k_1 \begin{bmatrix} 0.5 & 0 \\ 0 & 1.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.2 & 0 \\ 1 & 0.1 \end{bmatrix} + k_1 \begin{bmatrix} -0.2 & 0 \\ 0.1 & 0.5 \end{bmatrix} + k_2 \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}.$$

By using Theorem 2, it is found that $\|\tilde{A}_{ae}\|_s^{-2} = 1.3817^2$, and $\tau^2 = 0.9959$ when α and r are chosen as 0.7 and 2 respectively. Thus the system preserves its delay independent asymptotic stability if

$$k_1^2 + k_2^2 < 0.9959.$$

Conclusions

In this paper, we give a necessary and sufficient condition for checking whether a linear system with commensurate delays is asymptotically stable independent of delay. It is less conservative than previous results which are only sufficient conditions, while only needs finite computations. We also discuss the robust delay-independent asymptotic stability of such systems when there are structured uncertain parameters.

References

[1] Brierley, S. D., J. N. Chiasson, E. B. Lee, and S. H. Zak, "On stability independent of delay for linear systems," IEEE Trans. on Automat. Control, Vol. AC-27, pp. 252-254, 1982.

[2] Mori, T., N. Fukuma, and M. Kuwahara, "On an estimate of the decay rate for stable linear delay systems," Int. J. Control, Vol. 36, pp. 95-97, 1982.

[3] Hmamed, A., "On the stability of time-delay systems: new results," Int. J. Control, Vol. 43, pp. 321-324, 1986.

[4] Wang, S. S., C. H. Lee, and T. H. Hung, "Stability analysis of systems with multiple time delays," Proceedings of the 14th National Symposium on Automat. Contr., pp. 214-221, Hsinchu, Taiwan, 1990.

[5] Hmamed, A., "Further results on the delay-independent asymptotic stability of linear systems," Int. J. Systems Sci., Vol. 22, pp. 1127-1132, 1991.

[6] Tesi, A. and A. Vicino, "Robust stability of state-space models with structured uncertainties," IEEE Trans. on Automat. Control, Vol. AC-35, pp. 191-195, 1990.

[7] Bellman, R., Introduction to Matrix Analysis, New York: McGraw-Hill, 1974.