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CLOSED-FORM DESIGN OF MAXIMALLY FLAT R-REGULAR MTH-BAND FIR FILTERS

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ABSTRACT

In this paper, we derive some properties of maximally flat R-regular Mth-band FIR filters. We show that the R-regularity implies maximally flat frequency response at $\omega = 0$. The R-regular constraints are a set of linear equations with complex coefficients. We can convert these complex-value equations to equivalent ones with only real coefficients. We also show that it is possible to completely determine the filter coefficients by R-regularity. Design examples are presented to illustrate the R-regularity properties and the effectiveness of the proposed approach.

1. INTRODUCTION

Mth-band filters are often used to design efficient digital sampling rate conversion systems for real time operations [1]. The impulse response $h(n)$ of an Mth-band FIR filter of order N has the property that

$$h(L + M\ell) = h(L)\,\delta(\ell), \quad \ell = 0, \pm 1, \pm 2, \dots \tag{1}$$

where L is the center of symmetry. That is, one of the M polyphase components is equal to $h(L)z^{-L}$ which is just a delay. An Mth-band filter satisfying the constraint of Eq. (1) in time domain has the following equivalent property in frequency domain [3, 4, 2]

$$\sum_{k=0}^{M-1} e^{j(2\pi k/M)L} H(e^{j(\omega + 2\pi k/M)}) = Mh(L)e^{-j\omega L} \tag{2}$$

where $H(e^{j\omega})$ is the frequency response of the filter. Usually, $h(L)$ is assigned to be $1/M$ for normalization. In [3], constraint of the frequency response and bounds of passband and stopband ripples were derived. In [5], a linear-phase FIR Mth-band filter is decomposed into cascaded several FIR subfilters which is designed simultaneously by Remez-type algorithm. In [4], nonlinear-phase Mth-band FIR filters with reduced group delay were investigated and designed by using the eigenfilter approach.

Mth-band filter with R-regularity implies that there are R zeros at $\omega_k = 2\pi k/M, 1 \le k \le M-1$ on its frequency response. In [4], the R-regularity is transformed into the linear constraints in the impulse response. In this paper, we derive several properties of the R-regular Mth-band FIR filter. We will show that the normalization in Eq. (2) is achieved if the frequency response at $\omega = 0$ is unity. Moreover, the R-regularity also implies maximally flat frequency response at $\omega = 0$. Based on the properties, the linear equations constraining the impulse response can be split into several polyphase equations with fewer coefficients involved.

A lowpass FIR filter with frequency response $F(e^{j\omega})$ is maximally flat if its impulse response is determined by the flatness at $\omega = 0$ and $\omega = \pi$, in which the flatness is defined as the number of zeros of $dF(e^{j\omega})/d\omega$ [2]. Generally speaking, the maximally flat Mth-band FIR filter cannot be completely determined by the flatness at ω_k since the number of flatness does not match the filter coefficients in general . In this paper, we apply the concept of the maximally flatness to design of R-regular Mth-band FIR filters, and give a situation under which the impulse response of Mth-band FIR filter is solved by the flatness at ω_k.

2. PROPERTIES OF R-REGULAR MTH-BAND FIR FILTERS

Let $H(z)$ be the transfer function of an causal Nth order Mth-band FIR filter. In this paper, we consider the general case in which N and L can be arbitrary integers. Suppose $((L))_M = r$ where $((L))_M$ denotes L modulo M, the transfer function $H(z)$ can be expressed as

$$H(z) = \sum_{\substack{n=0 \\ ((n))_M \neq r}}^{N} h(n)z^{-n} + h(L)z^{-L}. \tag{3}$$

If the frequency response $H(e^{j\omega})$ is R-regular with R zeros at $\omega_k = 2\pi k/M$ for $1 \le k \le M-1$, then the transfer function $H(z)$ has R zeros at $z = W^k, 1 \le k \le M-1$, where $W = e^{-j2\pi/M}$ is a Mth root of unity. The location of these zeros implies that $H(z)$ can be factored as $H(z) = H_1(z)[1 + z^{-1} + z^{-2} + \dots + z^{-(M-1)}]^R$.

Since $H(e^{j\omega})$ has R zeros at $\omega_k = 2\pi k/M, 1 \le k \le M-1$, the following equations have to be satisfied:

$$\sum_{\substack{n=0 \\ ((n))_M \neq r}}^{N} h(n)n^q W^{kn} + h(L)L^q W^{kn} = 0$$

or, specifically,

$$\begin{aligned} &\sum_{\substack{n=0 \\ ((n))_M = 0}}^{N} h(n)n^q W^{kn} + \dots + \sum_{\substack{n=0 \\ ((n))_M = r-1}}^{N} h(n)n^q W^{kn} \\ &+ \sum_{\substack{n=0 \\ ((n))_M = r+1}}^{N} h(n)n^q W^{kn} + \dots + \sum_{\substack{n=0 \\ ((n))_M = M-1}}^{N} h(n)n^q W^{kn} \\ &+ h(L)L^q W^{kn} = 0 \end{aligned} \tag{4}$$

for $1 \le k \le M-1$ and $0 \le q \le R-1$. There are $R(M-1)$ equations in Eq. (4). In additional, the frequency response at $\omega =$

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0 is assigned to be unity, that is,

$$H(e^{j0}) = \sum_{\substack{n=0 \\ ((n))_M \neq r}}^{N} h(n) + h(L) = 1. \quad (5)$$

We will show that the R-regular Mth-band FIR filter constrained by Eqs. (4) and (5) has some interesting properties.

Generally speaking, Eqs. (4) and (5) can't be solved exactly since the number of equations is less than the number of unknowns. However, it is possible to simplify these equations. Consider the $R(M-1)$ equations in Eq. (4) and note that

$$W^{k(Mp+q)} = e^{-j2\pi k(Mp+q)/M} = e^{-j2\pi kq/M} = W^{kq}$$

if p and q are integers. Accordingly, Eq. (4) can be written as

$$A_0(q) + W^k A_1(q) + \cdots + W^{(M-1)k} A_{M-1}(q) = 0 \quad (6)$$

for $1 \leq k \leq M-1$ and $0 \leq q \leq R-1$ where

$$A_p(q) = \sum_{\substack{n=0 \\ ((n))_M = p}}^{N} h(n) n^q, \quad (7)$$

for $p = 0, 1, \ldots, r-1, r+1, \ldots, M-1$ and $A_r(q) = h(L)L^q$. By setting $q = 0$ in Eq. (6), we have $M-1$ equations which is represented as

$$A_0(0) + W^k A_1(0) + \cdots + W^{(M-1)k} A_{M-1}(0) = 0,$$

or, equivalently,

$$\sum_{\substack{n=0 \\ ((n))_M = 0}}^{N} h(n) + W^k \sum_{\substack{n=0 \\ ((n))_M = 1}}^{N} h(n) + \cdots + W^{kr} h(L) + \cdots + W^{k(M-1)} \sum_{\substack{n=0 \\ ((n))_M = M-1}}^{N} h(n) = 0 \quad (8)$$

for $1 \leq k \leq M-1$. Dividing Eq. (8) with W^{kr} and adding these $M-1$ equations together, we have

$$\alpha_0 \sum_{\substack{n=0 \\ ((n))_M = 0}}^{N} h(n) + \alpha_1 \sum_{\substack{n=0 \\ ((n))_M = 1}}^{N} h(n) + \cdots + (M-1)h(L) + \cdots + \alpha_{M-1} \sum_{\substack{n=0 \\ ((n))_M = M-1}}^{N} h(n) = 0 \quad (9)$$

where

$$\alpha_p = \sum_{k=1}^{M-1} W^{k(p-r)} = -1$$

for $p = 0, 1, \ldots, r-1, r+1, \cdots, M-1$. Since $p - r$ is not divisible by M, then we have $\alpha_p = -1$ [7]. Consequently, Eq. (9) is simplified as

$$- \sum_{\substack{n=0 \\ ((n))_M = 0}}^{N} h(n) - \sum_{\substack{n=0 \\ ((n))_M = 1}}^{N} h(n) - \cdots + (M-1)h(L) - \cdots - \sum_{\substack{n=0 \\ ((n))_M = M-1}}^{N} h(n) = 0$$

or, alternatively,

$$\sum_{\substack{n=0 \\ ((n))_M = 0}}^{N} h(n) + \sum_{\substack{n=0 \\ ((n))_M = 1}}^{N} h(n) + \cdots + h(L) + \cdots + \sum_{\substack{n=0 \\ ((n))_M = M-1}}^{N} h(n) = Mh(L).$$

However, because we constrain the unity DC gain at $\omega = 0$ and express the constraint by Eq. (5), we can obtain the following property.

Property 1. *For the R-regular Mth-band FIR filter whose frequency response is constrained to be unity at $\omega = 0$, the corresponding impulse response at the center of symmetry is equal to $1/M$, i.e.,*

$$h(L) = \frac{1}{M}, \quad (10)$$

for any filter order N and the center of symmetry L.

Property 1 is interesting and useful. In the design of the Mth-band FIR filters, it is an usual step to assign $1/M$ to $h(L)$. This property implies that this assignment can be replaced by the DC constraint if the transfer function has zeros at $z = W^k$, $1 \leq k \leq M-1$. However, more properties are revealed based on Property 1. Dividing Eq. (6) with W^{kr} and adding these $M-1$ equations together, the resulting R equations can be expressed as

$$-A_0(q) - \cdots + \frac{M-1}{M} L^q - \cdots - A_{M-1}(q) = 0,$$

or

$$A_0(q) + \cdots + \frac{1}{M} L^q + \cdots + A_{M-1}(q) = L^q,$$

or

$$\sum_{\substack{n=0 \\ ((n))_M \neq r}}^{N} h(n) n^q + \frac{1}{M} L^q = L^q \quad (11)$$

for $0 \leq q \leq R-1$. Since the above equations can be regarded as R "flatness" conditions for the frequency response at $\omega = 0$ and can be expressed as

$$\frac{d^q}{d\omega^q} H(e^{j\omega}) \Big|_{\omega=0} - \frac{d^q}{d\omega^q} e^{-jL\omega} \Big|_{\omega=0} = 0,$$

we have the following property.

Property 2. *For the R-regular Mth-band FIR filter with transfer function $H(z)$, $H(z) - z^{-L}$ has a factor of $(1 - z^{-1})^R$. In other words, $H(e^{j\omega})$ is approximated to $e^{-jL\omega}$ at $\omega = 0$ in maximally flat sense.*

We known that the transfer function $H(z)$ of a R-regular Mth-band FIR filter can be factored into the form of $H(z) = H_1(z)[1 + z^{-1} + z^{-2} + \cdots + z^{-(M-1)}]^R$. Property 2 implies a similar factorization of $H(z) - z^{-L} = H_2(z)(1 - z^{-1})^R$. In summary, the R zeros of $H(z)$ at W^k, $1 \leq k \leq M-1$ imply R zeros at unity of $H(z) - z^{-L}$, where W^k together with unity are the whole Mth roots of unity.

To solve the impulse response $h(n)$ constrained by Eqs. (4) or (6) will take more efforts and computing time since the coefficients in these equations are complex due to the factor W. However, it is possible to convert these equations to another ones with real coefficients only. If we regard $A_p(q)$ as unknowns in Eq. (6), there are $M-1$ equations (since $k = 1, 2, \ldots M-1$) and $M-1$

unknowns (since $p = 0, 1, \ldots, r-1, r+1, \ldots, M-1$). That is, $A_p(q)$ may be solved exactly. To find the values of $A_p(q)$, we rewrite Eq. (6) in matrix form as follow

$$\mathbf{Ax} = \mathbf{b} \tag{12}$$

where $\mathbf{A} =$

$$\begin{bmatrix} 1 & W & \cdots & W^{r-1} & W^{r+1} & \cdots & W^{M-1} \\ 1 & W^2 & \cdots & W^{2(r-1)} & W^{2(r+1)} & \cdots & W^{2(M-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{M-1} & \cdots & W^{(M-1)(r-1)} & W^{(M-1)(r+1)} & \cdots & W^{(M-1)^2} \end{bmatrix},$$

$$\mathbf{b} = \left[-\frac{1}{M}W^r L^q, -\frac{1}{M}W^{2r}L^q, \ldots, -\frac{1}{M}W^{(M-1)r}L^q\right]^t,$$

and

$$\mathbf{x} = [A_0(q), A_1(q), \ldots, A_{r-1}(q), A_{r+1}(q), \ldots, A_{M-1}(q)]^t,$$

for $1 \le k \le M-1; 0 \le q \le R-1$ where the superscript ^{t} denotes the transpose. It is easy to find out a solution for Eq. (12) by inspiration. In fact, $A_p(q) = L^q/M$ is an unique solution to Eq. (6) and (12). Note that $\mathbf{A}$ is nonsingular because $\det(\mathbf{A})$ is a *Vandermonde determinant* generated by distinct elements. Therefore, the solution to Eq. (12) is unique [6]. Since the above derivation is based on Property 1 which is deduced from Eq. (5), we obtain the following property.

Property 3. *The $R(M-1)+1$ equations expressed by Eq. (4) as well as Eq. (5) can be simplified to $R(M-1)$ equations represented as*

$$\sum_{\substack{n=0 \\ ((n))_M = p}}^{N} h(n)n^q = \frac{L^q}{M} \tag{13}$$

for $p = 0, 1, \ldots, r-1, r+1, \ldots, M-1$ and $0 \le q \le R-1$.

In summary, Eq. (4) and (5) can be reduced to Eq. (13) as well as $h(L) = 1/M$. However, the coefficients in Eq. (13) are real numbers while the coefficients in Eq. (4) are complex. The computation with real numbers involved needs much less memory than that with complex numbers.

3. DESIGN OF MAXIMALLY FLAT R-REGULAR MTH-BAND FIR FILTERS

In Eqs. (4) and (5), or equivalently, Eq. (13), it is obvious that the number of equations is determined by R and M. For a given M, we have more equations if R is increased. If the number of zeros at W^k is all the same, the number of unknowns is generally not equal to the number of equations, and consequently $h(n)$ can not be solved by Eqs. (4) and (5) only. However, it is possible to solve the impulse response exactly form Eqs. (4) and (5) for some specific M, N and L. That is, it is possible to completely determine the R-regular Mth-band FIR filter by the zeros at W^k for some M, N and L with a suitable choice of R. This design may be regarded as a generalization of the traditional maximally flat filter. In this section, we will indicate the situations to achieve the maximally flat design.

Let N_c denote the number of coefficients to be determined in Eq. (3) and N_p denote the number of filter coefficients in $A_p(q)$ defined by Eq. (7). Let $\lfloor x \rfloor$ denote the largest integer less than x.

It is easy to show that

$$N_c = N - \lfloor \frac{L}{M} \rfloor - \lfloor \frac{N-L}{M} \rfloor + 1, \tag{14}$$

$$N_p = \lfloor \frac{N-p}{M} \rfloor + 1, 0 \le p \le M-1, p \ne r. \tag{15}$$

The following property indicates the sufficient condition for the design of the maximally flat R-regular Mth-band filters.

Property 4. *Eqs. (4) and (5) can be solved exactly if*

$$((N))_M = M-2, ((L))_M = M-1,$$

and

$$R = \lfloor \frac{N}{M} \rfloor + 1.$$

In fact, based on the Property 4, the impulse response can be solved not only by Eqs. (4) and (5), but also by Eqs. (13). The closed form of the impulse response is

$$h(Mm+p) = \frac{(-1)^m \Pi_{i=0}^{R-1}(Mi+p-L)}{(Mm+p-L)m!(R-1-m)!)}$$

for $0 \le m \le R-1$ and $0 \le p \le M-1, p \ne r = ((L))_M$.

4. ILLUSTRATIVE EXAMPLES

In this section, we gives some design examples to illustrate the properties derived in Sec. 2 and 3.

Example 1. In the example, we will design the 14th order R-regular fourth-band FIR filter. The index of symmetry is 7. Thus, we have $M = 4$, $N = 14$, and $L = 7$. According Eq. 14, there is $N_c = 14 - \lfloor 7/4 \rfloor - \lfloor 7/4 \rfloor + 1 = 13$ coefficients to be solved which are

$\{h(0), h(4), h(8), h(12)\}$, $\{h(1), h(5), h(9), h(13)\}$,
$\{h(2), h(6), h(10), h(14)\}$, and $\{h(7)\}$

where each group denotes the unknown coefficients in corresponding polyphase components.

Since $((14))_4 = 4-2$ and $((7))_4 = 4-1$, according to Property 4 these impulse response can be solved by Eqs. (4) and (5) if $R = \lfloor 14/4 \rfloor + 1 = 4$. In fact, the coefficients of each polyphase components can be separated and solved by $(M-1)$ subequations of $A_p(q) = L^q/M$ in Eq. (13). The resulting transfer function is

$$\begin{aligned} H(z) &= \frac{1}{512}\left(-5 - 8z^{-1} - 7z^{-2} + 35z^{-4} + 72z^{-5}\right. \\ &+ 105z^{-6} + 128z^{-7} + 105z^{-8} + 72z^{-9} + 35z^{-10} \\ &+ \left. -7z^{-12} - 8z^{-13} - 5z^{-14}\right). \end{aligned}$$

It is easy to shown that

$$H(z) = \frac{-1}{512}\left(5 - 12z^{-1} + 5z^{-2}\right)\left(1 + z^{-1} + z^{-2} + z^{-3}\right)^4,$$

and $H(z) - z^{-L} =$

$$\begin{aligned} &\frac{-1}{512}\left(5 + 28z^{-1} + 89z^{-2} + 208z^{-3} + 370z^{-4} + 488z^{-5}\right. \\ &\left.+370z^{-6} + 208z^{-7} + 89z^{-8} + 28z^{-9} + 5z^{-10}\right)\left(1 - z^{-1}\right)^4 \end{aligned}$$

The designed filter has linear phase response since the impulse response is symmetric. Fig. 1(a) and (b) show the impulse response and magnitude response, respectively. The magnitude response is flat around $\omega = 0, \pi/2$ and π.

Example 2. In the example, we also design the R-regular fourth-band FIR filter of 14th order. But the index of symmetry is reduced to be 5. Thus, we have $M = 4$, $N = 14$, and $L = 5$. According Eq. 14, there is $N_c = 14 - \lfloor 5/4 \rfloor - \lfloor 9/4 \rfloor + 1 = 12$ coefficients to be solved which are

$$\{h(0), h(4), h(8), h(12)\}, \quad \{h(2), h(6), h(10), h(14)\},$$
$$\{h(3), h(7), h(11)\}, \qquad \text{and } \{h(5)\}$$

where each group denotes the unknown coefficients in corresponding polyphase components.

We assign one degrees of freedom to be the unity DC gain constraint. Since the remain 11 constraints cannot be assigned to the 3 frequencies of $2\pi/4$, $4\pi/4$ and $6\pi/4$ evenly, the maximally flat design cannot be solved. However, if let $R = 3$ in Eqs. (4) and (5) and put the additional 2 degrees of freedom on the flatness at $\omega = \pi$, we can solve the impulse response. The resulting transfer function is

$$\begin{aligned} H(z) &= \frac{1}{2048}\left(-9 + 73z^{-2} + 192z^{-3} + 363z^{-4} + 512z^{-5}\right. \\ &+ 501z^{-6} + 384z^{-7} + 197z^{-8} - 69z^{-10} - 64z^{-11} \\ &+ \left.-39z^{-12} + 7z^{-14}\right). \end{aligned}$$

It is easy to shown that

$$\begin{aligned} H(z) &= \frac{-1}{2048}\left(3 - z^{-1}\right)\left(3 - 14z^{-1} + 7z^{-2}\right) \\ &\times \left(1 + z^{-1}\right)^2 \left(1 + z^{-1} + z^{-2} + z^{-3}\right)^3, \end{aligned}$$

and $H(z) - z^{-L} =$

$$\begin{aligned} &\frac{1}{2048}\left(9 + 27z^{-1} - 19z^{-2} - 321z^{-3} - 1242z^{-4} - 1246z^{-5}\right. \\ &\left.-834z^{-6} - 390z^{-7} - 111z^{-8} + 3z^{-9} + 21z^{-10} + 7z^{-11}\right) \\ &\times \left(1 - z^{-1}\right)^3 \end{aligned}$$

The impulse response is not symmetric. Fig. 2(a) and (b) show the impulse response and magnitude response, respectively. The magnitude response in Fig. 2(b) falls slower than that in Fig. 1(b) since the degrees of flatness at DC in this example are less than the ones in Example 1.

5. CONCLUSIONS

In this paper, several properties of the R-regular Mth-band FIR filter are derived. We show that $h(L) = 1/M$ if the frequency response at $\omega = 0$ is unity. Moreover, if the Mth-band FIR filter is R-regular, then we can show that its frequency response have R degree of flatness at $\omega = 0$. Based on these properties, the linear equations constraining the impulse response can be split into several sub-group equations with fewer real polyphase coefficients involved. Generally speaking, the Mth-band FIR filter cannot be completely determined by the flatness at ω_k. In this paper, we apply the concept of the maximally flatness to the design of R-regular Mth-band FIR filters, and give a situation under which the impulse response of Mth-band FIR filter can be completely solved by the flatness at ω_k. Design examples are presented to verify these properties.

6. REFERENCES

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(a) (b)

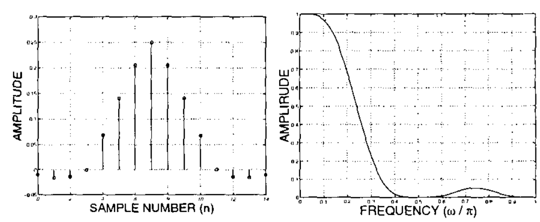

Figure 1: Design of the maximally flat 14th order R-regular fourth-band FIR filter with symmetry index $L = 7$. (a) Impulse response and (b) magnitude response.

(a) (b)

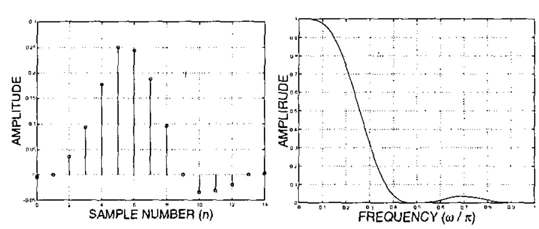

Figure 2: Design of the maximally flat 14th order R-regular fourth-band FIR filter with symmetry index $L = 5$. (a) Impulse response, and (b) magnitude response.