ANALYTIC CLOSED FORMS FOR DESIGNING HIGHER ORDER DIGITAL DIFFERENTIATORS BY EIGEN-APPROACH

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ABSTRACT: Eigenfilter method has been recently proposed for designing higher-order differentiators very effectively. The design method is based on the computation of an eigen-vector of an appropriate real, symmetric and positive-definite matrix. The elements of this matrix is usually evaluated by very time-consuming numerical integration. In this paper, we present simple analytic closed-form formulas to compute these matrix-elements very efficiently. Hence the eigenfilter approach for differentiators becomes much easier and more accurate than before, also the design time is reduced greatly with the larger filter length.

I. INTRODUCTION

Higher-order digital differentiators are very useful for calculation of geometric moments [1] and for biological signal processing [2]. Recently Rahenkamp and Vijaya Kumar [3] have modified the well-known McClellan-Parks program [4] for designing higher-order differentiators. However, this modified McClellan-Parks algorithm often leads to very large deviation or fails to converge especially for designing full-band higher-order differentiators. Pei and Shyu have extended Vaidyanathan and Nguyen's eigenfilter approach [5] to the design of FIR Hilbert transformers and differentiators [6], higher-order digital differentiators [7], and 2-D FIR filters [8]. As outlined in [7], the eigenfilter performance is much better and smoother than the McClellan-Parks algorithm. In this paper, we present a new closed form for designing higher-order differentiators, and we find that it's a very general recursive formula, which can be used to derive arbitrary high-order digital differentiators. Hence the eigenfilter approach becomes much easier and more accurate than before, also the design time is reduced greatly.

II. CLOSED FORM FOR DESIGNING EVEN-ORDER DIFFERENTIATORS

The eigenfilter method is based on minimizing a quadratic measure of the error in the frequency band. Let D(w) and H(w) denote the desired and designed frequency responses of the differentiator, the total error function can be formulated as

$$E=\frac{1}{\pi}\int_0^{w_p}[D(w)-H(w)]^2dw \tag{1}$$

where w_p is the highest radian frequency for which the differentiating action is required. The objective is to express the above error in the following quadratic eigen-formulation:

$$E=A^tQA \tag{2}$$

where t is the vector transpose operation, Q is a real, symmetric, and positive-definite matrix, and A is a real vector related to the filter coefficients in some manner. By Rayleigh principle [9], the eigenvector A associated with the smallest eigenvalue of matrix Q minimizes the total error E.

Assume the filter length is N and it has been shown in [7] that the elements of matrix Q minimizes the total error E.

Assume the filter length is N and it has been shown in [7] that the elements of matrix Q for even-order differentiator are given by

$$q(n,m)=\frac{1}{\pi}\int_0^{w_p}[\frac{w^{2k}}{w_0^{2k}}\cos nw_0\cos mw_0-\frac{w^k}{w_0^k}\cos nw_0\cos mw$$
$$-\frac{w^k}{w_0^k}\cos nw\cos mw_0+\cos nw\cos mw]dw$$
$$N\text{ odd},\ 0\le n,m\le\frac{N-1}{2} \tag{3a}$$

and

$$q(n,m)=\frac{1}{\pi}\int_0^{w_p}[\frac{w^{2k}}{w_0^{2k}}\cos(n-\tfrac{1}{2})w_0\cos(m-\tfrac{1}{2})w_0$$
$$-\frac{w^k}{w_0^k}\cos(n-\tfrac{1}{2})w_0\cos(m-\tfrac{1}{2})w$$
$$-\frac{w^k}{w_0^k}\cos(n-\tfrac{1}{2})w\cos(m-\tfrac{1}{2})w_0$$
$$+\cos(n-\tfrac{1}{2})w\cos(m-\tfrac{1}{2})w]dw$$
$$N\text{ even},\ 1\le n,m\le\frac{N}{2} \tag{3b}$$

where k is even and denotes the order of the

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differentiator, and w_0 is the reference frequency point which is generally chosen at the center of the band in our eigenfilter approach [7]. It is noted that the first term and the last term in Eqs.(3a) and (3b) are easy to obtain closed-form expressions for the integrals, but the other terms are very difficult, especially for higher-order k. Hence numerical integration for evaluating (3) is usually taken by Simpson's rule, this not only results in larger numerical error and also takes much more computation time to design. In the following description, we will derive the closed-form formula for Eq.(3). Let

$$q(n,m)=q_1(n,m)-q_2(n,m)-q_3(n,m)+q_4(n,m) \tag{4}$$

where q_i, i=1,2,3,4 represents the respective result of the four terms in Eq.(3). First suppose N is odd, it is easy to obtain

$$q_1(n,m)=\frac{\cos w_0 \cdot \cos mw_0}{(2k+1)\pi \cdot w_0^{2k}}w_p^{2k+1} \tag{5}$$

and

$$q_4(n,m)=\frac{1}{2\pi}\left[\frac{\sin(n-m)w_p}{n-m}+\frac{\sin(n+m)w_p}{n+m}\right] \tag{6}$$

As for evaluating $q_2(n,m)$ and $q_3(n,m)$, the following equations [10] are used:

$$\int x^n \sin ax\,dx=-\frac{x^n}{a}\cos ax+\frac{n}{a}\int x^{n-1}\cos ax\,dx \tag{7}$$

$$\int x^n \cos ax\,dx=\frac{x^n}{a}\sin ax-\frac{n}{a}\int x^{n-1}\sin ax\,dx \tag{8}$$

Let

$$q_2(n,m)=\frac{\cos nw_0}{\pi \cdot w_0^{k}}\hat{q}_c(m) \tag{9}$$

and by Eqs.(7) and (8), we can get

$$\hat{q}_c(m)=\int_0^{w_p} w^k \cos mw\,dw=\frac{w_p^k \sin w_p}{m}+\sum_{\ell=1}^{\frac{k}{2}}\left[(-1)^{\ell+1}\frac{k(k-1)\dots(k-2\ell+2)w_p^{k-2\ell+1}\cos w_p m}{m^{2\ell}}+(-1)^{\ell+2}\frac{k(k-1)\dots(k-2\ell+1)w_p^{k-2\ell}\sin mw_p}{m^{2\ell+1}}\right],\ m\neq 0 \tag{10}$$

Notice that $q_3(n,m)$ is the same is $q_2(n,m)$ in form by interchanging the variables m and n, we get

$$q_3(n,m)=q_2(m,n)=\frac{\cos mw_0}{\pi \cdot w_0^{k}}\hat{q}_c(n) \tag{11}$$

For N even, the expressions for $q_i(n,m)$, i=1,2,3,4 are very similar to the above ones for N odd, if we replace the variables n, m, n+m by $(n-\frac{1}{2})$, $(m-\frac{1}{2})$ and $(n+m-1)$ respectively as below

$$q_1(n,m)=\frac{\cos(n-\frac{1}{2})w_0 \cdot \cos(m-\frac{1}{2}w_0)}{(2k+1)\pi \cdot w_0^{2k}}w_p^{2k+1} \tag{12a}$$

$$q_2(n,m)=\frac{\cos(n-\frac{1}{2})w_0}{\pi \cdot w_0^{k}}\hat{q}_c(m-\tfrac{1}{2}) \tag{12b}$$

$$q_3(n,m)=\frac{\cos(m-\frac{1}{2})w_0}{\pi \cdot w_0^{k}}\hat{q}_c(n-\tfrac{1}{2}) \tag{12c}$$

and

$$q_4(n,m)=\frac{1}{2\pi}\left[\frac{\sin(n-m)w_p}{n-m}+\frac{\sin(n+m-1)w_p}{n+m-1}\right] \tag{12d}$$

III. CLOSED FORM FOR DESIGNING ODD-ORDER DIFFERENTIATORS

The elements of matrix Q for odd-order differentiator design can be obtained by the similar procedures as above.

$$q(n,m)=q_1(n,m)-q_2(n,m)-q_3(n,m)+q_4(n,m) \tag{13}$$

For N odd,

$$q_1(n,m)=\frac{\sin nw_0 \cdot \sin mw_0}{(2k+1)\pi \cdot w_0^{2k}}w_p^{2k+1} \tag{14}$$

and

$$q_4(n,m)=\frac{1}{2\pi}\left[\frac{\sin(n-m)w_p}{n-m}-\frac{\sin(n+m)w_p}{n+m}\right] \tag{15}$$

Also let

$$q_2(n,m)=\frac{\sin nw_0}{\pi \cdot w_0^{k}}\hat{q}_s(m) \tag{16}$$

and by Eqs.(7) and (8), we can get

$$\hat{q}_s(m)=\int_0^{w_p} w^k \sin mw\,dw=\sum_{\ell=1}^{\frac{k+1}{2}}\left[(-1)^{\ell}\frac{r \cdot w_p^{k-2\ell+2}\cos mw_p}{m^{2\ell-1}}+(-1)^{\ell+1}\frac{k(k-1)\dots(k-2\ell+2)w_p^{k-2\ell+1}\sin mw_p}{m^{2\ell}}\right]\ m\neq 0 \tag{17}$$

where

$$r=\begin{cases}1, & \ell=1\\ k(k-1)(k-2)\dots(k-2\ell+3), & \ell=2,3,\dots,\frac{k+1}{2}\end{cases} \tag{18}$$

Similarly by Eq.(17),

$$q_3(n,m)=q_2(m,n)=\frac{\sin mw_0}{\pi \cdot w_0^{k}}\hat{q}_s(n) \tag{19}$$

For N even, as above if we replace the variables n, m, n+m by $(n-\frac{1}{2})$, $(m-\frac{1}{2})$ and $(n+m-1)$ respectively, we can get

$$q_1(n,m)=\frac{\sin(n-\frac{1}{2})w_0 \cdot \sin(m-\frac{1}{2})w_0}{(2k+1)\,\pi \cdot w_0^{2k}}\, w_p^{2k+1} \qquad (20a)$$

$$q_2(n,m)=\frac{\sin(n-\frac{1}{2})w_0}{\pi \cdot w_0^{k}}\, \hat{q}_s(m-\tfrac{1}{2}) \qquad (20b)$$

$$q_3(n,m)=\frac{\sin(m-\frac{1}{2})w_0}{\pi \cdot w_0^{k}}\, \hat{q}_s(n-\tfrac{1}{2}) \qquad (20c)$$

and

$$q_4(n,m)=\frac{1}{2}\left[\frac{\sin(n-m)w_p}{n-m} - \frac{\sin(n+m-1)w_p}{n+m-1}\right] \qquad (20d)$$

IV. EXAMPLES AND COMPARISONS

In this section, two higher-order differentiators are designed by the closed-form expressions and the numerical integration respectively. We have taken the general rectangular Simpson's rule done on VAX 11/780 computer in Fortran program.

For the fourth-order full-band differentiator design, Figs.1(a) and (b) show the magnitude response and the error curves of a full-band, fourth-order differentiator with length N=31 respectively. Notice that the error by the closed-form method (solid line) is smaller than the numerical integration method (dotted line) in all of the band, especially near the band edge. Fig.1(c) illustrates the design time for various length fourth-order full-band differentiator design from length 11 to 49, the amount of saving time by the closed-form method is greatly increased with the larger filter length, for example with length N=49, the closed-form method is about 4 times faster than the numerical method. As for its accuracy and total error measure in the frequency band, their corresponding minimum eigenvalues for two methods are illustrated in Fig.1(d), obviously the closed-form method is much faster, and its error is also smaller than that of the numerical method.

V. CONCLUSIONS

We have presented a closed-form method for designing arbitrary high-order differentiators by the eigenfilter approach, and have shown that the closed-form method is much faster and gives better performance than the numerical method. Several design examples are used to illustrate the effectiveness of this approach.

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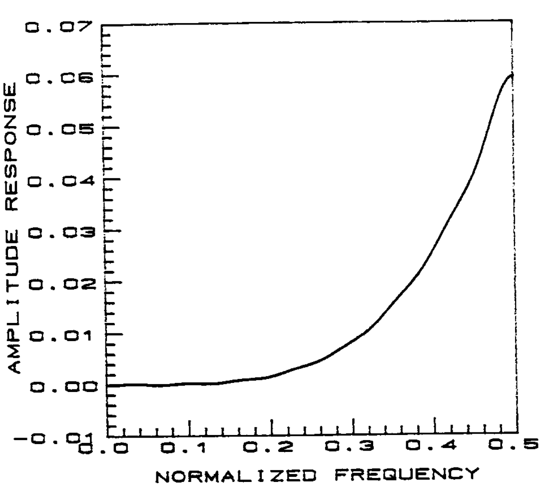

Fig.1.(a)

Comparison of fourth-order, full-band differentiator design with closed-form method and numerical method. (a) the magnitude response of differentiator with length N=31. (b) the error curves for closed-form (solid line) and numerical method (dotted line) (c) design time (d) minimum eigenvalues.

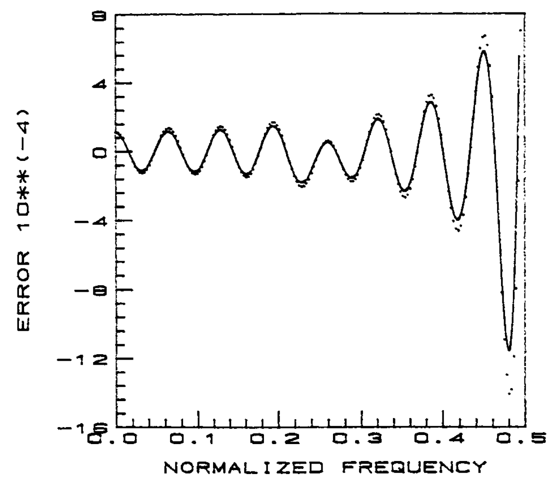

Fig.1.(b)

Comparison of fourth-order, full-band differentiator design with closed-form method and numerical method. (a) the magnitude response of differentiator with length N=31. (b) the error curves for closed-form (solid line) and numerical method (dotted line) (c) design time (d) minimum eigenvalues.

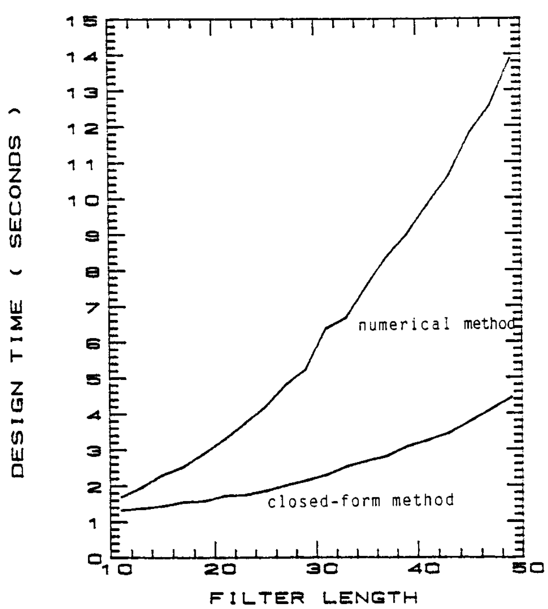

Fig.1.(c)

Comparison of fourth-order, full-band differentiator design with closed-form method and numerical method. (a) the magnitude response of differentiator with length N=31. (b) the error curves for closed-form (solid line) and numerical method (dotted line) (c) design time (d) minimum eigenvalues.

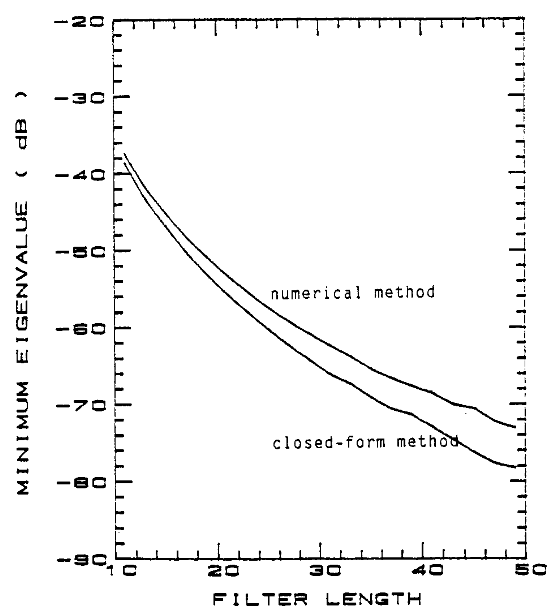

Fig.1.(d)

Comparison of fourth-order, full-band differentiator design with closed-form method and numerical method. (a) the magnitude response of differentiator with length N=31. (b) the error curves for closed-form (solid line) and numerical method (dotted line) (c) design time (d) minimum eigenvalues.