# Early detection of target manoeuvres under a specific false alarm rate

W.-C. Yang, F.-R. Chang, Y.-H. Tsai and C.-M. Fan

**Abstract:** Two improvements in fading memory average for manoeuvre detection are given. First, according to the characteristics of fading memory average, the exact detection threshold is derived by a Markov chain approach. In conventional algorithms, which ignore the dynamic and dependent properties of fading memory average, only approximate threshold values are computed for a given false alarm rate. Secondly, the univariate fading memory average is further extended to the multivariate form to increase the speed of detection. The multivariate Markov chain approach, which is an extension of the Markov chain approach, can provide exact thresholds for the multivariate fading memory average. Monte Carlo simulation results verify not only the accuracy of the thresholds through both the univariate and multivariate Markov chain approaches, but also the superior detection speed of the proposed multivariate fading memory average. Under the same false alarm rates, the mean time to detection in the multivariate algorithm will be 26% less than that in the univariate algorithm.

## 1 Introduction

Target tracking can be successfully accomplished through Kalman filtering as long as the system is modelled accurately. Target manoeuvres, referring to unpredictable changes in target motion, may cause serious inaccuracies in modelling the system. As a result, loss of tracking may take place. One of the important criteria of surveillance is to detect manoeuvres as soon as possible under a specified false alarm rate. By accumulating all past and present data of the normalised innovations squared, Bar-Shalom and Birmiwal [1] employed a sliding window, or a (univariate) fading memory average, to improve the detection speed [2]. Chan and Couture [3] used a normalised sum of the differences between the measurements and the Kalman filter projections of positions as the detector. Applying the sliding window to manoeuvre detection, Wang and Varshney [4] suggested a method for selecting an optimal window length by means of the Neyman–Pearson criterion. Korn *et al.* [5] proposed a generalised likelihood ratio method for manoeuvre detection and estimation. In literature, there have been various algorithms for tracking targets after a manoeuvre is detected, such as the input estimation method [6] and the variable dimension filter [1]. Nevertheless, this paper does not focus on this aspect, but on manoeuvre detection instead.

Two improvements in fading memory average are given in this paper. First, according to the characteristics of fading memory (FM) average, the exact detection threshold is derived by a Markov chain (MC) approach. In conventional algorithms, which ignore the dynamic and dependent properties of FM average, only approximate threshold values are computed for a given false alarm rate. Secondly, to achieve the goal of fast detection, the fading memory average is extended to the multivariate fading memory average by accumulating the normalised innovation vector. The thresholds of the multivariate fading memory (MFM) average can also be found through the multivariate Markov chain (MMC) approach, which is an extension of the MC approach. During extension, the non-central chi-square distribution is considered. Simulation results verified the two improvements. In comparison to conventional methods, the proposed MC and MMC approaches provide accurate thresholds for FM and MFM through Monte Carlo simulation. In addition, by modelling the manoeuvres as a target moving with uniform circular motion in a two-dimensional plane, the FM and MFM methods, with various values of smoothing parameters, were also compared. The results showed that the MFM method could reduce the mean time to detection by 26% in comparison with the FM method.



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## 2 Background on manoeuvre detection

Based on the position–velocity (PV) model, a target moving in a two-dimensional plane can be described by the following kinematic and measurement equations [1]:

$$\boldsymbol{x}(k+1) = \boldsymbol{F}\boldsymbol{x}(k) + \boldsymbol{G}\boldsymbol{w}(k) \tag{1}$$

and

$$\boldsymbol{z}(k) = \boldsymbol{H}\boldsymbol{x}(k) + \boldsymbol{v}(k) \tag{2}$$

where the state is $\boldsymbol{x} = [p_x \quad v_x \quad p_y \quad v_y]^T$, representing the positions and velocities in $x$ and $y$ directions, $\boldsymbol{z}(k)$ is the measurement vector,

$$\boldsymbol{F} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} T/2 & 0 \\ 1 & 0 \\ 0 & T/2 \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{w} = [w_1 \quad w_2]^T, \quad \boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $T$ is the sampling period. Furthermore, $\boldsymbol{w}(k)$ and $\boldsymbol{v}(k)$ are assumed to be mutually independent and zero-mean white noise with covariance $\boldsymbol{Q}(k)$ and $\boldsymbol{R}(k)$, respectively.

By applying a Kalman filter, a fading memory average of the innovations is proposed by Bar-Shalom and Birmiwal [1] as follows:

$$y(k) = \eta y(k-1) + e(k) \tag{3}$$

with the normalised innovation squared $e(k) \equiv \tilde{z}^T(k)\boldsymbol{S}^{-1}(k)\tilde{z}(k)$, where $\eta$ is a smoothing factor satisfying $0 \le \eta < 1$, $\tilde{z}(k)$ is the innovation, and $\boldsymbol{S}(k)$ its covariance.

For convenience, the notation $x \sim \chi^2(v)$ is introduced to indicate that the variable $x$ has the chi-square distribution with $v$ degrees of freedom. Under the Gaussian assumption and the hypothesis that no manoeuvre is taking place, it can then be proved that $e(k) \sim \chi^2(n)$, where $n$ is the dimension of the measurement vector. The initial condition is given by $y(0) = \mathcal{E}[e(k)]/(1-\eta) = n/(1-\eta)$.

The distribution of $y(k)$ is not chi-square and is rather complicated. The mean and the asymptotical variance of $y(k)$ can be found as follows:

$$\mathcal{E}[y(k)] = n/(1-\eta), \tag{4}$$

and

$$\mathrm{Var}[y(k)] = 2n/(1-\eta^2) \tag{5}$$

It is often convenient to approximate $y(k)$ with a simpler random variable whose moments are matched to those of $y(k)$. Approximate $y(k)$ with a 'chi-square' random variable and match the first moment to (4) [1], it then follows that

$$y(k) \sim \chi^2(n/(1-\eta)) \tag{6}$$

An alternative method is to approximate $y(k)$ with the 'scaled chi-square' random variable [2]. By equating the first and the second moments of the 'scaled chi-square' random variable to (4) and (5), the distribution of $y(k)$ can be approximated as

$$y(k) \sim \frac{1}{1+\eta}\chi^2(n_1) \tag{7}$$

where

$$n_1 = n\frac{(1+\eta)}{(1-\eta)}$$

Under a specific false alarm rate, the thresholds obtained by means of the two methods above are insufficiently accurate for two reasons. First, the distribution of $y(k)$ is complicated. Ignorance of higher order moments matching can cause large errors. Second, the distribution obtained via the moment matching approach is time invariant. However, the distribution of $y(k)$ will change with time, especially when $k$ is small. Consequently, taking advantage of the Markov characteristic of $y(k)$, these authors propose another approach to evaluating the thresholds of the fading memory average more accurately.

Before going further, the terminology of detection is introduced for later use [7].

A *false alarm* is sounded when the test statistic exceeds the threshold in the absence of manoeuvres.

*Time to false alarm* is the time from the beginning of detection to the declaration of a false alarm, the mean of which is called *mean time to false alarm*.

*False alarm rate* is the inverse of mean time to false alarm if the sampling period $T = 1$ s; otherwise, it equals $T$ divided by mean time to false alarm.

*Time to detection* is the time required from the onset of the manoeuvre to the announcement of the alarm.

*Mean time to detection* with respect to a particular manoeuvre is defined as the mean of time to detection under the occurrence of this manoeuvre.

*Probability of detection* is the conditional probability that the detection algorithm declares the presence of a manoeuvre when, in fact, a manoeuvre occurs.

## 3 Markov chain approach for calculating the thresholds of fading memory average

In view of (3), each test statistic $y(k)$ is a weighted sum of its former value $y(k-1)$ and the present $e(k)$, which is a $\chi^2(n)$. Hence, $y(k)$ can be modelled as a discrete Markov chain with a stationary transition probability matrix and countable state space. Let $Td_{FM}$ be the threshold corresponding to the fading memory average. Then $y(k)$ is in a transient state as long as it lies within $[0, Td_{FM})$. To approximate $y(k)$, the interval $[0, Td_{FM})$ is partitioned into $m$ equal subintervals, each with a width $2d = Td_{FM}/m$. Figure 1 depicts the above. $y(k)$ is in transient state $j$ at time $k$ if

$$M_j - d \le y(k) < M_j + d, \quad 0 < j \le m \tag{8}$$

where $M_j$ represents the midpoint of the $j$th transient state.

Let the value of $y(k-1)$ in the $j$th state be approximated by $M_j$, and by using (3) and (8), then the transition matrix among transition states, $\boldsymbol{T} = [T_{ij}]$, can be obtained through the following calculations:

$$\begin{aligned} T_{ij} &= \Pr[y(k) \text{ in state } i | y(k-1) \text{ in state } j] \\ &= \Pr[(M_i - d) \le \eta y(k-1) + e(k) \\ &\qquad < (M_i + d) | y(k-1) = M_j] \\ &= \Pr[(M_i - d) - \eta M_j \le e(k) < (M_i + d) - \eta M_j] \\ &= \Pr[(M_i - d) - \eta M_j \le \chi^2(n) < (M_i + d) - \eta M_j] \end{aligned} \tag{9}$$

Thus, $T_{ij}$ can be found through the chi-square cumulative distribution function. From the fundamental matrix $(\boldsymbol{I} - \boldsymbol{T})^{-1}$, the mean time to false alarm (MTFA) is [8–10]

$$\mathrm{MTFA} = \boldsymbol{L}(\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{p}(0) \tag{10}$$

where $\boldsymbol{L} = [1 \cdots 1]$ is a $1 \times m$ row vector, $\boldsymbol{T}$ is the transition probability matrix given by $\boldsymbol{T} = [T_{ij}]$, where $T_{ij}$ are computed from (9) with $1 \le i, j \le m$, and $\boldsymbol{p}(0)$ is the initial

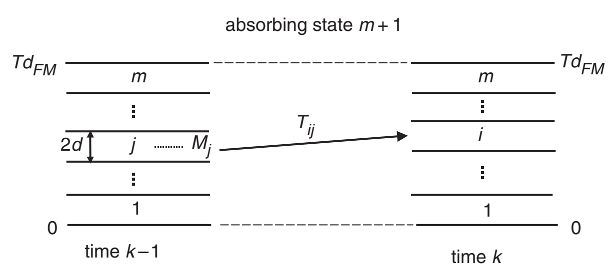


**Fig. 1** *Calculation of thresholds for FM using Markov chain approach*

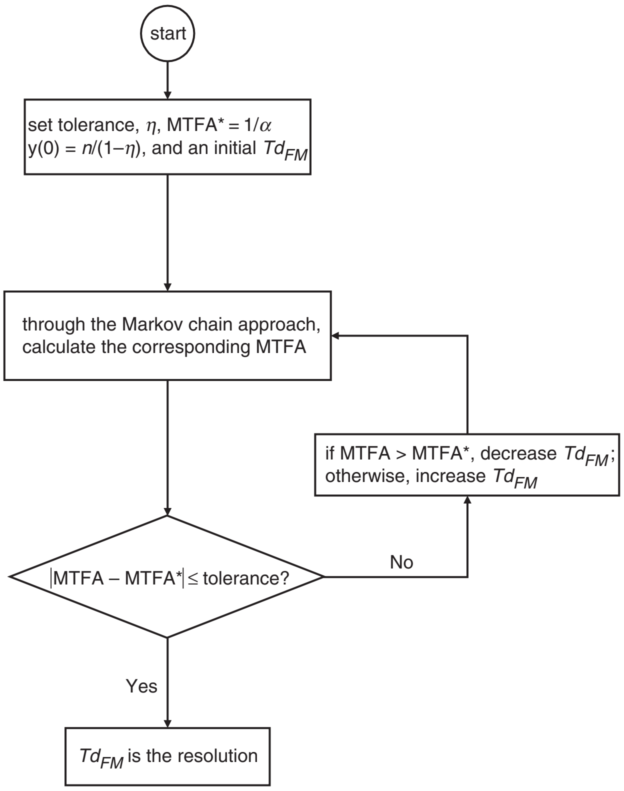


**Fig. 2** *Flow diagram for calculating threshold of FM*

probability vector. $\boldsymbol{p}(0)$ consists of a single element equal to one, which corresponds to the state containing $\mathcal{E}[y(k)]$, and zeros elsewhere.

The procedures for obtaining $Td_{FM}$ under a given false alarm rate $\alpha$ can be depicted as a flow diagram (Fig. 2). The threshold can be calculated offline and can be tabulated in advance. The number of state divisions was chosen as 100

**Table 1: Thresholds of the FM $y(k)$ and MFM $\|\boldsymbol{Y}(k)\|$ under $\alpha = 1/100$**

| $\eta$ | FM | MFM |
|---|---|---|
| 0.95 | 46.2810 | 7.6060 |
| 0.90 | 28.1803 | 6.0620 |
| 0.85 | 21.5347 | 5.2553 |
| 0.80 | 18.0469 | 4.7390 |
| 0.75 | 15.8858 | 4.3744 |
| 0.70 | 14.4166 | 4.1010 |
| 0.65 | 13.3488 | 3.8880 |
| 0.60 | 12.5391 | 3.7175 |
| 0.55 | 11.9096 | 3.5787 |
| 0.50 | 11.4044 | 3.4644 |
| 0.45 | 10.9879 | 3.3696 |
| 0.40 | 10.6435 | 3.2908 |
| 0.35 | 10.3561 | 3.2255 |
| 0.30 | 10.1114 | 3.1717 |
| 0.25 | 9.9016 | 3.1281 |
| 0.20 | 9.7212 | 3.0938 |
| 0.15 | 9.5658 | 3.0675 |
| 0.10 | 9.4323 | 3.0493 |
| 0.05 | 9.3147 | 3.0384 |
| 0 | 9.2114 | 3.0349 |

and the tolerance as 0.1. Under $\alpha = 1/100$, the resulting thresholds are shown in the second column of Table 1 with $\eta$ ranging from 0 to 0.95. Note that with $\eta = 0$, the fading memory average is exactly the normalised innovation squared $e(k)$.

## 4 Multivariate fading memory average

In this Section, the fading memory (FM) average is further extended to the multivariate case. Define the vector $\boldsymbol{E}(k) \equiv \boldsymbol{S}(k)^{-\frac{1}{2}}\tilde{\boldsymbol{z}}(k)$, and then the $n$-dimensional random vector $\boldsymbol{E}(k)$ is multivariate normal with zero mean and an identity covariance matrix, since $\tilde{\boldsymbol{z}}(k)$ originally has zero mean and covariance matrix $\boldsymbol{S}(k)$. According to (3), the multivariate fading memory (MFM) can be defined in terms of $\boldsymbol{E}(k)$ as follows:

$$\boldsymbol{Y}(k) = \eta \boldsymbol{Y}(k-1) + \boldsymbol{E}(k) \tag{11}$$

where the initial condition $\boldsymbol{Y}(0)$ is chosen as the zero vector, and $0 \le \eta < 1$.

$\|\boldsymbol{Y}(k)\|$ is selected as the test statistic. Runger and Prabhu [11] showed that the $\|\boldsymbol{Y}(k)\|$ process is a Markov chain. Therefore, the calculation of the thresholds of the statistic $\|\boldsymbol{Y}(k)\|$ can be modeled as a discrete Markov chain. Let $\mathrm{Sp}(r)$ denote the $n$-dimensional sphere of radius $r > 0$. To approximate $\|\boldsymbol{Y}(k)\|$, $\mathrm{Sp}(Td_{MFM})$ is partitioned into $m$ concentric spherical rings of an equal width $2d = Td_{MFM}/m$, where $Td_{MFM}$ is the threshold corresponding to the MFM. Figure 3 illustrates the divisions of Markov states in modelling the MFM for the two-dimensional case. The MFM is in transient state $j$ at time $k$ if

$$M_j - d \le \|\boldsymbol{Y}(k)\| < M_j + d, \quad 0 < j \le m \tag{12}$$

where $M_j$ represents the midpoint of the $j$th subinterval.

Use (11) and approximate $\boldsymbol{Y}(k-1)$ to $M_j$. Then the transition probability matrix among transient states for the MFM is given by

$$\begin{aligned} T_{ij} &= \Pr[\|\boldsymbol{Y}(k)\| \text{ in state } i \,|\, \|\boldsymbol{Y}(k-1)\| \text{ in state } j] \\ &= \Pr[M_i - d \le \|\eta \boldsymbol{Y}(k-1) + \boldsymbol{E}(k)\| \\ &\quad < \boldsymbol{M}_i + d \,|\, \|\boldsymbol{Y}(k-1)\| = M_j] \end{aligned} \tag{13}$$

From the Appendix, the transition probability matrix can be found through the non-central chi-square distribution as below.

$$T_{ij} = \Pr[(M_i - d)^2 \le \chi^2(n,c) < (M_i + d)^2] \tag{14}$$

where $\chi^2(n,c)$ denotes the non-central chi-square random variable with $n$ degrees of freedom and with non-centrality

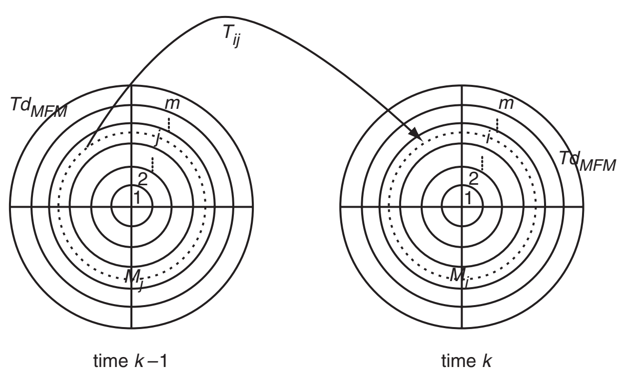


**Fig. 3** *Calculation of thresholds for MFM using Markov chain approach*

parameter $c = (\eta M_j)^2$. Similar to the case of the FM, the mean time to false alarm (MTFA) is

$$\mathrm{MTFA} = \boldsymbol{L}(\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{p}(0) \tag{15}$$

where $\boldsymbol{L} = [1 \cdots 1]$ is a $1 \times m$ row vector, $\boldsymbol{T}$ is the transition probability matrix given by $\boldsymbol{T} = [T_{ij}]$, where $T_{ij}$ are computed from (14) with $1 \le i, j = m$, and $\boldsymbol{p}(0)$ is the initial probability vector. Because $\boldsymbol{Y}(0)$ is chosen as the zero vector, $p(0)$ consists of a single element equal to one, which corresponds to the state containing the value 0, and zeros elsewhere.

The procedures to obtain $Td_{MFM}$ under a specific false alarm rate $\alpha$ resemble the case of the FM. Hence they are not repeated here. Similarly, the thresholds can be calculated offline and made into a table in advance. The results are also shown in the third column of Table 1 with $\eta$ ranging from 0 to 0.95 and $\alpha = 1/100$. The number of state divisions $m$ is chosen as 100. Note that under $\eta = 0$, the MFM is exactly the normalised innovation squared $e(k)$.

## 5 Simulation results and analysis

Two examples are given to verify the proposed methods. The first one examines the accuracy of thresholds. The second one verifies the superior detection speed of MFM over FM.

*Example 1:* Bivariate Gaussian vectors $\tilde{\boldsymbol{z}}(k)$, with zero mean and identity covariance matrices, are independently generated as the innovation vectors. Given three different false alarm rates, 1/20, 1/100, and 1/1000, find the corresponding detection thresholds using the four methods and examine the accuracy of these thresholds.

*Solution:* For convenience, the moment matching to the first order in (6) is abbreviated to MM1, the moments matching to the first and second orders in (7) to MM2, the Markov chain approach proposed in Section 3 to MC, and the multivariate Markov chain approach proposed in Section 4 to MMC.

The MM1, MM2 and MC were used to compute the FM thresholds, and the MMC was used to compute the MFM thresholds. We chose $\eta = 0.8$, and then $y(k) \sim \chi^2(10)$ using MM1 and $y(k) \sim \chi^2(18)/1.8$ using MM2. The number of state divisions was chosen as 100 and the tolerance as 0.1 when using MC and MMC. The thresholds evaluated from the above four methods are recorded in Table 2 under given false alarm rates $\alpha = 1/20$, 1/100 and 1/1000.

In the absence of a manoeuvre, the thresholds in Table 2 were examined. The simulated values of false alarm rate $\hat{\alpha}$ can be estimated as the inverse of the mean time to false alarm, which was obtained through 10 000 simulation replications. The resulting $\hat{\alpha}/\alpha$ was recorded in Table 3. The closer the value of $\hat{\alpha}/\alpha$ to 1, the more accurate the algorithm. Compared with MM1 and MM2, MC and MMC provide more accurate thresholds. The reason is because the truncation errors in MM1 and MM2 are larger than the quantisation errors in MC and MMC. This verifies the accuracy of the proposed Markov chain approach.

**Table 2: Resulting thresholds using MM1, MM2, MC and MMC with $\eta = 0.8$**

| Threshold | $\alpha = 1/20$ | $\alpha = 1/100$ | $\alpha = 1/1000$ |
|---|---|---|---|
| MM1 | 18.3070 | 23.2093 | 29.5883 |
| MM2 | 16.0385 | 19.3363 | 23.5069 |
| MC | 12.5098 | 18.0469 | 23.9998 |
| MMC | 3.4965 | 4.7390 | 6.0475 |

**Table 3: Resulting $\hat{\alpha}/\alpha$ applying MM1, MM2, MC and MMC with $\eta = 0.8$**

| $\hat{\alpha}/\alpha$ | $\alpha = 1/20$ | $\alpha = 1/100$ | $\alpha = 1/1000$ |
|---|---|---|---|
| MM1 | 0.1987 | 0.1435 | 0.0803 |
| MM2 | 0.4514 | 0.6788 | 1.2527 |
| MC | 1.0045 | 1.0035 | 1.0067 |
| MMC | 1.0061 | 1.0064 | 1.0088 |

*Example 2:* A moving target described by (1) and (2) with $\boldsymbol{Q} = \boldsymbol{0}$, $T = 1$ s, and

$$\boldsymbol{R} = \begin{bmatrix} 100000 & 5000 \\ 5000 & 100000 \end{bmatrix}$$

were used to compare the detection speeds of the FM and MFM, which is similar to the example in [1], with the exception that $T$ is changed from 10 s to 1 s. The target followed a constant course with a constant velocity −15 m/s along the y-axis until time $k = 300$ s, at which point it began to manoeuvre in uniform circular motion with a radius of 100 m and a centripetal acceleration of 5 m/s$^2$. For $\alpha = 1/100$, compare the detection speeds and probabilities of detection between FM and MFM.

*Solution:* The thresholds listed in Table 1 are adopted for the FM and MFM since Example 1 verifies the correctness of the proposed Markov chain approach. The initial conditions of the target are given by $\boldsymbol{x}(0) = [2000\ 0\ 13000\ \ -15]^T$, and the initial covariance matrix is

$$\boldsymbol{P}(0|0) = \begin{bmatrix} R_{11} & R_{11}/T & 0 & 0 \\ R_{11}/T & 2R_{11}/T^2 & 0 & 0 \\ 0 & 0 & R_{22} & R_{22}/T \\ 0 & 0 & R_{22}/T & 2R_{22}/T^2 \end{bmatrix}$$

with $R_{11} = R_{22} = 100\,000$. The resulting mean time to detection of FM and MFM, obtained through 10 000 simulation replications, are depicted in Fig. 4, which shows that the MFM has superior performance to the FM. The optimal mean time to detection is 21.187 s in FM with $\eta = 0.8$, and 15.685 s in MFM with $\eta = 0.8$. The percentage improvement of MFM over FM is 26%. For a more robust selection of the optimal $\eta$, methods based on the worst-case design techniques can be used. First, the types or the ranges of manoeuvres must be specified. Next, the optimal $\eta$ can be

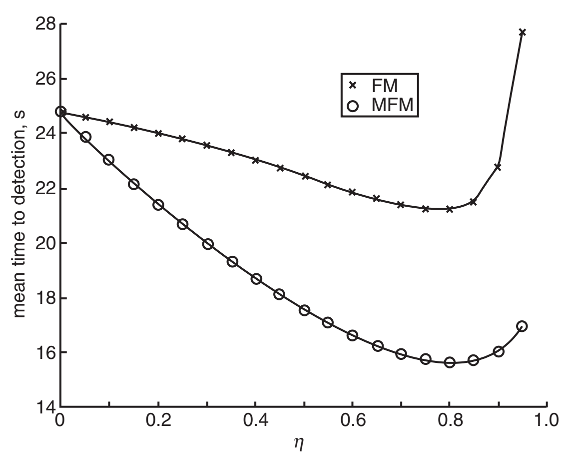


**Fig. 4** *Mean time to detection for FM and MFM with $\eta = 0$ to 0.95*

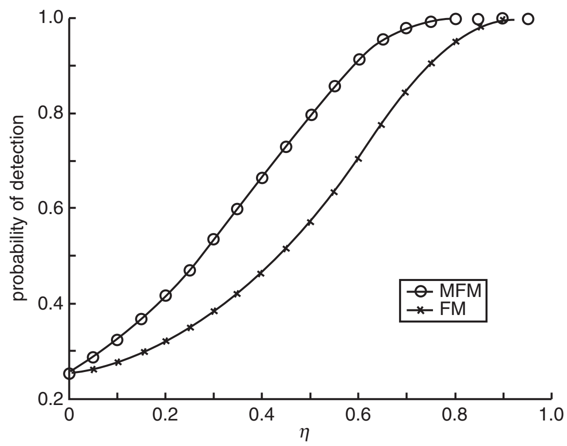


**Fig. 5** *Probability of detection for FM and MFM with* $\eta = 0$ *to 0.95*

determined by analysing the table of resulting mean times to detection.

Next, the probabilities of detection of FM and MFM at $k = 350$ s (i.e. 50 s after the occurrence of the manoeuvre) were simulated under $\alpha = 1/100$. 10 000 simulation replications were performed; the results are shown in Fig. 5. Note that MFM always gives a higher probability of detection than FM.

## 6 Conclusions

A new algorithm is proposed to determine the threshold corresponding to a specific false alarm rate when the 'fading memory' average is applied to manoeuvre detection. Compared to the conventional moment matching approach, the proposed algorithm, by means of the Markov chain approach, can evaluate more accurate thresholds that meet the requirement of a fixed false alarm rate. These different algorithms are simulated and compared under various values of false alarm rate. Moreover, in this paper, the fading memory average is extended to the multivariate case, the threshold of which can also be obtained via the Markov chain approach. Although calculations of thresholds in both univariate and multivariate cases are lengthy, the thresholds can be calculated offline and tabulated in advance. From the simulation results, it appears that the multivariate fading memory average has a performance superior to that of the univariate case in detecting target manoeuvres. In fact, the multivariate fading memory average can further improve the detection speed of the univariate case by 26%. The proposed methods can be applied to the detection of abrupt changes as long as the normalised innovation squared is chi-square distributed, not limited to the detection of target manoeuvres. Furthermore, in view of the fact that many dynamic systems and sensors are not absolutely linear, extensions of the proposed methods to the nonlinear problems are of great value. Techniques like the extended Kalman filter or the linearised Kalman filter can be used for nonlinear systems. The main difficulty of detecting target manoeuvres in nonlinear systems lies in analysing the characteristic of the innovation vector.

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## 9 Appendix: Transition probability matrix for MFM

Let $N(\mu, \sigma^2)$ denote the normal distribution with mean $\mu$ and variance $\sigma^2$, $\chi^2(n)$ denote the central chi-square distribution with $n$ degrees of freedom, and $\chi^2(n, c)$ denote the non-central chi-square distribution with $n$ degrees of freedom and non-centrality parameter $c$. The following proposition from [12] is required for derivation of the transition probability matrix of MFM.

*Proposition A1:* Let $z_1, z_2, \ldots, z_n$ be independent random variables each with distribution $N(c_i, 1)$. Then the random variable $Y = z_1^2 + z_2^2 + \cdots + z_n^2$ has the distribution $\chi^2(n, c)$, where $c = \sum_{i=1}^{n} c_i^2$.

Recall (13)

$$T_{ij} = \Pr[M_i - d \leq \|\eta \boldsymbol{Y}(k-1) + \boldsymbol{E}(k)\| < M_i + d | \, \|\boldsymbol{Y}(k-1)\| = M_j] \quad (16)$$

The $n$-dimensional random vector $\boldsymbol{E}(k)$ is multivariate normal with zero mean vector and an identity covariance matrix. By accumulating historical $\boldsymbol{E}(\bullet)$ up to current, $\boldsymbol{Y}(k-1)$ is also multivariate normal with zero mean vector and a diagonal covariance matrix with identical values along the diagonal. This implies that $\boldsymbol{Y}(k-1)$ has a spherical distribution, whose contours of constant density are thus spheres. Conditionally on $\|\boldsymbol{Y}(k-1)\| = M_j$, the distribution of $\boldsymbol{Y}(k-1)$ is the same as $M_j\boldsymbol{U}$, where $\boldsymbol{U}$ represents the uniform random vector on the $n$-dimensional sphere of radius 1, Sp(1) [11]. In summary, (16) can be written as

$$T_{ij} = \Pr[M_i - d \leq \|\eta \mathrm{M}_j \boldsymbol{U} + \boldsymbol{E}(k)\| < M_i + d] \quad (17)$$

Let $\boldsymbol{z} = \eta M_j \boldsymbol{U} + \boldsymbol{E}(k)$. Assume that $\boldsymbol{z} = [z_1\, z_2\, \cdots\, z_n]^T$, $\boldsymbol{U} = [u_1\, u_2\, \cdots\, u_n]^T$, and $\boldsymbol{E}(k) = [e_1\, e_2\, \cdots e_n]^T$. Since $\boldsymbol{U}$ represents the uniform random vector on Sp(1), we have $\sum_{i=1}^{n} u_i^2 = 1$. The distribution of $z_i$ can be found as

$$z_i = \eta M_j u_i + e_i \sim N(\eta M_j u_i,\, 1) \tag{18}$$

By applying proposition A1, then

$$\|\eta M_j \boldsymbol{U} + \boldsymbol{E}(k)\|^2 = \boldsymbol{z}^T \boldsymbol{z} = \sum_{i=1}^{n} z_i^2 \sim \chi^2(n, c) \tag{19}$$

and

$$c = \sum_{i=1}^{n} (\eta M_j u_i)^2 = \eta^2 M_j^2 \sum_{i=1}^{n} u_i^2 = \eta^2 M_j^2 \tag{20}$$

Substitute (19) and (20) into (17), and the transition probability matrix can be obtained as

$$T_{ij} = \Pr[(M_i - d)^2 \leq \chi^2(n, c) < (M_i + d)^2] \tag{21}$$

where $c = (\eta M_j)^2$.