RELATIONSHIPS AMONG DIGITAL ONE/HALF BAND FILTERS, LOW/HIGH ORDER DIFFERENTIATORS AND DISCRETE/DIFFERENTIATING HILBERT TRANSFORMERS

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ABSTRACT

There exists a close relationship among digital one/half band filters, low/high order differentiators and discrete/differentiating Hilbert transformers. The purpose of this paper is to present a complete picture of their interrelationships and conversions between each other. A useful table and some block diagrams have been developed for their impulse response's connections, and a design example is also given for illustration.

I. INTRODUCTION

Conventionally, we often use the well-known McClellan-Parks program [1][2] to design the FIR digital filters, differentiators and Hilbert transformers independently; Also recently the eigenfilter approach [3][4][5] has been developed to design these filters, differentiators and Hilbert transformers differently in each independent case. However, these designs can be closely related to each other. This paper is to present their relationships and unifies them by means of a useful table and some block diagram.

The conversions between digital Hilbert transformers and one/half band filters have been developed by Jackson [6] , Vaidyanathan and Nguyen [7]. This paper further exploits their relationships to differentiating Hilbert transformers [8], low/high order differentiators and the conversions between each other.

II. INTERRELATIONS AMONG ONE/HALF BAND FILTERS AND HILBERT TRANSFORMERS

The overall block diagram for illustrating the relationships among one-band filter (B_1), half-band filter ($B_{\frac{1}{2}}$), Case 3 symmetric Hilbert transformer (H_3), Case 4 Hilbert transformer (H_4), differentiating Hilbert transformer (H_d) and first-order differentiator (D_1) are given in Fig.1(a) in which the transfer functions and its impulse responses for each filter are also characterized in this block diagram; and the derivations of which are described as below. In [6], Jackson had shown that

$$h_3(n)=\begin{cases}(-1)^{\frac{n-N-1}{2}}\,2b_{\frac{1}{2}}(n) & n \text{ even}\\ 0 & n \text{ odd}\end{cases} \qquad (1)$$

and

$$b_{\frac{1}{2}}(n)=\begin{cases}(-1)^{\frac{n-N-1}{2}}\,\tfrac{1}{2}h_3(n) & n \text{ even}\\ 0 & n \text{ odd}\\ \tfrac{1}{2} & n=N\end{cases} \qquad (2)$$

And Vaidyanathan and Nguyen [7] found that

$$b_{\frac{1}{2}}(n)=\begin{cases}\tfrac{1}{2}b_1(\tfrac{n}{2}) & n \text{ even}\\ 0 & n \text{ odd}\neq N\\ \tfrac{1}{2} & n=N\end{cases} \qquad (3)$$

and

$$b_1(n)=2b_{\frac{1}{2}}(2n) \qquad n=0,1,\ldots,N \qquad (4)$$

In the following, we will further exploit the relationship between differentiating Hilbert transformers and Case 3 symmetric Hilbert transformers, and the conversions of Cases 3 and 4 Hilbert transformers [9].

Recently Cizek proposed a new differentiating Hilbert transformer [8], the output of which is the derivative of the Hilbert transform of the input signal. This signal is useful for the evaluation of the instantaneous frequency by means of an analytic signal.

The amplitude responses of a Case 3 symmetric Hilbert transformer and differentiating Hilbert transformer can be represented by

$$\hat{H}_3(w)=\sum_{n=1}^{N}\hat{h}_3(n)\sin nw \qquad (5)$$

and

$$\hat{H}_d(w)=\sum_{n=0}^{N}\hat{h}_d(n)\cos nw \simeq |w| = (jw)\cdot(-j) \qquad (6)$$

respectively, where

$$\hat{h}_3(n)=2h_3(n) \qquad n=1,\ldots,N \qquad (7)$$

and

$$\hat{h}_d(n)=\begin{cases}h_d(n), & n=0\\ 2h_d(N-n), & n=1,\ldots,N\end{cases} \qquad (8)$$

From Eq.(6), the differentiating Hilbert transformer can be implemented by cascade connection of a differentiator and a Hilbert transformer; However, the direct use of differentiating Hilbert transformer is more accurate and efficient

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than the above cascade scheme.

The integration of Eq.(5) along w-axis will approximately lead to the amplitude response $-\hat{H}_d(w)+\pi/2$, hence we can get the following conversions:

$$h_d(n)=\begin{cases}\frac{h_3(n)}{N-n} & n \text{ even}\\ 0 & n \text{ odd}\neq N\\ \frac{\pi}{2} & n=N\end{cases} \qquad (9)$$

and

$$h_3(n)=\begin{cases}(N-n)h_d(n) & n \text{ even}\\ 0 & n \text{ odd}\end{cases} \qquad (10)$$

If we reduce the sampling rate on the Case 3 symmetric Hilbert transformer's impulse responses by a factor of 2, this decimation process [10] would correspond to extracting every 2nd sample of the discrete sequences. Since every other impulse response sample of the symmetric Hilbert transformer is equal to zero, by taking out these zero-valued impulse response samples, the decimated version of Case 3 symmetric Hilbert transformers will become the Case 4 Hilbert transformers with even length very interestingly. If we reverse the above process by inserting a zero-valued sample between each impulse response sequence of the Case 4 Hilbert transformers; then the Case 4 Hilbert transformers will become Case 3 symmetric Hilbert transformers after this interpolation process [10]. So there exists

$$h_3(n)=\begin{cases}h_4(\frac{n}{2}) & n \text{ even}\\ 0 & n \text{ odd}\end{cases} \qquad (11)$$

and

$$h_4(n)=h_3(2n) \qquad n=0,1,\ldots,N \qquad (12)$$

The transition bandwidth of Case 4 Hilbert transformer is equal to twice of that of Case 3 symmetric Hilbert transformer, and the ripple is the same in these two type Hilbert transformers.

III. DERIVATIONS OF EVEN LENGTH LOW/HIGH ORDER DIFFERENTIATORS FROM ONE-BAND FILTERS

Low/high order differentiators are very useful for calculation of geometric moments and for biological signal processing. Recently the modified McClellan-Parks program [2] and the eigen-approach [5] are proposed for designing these digital differentiators. This section is to point out that these designs can be easily derivated from the one-band filters without involving any complicated time consuming optimization procedures.

Suppose the transfer function of even length, i-th order differentiator is characterized as

$$D_i(Z)=\sum_{i=0}^{N} d_i(n)Z^{-n} \qquad (13)$$

where N is an odd integer.
For an odd-order Case 4 differentiator, the amplitude response of Eq.(13) is

$$\hat{D}_i(w)=\sum_{n=1}^{\frac{N+1}{2}} \hat{d}_i(n)\sin(n-\tfrac{1}{2})w\simeq(jw)^i, \quad i \text{ odd} \qquad (14)$$

and for an even-order Case 2 differentiator, the amplitude response is

$$\hat{D}_i(w)=\sum_{n=1}^{\frac{N+1}{2}} \hat{d}_i(n)\cos(n-\tfrac{1}{2})w\simeq(jw)^i, \quad i \text{ even} \qquad (15)$$

$$\text{where } \hat{d}_i(n)=2d_i(\tfrac{N+1}{2}-n), \quad n=1,2,\ldots,\tfrac{N+1}{2} \qquad (16)$$

Also, suppose the amplitude response of an even length one-band filter is represented by

$$\hat{B}_1(w)=\sum_{n=1}^{\frac{N+1}{2}} \hat{b}_1(n)\cos(n-\tfrac{1}{2})w \qquad (17)$$

where

$$\hat{b}_1(n)=2b_1(\tfrac{N+1}{2}-n) \quad n=1,2,\ldots,\tfrac{N+1}{2} \qquad (18)$$

1) FIRST-ORDER DIFFERENTIATOR:

The integration of Eq.(17) along the w-axis will lead to the amplitude response of the corresponding first-order differentiator, then we can get

$$\hat{d}_1(n)=\frac{\hat{b}_1(n)}{n-\frac{1}{2}} \qquad n=1,2,\ldots,\tfrac{N+1}{2} \qquad (19)$$

From Eqs.(16), (18) and (19), it is easy to derive the conversions

$$b_1(n)=(\tfrac{N}{2}-n)d_1(n) \quad n=0,1,\ldots,N \qquad (20)$$

and

$$d_1(n)=\frac{b_1(n)}{\frac{N}{2}-n} \qquad n=0,1,\ldots,N \qquad (21)$$

Due to the inherent zero magnitude response at folding frequency for Case 2 one-band filters [9], this will be suitable for designing the nonfull-band differentiators.

By the above description and those conversions in Section II, it is easy to establish the relationships among one/half band filters, first-order differentiators and the discrete/differentiating Hilbert transformers. Fig.1(a) shows the block diagram for connecting these relationships.

2) SECOND-ORDER DIFFERENTIATOR

Similarly, the amplitude response of a second-order differentiator can be derived by integrating the amplitude response of the first-order differentiator (see Eq.(14)) along w-axis, i.e.

$$\hat{D}_2(w)=-2\int_0^w \hat{D}_1(w)dw=-2\int_0^w \{ \sum_{n=1}^{\frac{N+1}{2}} \hat{d}_1(n)\sin(n-\tfrac{1}{2})w\}dw$$

$$= \sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{d}_1(n)}{n-\frac{1}{2}}\cos(n-\tfrac{1}{2})w - \sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{d}_1(n)}{n-\frac{1}{2}}\cdot 1 \qquad (22)$$

Substitute $\hat{d}_1(n)=\frac{\hat{b}_1(n)}{n-\frac{1}{2}}$ (see Eq.(19)) and $\sum_{n=0}^{\frac{N+1}{2}} \hat{b}_1(n)\cdot\cos(n-\frac{1}{2})w \approx 1$ (one-band frequency response is approximately equal to 1 except near the folding frequency) in the above equation, we get

$$\hat{D}_2(w)\approx \sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{b}_1(n)}{(n-\frac{1}{2})^2}\cos(n-\tfrac{1}{2})w-[\sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{b}_1(n)}{(n-\frac{1}{2})^2}][\sum_{n=0}^{\frac{N+1}{2}} \hat{b}_1(n)\cdot\cos(n-\tfrac{1}{2})w]$$

$$= \sum_{n=1}^{\frac{N+1}{2}} [\frac{2\hat{b}_1(n)}{(n-\frac{1}{2})^2} - (\sum_{n'=1}^{\frac{N+1}{2}} \frac{2\hat{b}_1(n')}{(n'-\frac{1}{2})^2})\hat{b}_1(n)]\cos(n-\tfrac{1}{2})w \qquad (23)$$

Using Eqs.(15), (16) and (18), we can get the filter coefficients of the second-order differentiator from the impulse response of one-band filter by the following relation

$$d_2(n)=\frac{-2b_1(n)}{(\frac{N}{2}-n)^2}+[\sum_{n'=0}^{N} \frac{2b_1(n')}{(\frac{N}{2}-n')^2}]b_1(n), \quad n=0,1,\ldots,N \qquad (24)$$

3) THIRD-ORDER DIFFERENTIATOR

For third-order differentiator, its amplitude response can be obtained as

$$\hat{D}_3(w)=3\int_0^w \hat{D}_2(w)dw$$

$$= \sum_{n=1}^{\frac{N+1}{2}} [\frac{3!\hat{b}_1(n)}{(n-\frac{1}{2})^3} - (\sum_{n'=1}^{\frac{N+1}{2}} \frac{3!\hat{b}_1(n')}{(n'-\frac{1}{2})^2})\cdot \frac{\hat{b}_1(n)}{n-\frac{1}{2}}]\sin(n-\tfrac{1}{2})w \qquad (25)$$

and its impulse response is

$$d_3(n)=\frac{-3!b_1(n)}{(\frac{N}{2}-n)^3}+[\sum_{n'=0}^{N} \frac{3!b_1(n')}{(\frac{N}{2}-n')^2}]\frac{b_1(n)}{\frac{N}{2}-n}, \quad n=0,1,\ldots,N \qquad (26)$$

The relationships and the formulations between the one-band filters and these high-order differentiators are illustrated in Fig.1(b).

For i>3, the impulse response coefficients of higher-order differentiator can be derived by similar procedures described above. Observe that the deviations of even-order differentiators (a one-band filter can be taken as a zero-order differentiators) is larger than that of odd-order differentiators, because we approximate directly a unit constant by the magnitude response of a one-band filters, for example as in Eq.(23). So the even-order differentiators from this method are generally inferior to the odd-order ones.

IV. DESIGN EXAMPLE

A numerical example is given here for clear illustration. First we can design a length 30 one-band filter with cutoff frequency 0.46 by the McClellan-Parks program [1]. From TABLE I we can easily get the filter coefficients of the half-band filter, first-order differentiator, discrete/differentiating Hilbert transformers and first to third order differentiators by this one-band filter's impulse response. Their frequency responses are shown in Fig. 2 for illustration.

V. CONCLUSIONS

In this paper, digital one/half band filters, low/high order differentiators and discrete/differentiating Hilbert transformer have been shown in close relation to each other. A useful table and block diagrams have been developed for unifying their impulse's connections, and a design example is also given for illustration.

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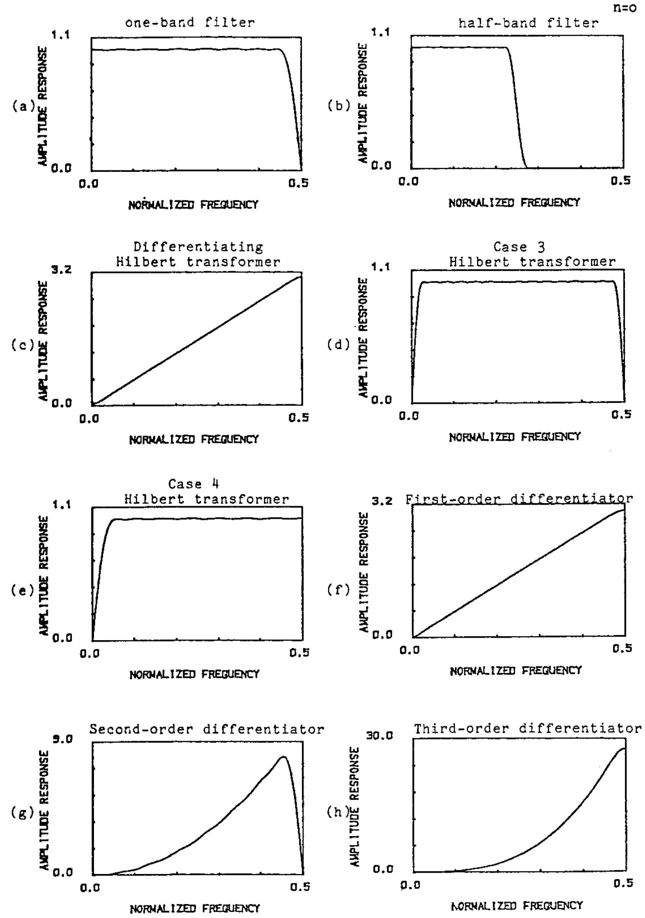

Fig. 2 Amplitude responses of (a) one-band prototype filter, and (b) - (h): its derivated filters.

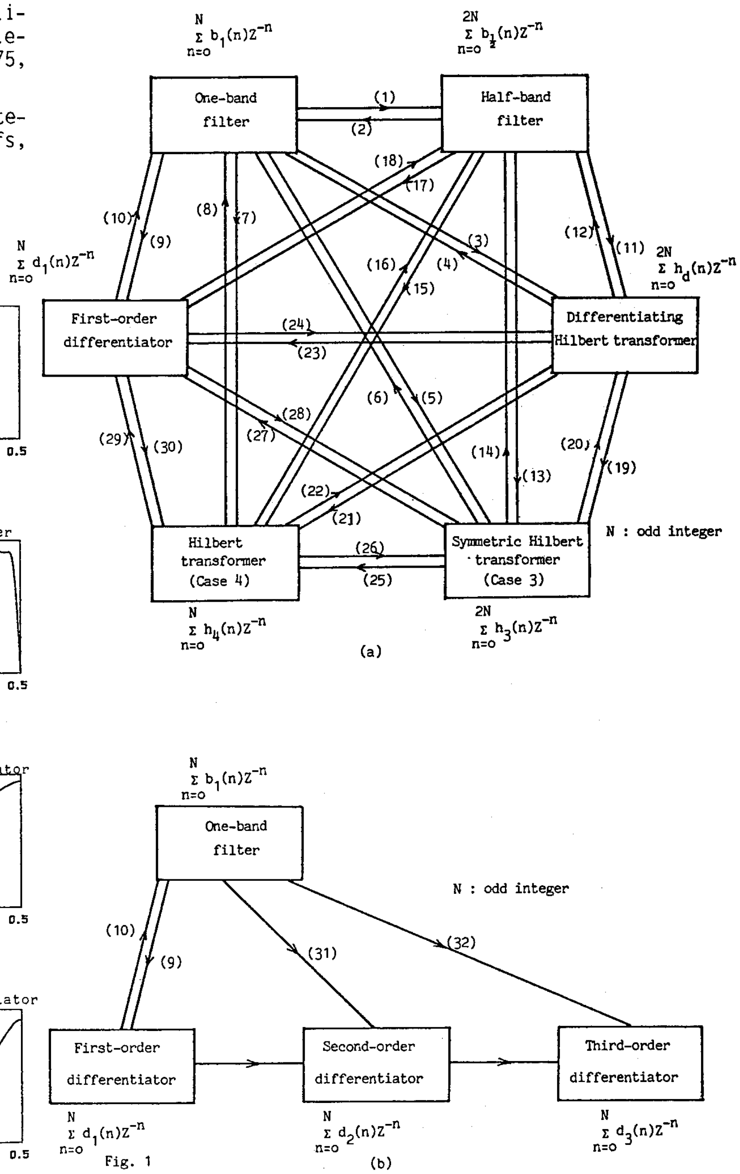

Fig. 1 Block diagram for illustrating the relationships among one/half band filters, lower/higher order differentiators, and discrete/differentiating Hilbert transformers, in which the representations of lable numbers are tabulated in TABLE I.