Symmetric Properties of 2-D Sequences and Their Applications for Designing Quadrantally Symmetric Linear-Phase 2-D FIR Digital Filters

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Abstract

In this paper, various symmetric and antisymmetric 2-D sequences are used to design quadrantally symmetric/antisymmetric 2-D filters. It is shown that there are sixteen types cases to be considered according to the symmetry/antisymmetry of 2-D sequences in both directions and their filter lengths (even or odd).

I. Introduction

Conventionally, the design of linear-phase 2-D FIR digital filters is concentrated on the class of quadrantally symmetric filters, such as circular filters, fan-type filters etc [1]. A 2-D sequence, which is symmetric in both directions, is required to realize such quadrantally symmetric filters.

In this paper, we will start from the discussion of the symmetric properties of 2-D sequences to disclose their applications for designing quadrantally symmetric/antisymmetric linear-phase 2-D FIR digital filters by the eigenfilter approach, which has been used successfully to design linear-phase 1-D filters [2][3] and 2-D quadrantally symmetric filters [4]. It is shown that there are sixteen types cases to be considered according to the symmetry/antisymmetry of 2-D sequences in both directions and their filter lengths (even or odd). The corresponding types of amplitude responses are tabulated into a complete table if these 2-D sequences are used to realize 2-D FIR filters. Also, the definitions of quadrantal-plane, half-plane and full-plane filters are described.

II. Symmetric Properties of 2-D Sequences

Let $\mathbf{X}$ be an $N_1 \times N_2$ 2-D sequence which is represented in a matrx form with its elements being denoted by $x(n_1, n_2)$, $n_1 = 0, 1, \cdots, N_1 - 1$,

$n_2 = 0, 1, \cdots, N_2 - 1$, i.e.

$$\mathbf{X} = \begin{bmatrix} x(0,0) & x(0,1) & \cdots & x(0,N_2-1) \\ x(1,0) & x(1,1) & \cdots & x(1,N_2-1) \\ \vdots & \vdots & \ddots & \vdots \\ x(N_1-1,0) & x(N_1-1,1) & \cdots & x(N_1-1,N_2-1) \end{bmatrix}. \quad (1)$$

If

$$x(n_1, n_2) = x(N_1 - 1 - n_1, n_2) \quad 0 \le n_1 \le N_1 - 1, \; 0 \le n_2 \le N_2 - 1, \quad (2)$$

we call $\mathbf{X}$ an even-symmetric 2-D sequence in n_1-direction; and if

$$x(n_1, n_2) = -x(N_1 - 1 - n_1, n_2) \quad 0 \le n_1 \le N_1 - 1, \; 0 \le n_2 \le N_2 - 1, \quad (3)$$

the sequence is called an odd-symmetric 2-D sequence in n_1-direction. Similarly, the same case exists for n_2-direction. Then quadrantally symmetric or antisymmetric 2-D sequences can be divided into four major types:

Type I: even symmetry in both n_1- and n_2-directions, i.e.

$$x(n_1, n_2) = x(N_1 - 1 - n_1, n_2) = x(n_1, N_2 - 1 - n_2) \quad 0 \le n_1 \le N_1 - 1, \; 0 \le n_2 \le N_2 - 1. \quad (4)$$

Such an evev-even sequence is denoted by $\mathbf{X}_{ee}$.

Type II: even symmetry in n_1-direction and odd symmetry in n_2-direction, i.e.

$$x(n_1, n_2) = x(N_1 - 1 - n_1, n_2) = -x(n_1, N_2 - 1 - n_2) \quad 0 \le n_1 \le N_1 - 1, \; 0 \le n_2 \le N_2 - 1. \quad (5)$$

We denote such an even-odd sequence by $\mathbf{X}_{eo}$.

Type III: odd symmetry in n_1-direction and even symmetry in n_2-direction, i.e.

$$x(n_1, n_2) = -x(N_1 - 1 - n_1, n_2) = x(n_1, N_2 - 1 - n_2) \quad 0 \le n_1 \le N_1 - 1, \; 0 \le n_2 \le N_2 - 1. \quad (6)$$

We denote such an odd-even sequence by $\mathbf{X}_{oe}$.

Type IV: odd symmetry in both n_1- and n_2-directions, i.e.

$$x(n_1, n_2) = -x(N_1 - 1 - n_1, n_2) = -x(n_1, N_2 - 1 - n_2) \quad 0 \le n_1 \le N_1 - 1, \; 0 \le n_2 \le N_2 - 1. \quad (7)$$

We denote such an odd-odd sequence by $\mathbf{X}_{oo}$.

For any real $N_1 \times N_2$ 2-D sequence $\mathbf{X}$, it always can be decomposed into the above four type 2-D sequences, i.e.

$$\mathbf{X} = \mathbf{X}_{ee} + \mathbf{X}_{eo} + \mathbf{X}_{oe} + \mathbf{X}_{oo}, \quad (8)$$

and $\mathbf{X}_{ee}$, $\mathbf{X}_{eo}$, $\mathbf{X}_{oe}$ and $\mathbf{X}_{oo}$ can be calculated from $\mathbf{X}$ by

$$\begin{aligned} &x_{ee}(n_1, n_2) \\ &= \tfrac{1}{4}[x(n_1, n_2) + x(N_1 - 1 - n_1, n_2) \\ &+ x(n_1, N_2 - 1 - n_2) + x(N_1 - 1 - n_1, N_2 - 1 - n_2)], \end{aligned} \quad (9)$$

$$\begin{aligned} &x_{eo}(n_1, n_2) \\ &= \tfrac{1}{4}[x(n_1, n_2) + x(N_1 - 1 - n_1, n_2) \\ &- x(n_1, N_2 - 1 - n_2) - x(N_1 - 1 - n_1, N_2 - 1 - n_2)], \end{aligned} \quad (10)$$

$$\begin{aligned} &x_{oe}(n_1, n_2) \\ &= \tfrac{1}{4}[x(n_1, n_2) - x(N_1 - 1 - n_1, n_2) \\ &+ x(n_1, N_2 - 1 - n_2) - x(N_1 - 1 - n_1, N_2 - 1 - n_2)] \end{aligned} \quad (11)$$

and

$$\begin{aligned} &x_{oo}(n_1, n_2) \\ &= \tfrac{1}{4}[x(n_1, n_2) - x(N_1 - 1 - n_1, n_2) \\ &- x(n_1, N_2 - 1 - n_2) + x(N_1 - 1 - n_1, N_2 - 1 - n_2)] \end{aligned} \quad (12)$$

where $x_{ee}(n_1, n_2)$, $x_{eo}(n_1, n_2)$, $x_{oe}(n_1, n_2)$ and $x_{oo}(n_1, n_2)$ are the elements of $\mathbf{X}_{ee}$, $\mathbf{X}_{eo}$, $\mathbf{X}_{oe}$ and $\mathbf{X}_{oo}$ respectively.

III. Sixteen Types of Quadrantally Symmetric Linear-phase FIR 2-D Filters

The frequency response of a 2-D FIR digital filter with its impulse response $h(n_1, n_2)$, $n_1 = 0, 1, \cdots, N_1 - 1$, $n_2 = 0, 1, \cdots, N_2 - 1$ can be characterized as

$$H(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) e^{-jn_1\omega_1} e^{-jn_2\omega_2}. \quad (13)$$

If $h(n_1, n_2)$ is one of four types 2-D sequences, Eq.(13) can be rewritten as

$$H(\omega_1, \omega_2) = e^{-j\frac{N_1-1}{2}\omega_1} e^{-j\frac{N_2-1}{2}\omega_2} e^{j\frac{M\pi}{2}} \hat{H}(\omega_1, \omega_2) \quad (14)$$

where

$$M = \begin{cases} 0 & \textbf{Type I}, \\ 1 & \textbf{Type II} \text{ and } \textbf{Type III}, \\ 2 & \textbf{Type IV}, \end{cases} \quad (15)$$

and $\hat{H}(\omega_1, \omega_2)$ is a real-valued function. Notice that by excluding the linear-phase part in (14), the frequency responses are real-valued functions for **Type I** and **Type IV** sequences, and are imaginary-valued functions for **Type II** and **Type III** sequences. For example, if $h(n_1, n_2)$ is a **Type I** 2-D sequence and N_1, N_2 are odd integers, then

$$\hat{H}(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_1-1}{2}} \sum_{n_2=0}^{\frac{N_2-1}{2}} a(n_1, n_2) cos(n_1\omega_1) cos(n_2\omega_2) \quad (16)$$

which is a real-valued function and $a(n_1, n_2)$ are related to $h(n_1, n_2)$ by

$$\begin{cases} a(0,0) = h(\frac{N_1-1}{2}, \frac{N_2-1}{2}) \\ a(0,n_2) = 2h(\frac{N_1-1}{2}, \frac{N_2-1}{2} - n_2) \\ \qquad n_2 = 1, \cdots, \frac{N_2-1}{2} \\ a(n_1,0) = 2h(\frac{N_1-1}{2} - n_1, \frac{N_2-1}{2}) \\ \qquad n_1 = 1, \cdots, \frac{N_1-1}{2} \\ a(n_1,n_2) = 4h(\frac{N_1-1}{2} - n_1, \frac{N_2-1}{2} - n_2) \\ \qquad n_1 = 1, \cdots, \frac{N_1-1}{2}, \\ \qquad n_2 = 1, \cdots, \frac{N_2-1}{2}. \end{cases} \quad (17)$$

Therefore, according to the four types of 2-D sequences discussed above and their even/odd lengths ($N_1 \times N_2$), there are sixteen different kinds of $\hat{H}(\omega_1, \omega_2)$ which are tabulated in Table I. The relationships between the coefficients $a(n_1, n_2)$s in $\hat{H}(\omega_1, \omega_2)$ and $h(n_1, n_2)$ can be derived easily.

As in the spatial-domain case, any magnitude response $\hat{H}(\omega_1, \omega_2)$ can be similiarly descomposed into four parts in frequency domain as below:

$$\begin{aligned} \hat{H}(\omega_1, \omega_2) = \hat{H}_{ee}(\omega_1, \omega_2) + \hat{H}_{eo}(\omega_1, \omega_2) \\ + \hat{H}_{oe}(\omega_1, \omega_2) + \hat{H}_{oo}(\omega_1, \omega_2) \end{aligned} \quad (18)$$

where

$$\begin{aligned} \hat{H}_{ee}(\omega_1, \omega_2) = \hat{H}_{ee}(-\omega_1, \omega_2) = \hat{H}_{ee}(\omega_1, -\omega_2) \\ = \hat{H}_{ee}(-\omega_1, -\omega_2), \end{aligned} \quad (19)$$

$$\begin{aligned} \hat{H}_{eo}(\omega_1, \omega_2) = \hat{H}_{eo}(-\omega_1, \omega_2) = -\hat{H}_{eo}(\omega_1, -\omega_2) \\ = -\hat{H}_{eo}(-\omega_1, -\omega_2), \end{aligned} \quad (20)$$

$$\begin{aligned} \hat{H}_{oe}(\omega_1, \omega_2) = -\hat{H}_{oe}(-\omega_1, \omega_2) = \hat{H}_{oe}(\omega_1, -\omega_2) \\ = -\hat{H}_{oe}(-\omega_1, -\omega_2) \end{aligned} \quad (21)$$

and

$$\begin{aligned} \hat{H}_{oo}(\omega_1, \omega_2) = -\hat{H}_{oo}(-\omega_1, \omega_2) = -\hat{H}_{oo}(\omega_1, -\omega_2) \\ = \hat{H}_{oo}(-\omega_1, -\omega_2). \end{aligned} \quad (22)$$

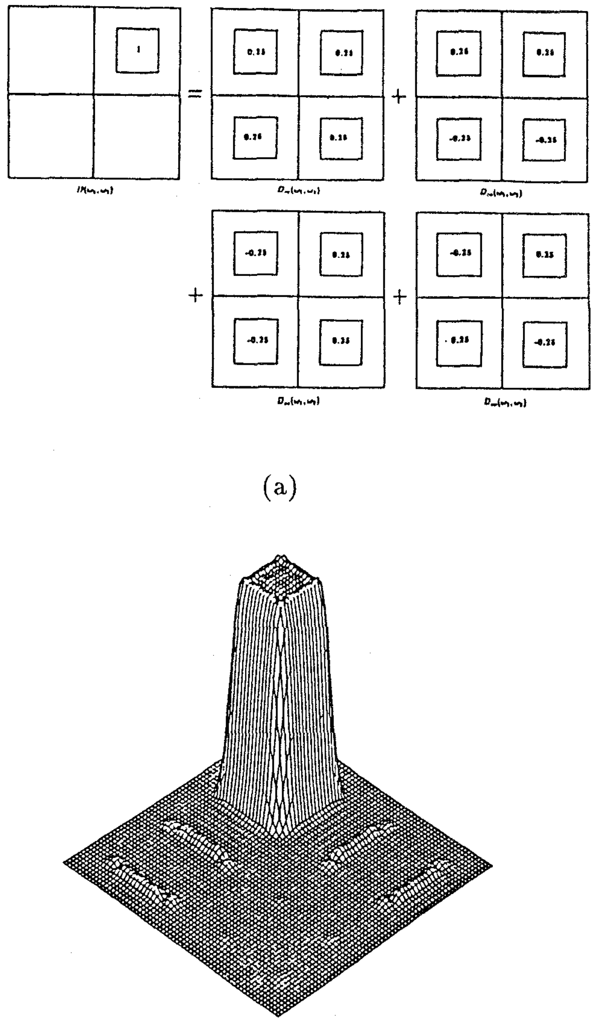
It is noted that $\hat{H}_{ee}(\omega_1, \omega_2)$, $\hat{H}_{eo}(\omega_1, \omega_2)$, $\hat{H}_{oe}(\omega_1, \omega_2)$ and $\hat{H}_{oo}(\omega_1, \omega_2)$ can be realized by **Type I, II, III** and **IV** 2-D sequences, respectively. Hence given a desired magnitude response, we can realize it by either a single type 2-D sequence or several mixed type 2-D sequences. In this paper, the linear-phase 2-D FIR filters are divided into three classes as below:

- **Quadrantal-plane filter**: the filters which can be realized by only a single type 2-D sequence.
- **Half-plane filter**: the filters which can be realized by synthesizing two types 2-D sequences.
- **Full-plane filter**: The filters which can be realized by synthesizing three or four types 2-D sequences.

For example, a full-plane filter with the desired response and its ingredients as shown in Fig.1(a). D_{ee} and D_{oo} can be synthesized by using **Type I** and **Type IV** 2-D sequences respectively, and D_{eo} and D_{oe} can be approached by using **Type II** and **Type III** 2-D sequences. Fig.1(b) shows the resultant amplitude response with filter length 27 × 27, which is designed by the eigenfilter approach [4].

IV. Conclusions

In this paper, we have used the symmetric/antisymmetric 2-D sequences to design quadrantally symmetric linear-phase 2-D FIR filters. It is shown that there are sixteen type cases to be considered according to the symmetry/antisymmetry of 2-D sequences in both directions and their filter lengths (even or odd). The definitions of quadrantal-plane, half-plane and full-plane filters are also defined.



(a)

(b)

Figure 1: Design of a full-band 2-D filter. (a) Desired amplitude response and its ingredients, (b) the resultant amplitude response.

References

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Table I: $\hat{H}(\omega_1, \omega_2)$ of 2-D Sequences with Length $N_1 \times N_2$. ($L_i = \frac{N_i - 1}{2}$ for odd N_i and $L_i = \frac{N_i}{2}$ for even N_i, $i = 1, 2$.)

Type	Sub-type	N_1, N_2	$\hat{H}(\omega_1, \omega_2)$
I	1	N_1:odd, N_2:odd	$\sum_{n_1=0}^{L_1} \sum_{n_2=0}^{L_2} a(n_1,n_2) cos(n_1\omega_1) cos(n_2\omega_2)$
	2	N_1:odd, N_2:even	$\sum_{n_1=0}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) cos(n_1\omega_1) cos((n_2 - \frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1} \sum_{n_2=0}^{L_2} a(n_1,n_2) cos((n_1 - \frac{1}{2})\omega_1) cos(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) cos((n_1 - \frac{1}{2})\omega_1) cos((n_2 - \frac{1}{2})\omega_2)$
II	1	N_1:odd, N_2:odd	$\sum_{n_1=0}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) cos(n_1\omega_1) sin(n_2\omega_2)$
	2	N_1:odd, N_2:even	$\sum_{n_1=0}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) cos(n_1\omega_1) sin((n_2 - \frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) cos((n_1 - \frac{1}{2})\omega_1) sin(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) cos((n_1 - \frac{1}{2})\omega_1) sin((n_2 - \frac{1}{2})\omega_2)$
III	1	N_1:odd, N_2:odd	$\sum_{n_1=1}^{L_1} \sum_{n_2=0}^{L_2} a(n_1,n_2) sin(n_1\omega_1) cos(n_2\omega_2)$
	2	N_1:odd, N_2:even	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) sin(n_1\omega_1) cos((n_2 - \frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1} \sum_{n_2=0}^{L_2} a(n_1,n_2) sin((n_1 - \frac{1}{2})\omega_1) cos(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) sin((n_1 - \frac{1}{2})\omega_1) cos((n_2 - \frac{1}{2})\omega_2)$
IV	1	N_1:odd, N_2:odd	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) sin(n_1\omega_1) sin(n_2\omega_2)$
	2	N_1:odd, N_2:even	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) sin(n_1\omega_1) sin((n_2 - \frac{1}{2})\omega_2)$
	3	N_1:even, N_2:odd	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) sin((n_1 - \frac{1}{2})\omega_1) sin(n_2\omega_2)$
	4	N_1:even, N_2:even	$\sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} a(n_1,n_2) sin((n_1 - \frac{1}{2})\omega_1) sin((n_2 - \frac{1}{2})\omega_2)$