Multi-port Scattering Matrix Measurement Using a Reduced-port Network Analyzer

Hsin-Chia Lu and Tah-Hsiung Chu
Department of Electrical Engineering,
National Taiwan University, Taipei, Taiwan, R.O.C.

Abstract—**A novel method to acquire the scattering matrix of an *n*-port network from the measurements using a reduced-port network analyzer is developed. This method can obtain the scattering matrix of a non-reciprocal or reciprocal *n*-port network with the use of a three-port or two-port network analyzer. The formulation of this method considers the imperfection of terminators used in the measurement, and only two of the terminators are required to be known. Experimental results of a four-port microstrip circuit show the good accuracy using the developed method.**

I. INTRODUCTION

In practice, the multiport scattering matrix of an n-port network is measured with a two-port network analyzer based on the definition of scattering matrix by assuming perfect terminators connected. However, those imperfect terminators used may degrade the measurement performance. Several rigorous methods for solving the scattering matrix of a multiport network from the two-port measurements with imperfect terminators were described in [1-3]. In [4], Lin and Ruan proposed an approach from the port reduction point of view. With their method, the n-port scattering matrix can be reconstructed from n sets of the reduced (n-1)-port scattering matrix by connecting n known terminators at each port at a time. This port reduction process can be continued and the resulting minimal port order for the reduced network is three.

In this paper, we present a novel formulation of the port reduction method (PRM) for solving the n-port scattering matrix from the reduced port measurements. With this method, the minimal order of measured port can be further reduced to be two for a reciprocal n-port network. In addition, only two of the n imperfect terminators used to reduce the measured ports are required to be known instead of n in [4]. Experimental results of four-port non-reciprocal and reciprocal circuits using the developed PRM are given.

This work was supported by the National Science Council of ROC under Grant NSC 88-2213-E-002-056.

II. FORMULATION

For an n-port network, as a terminator with reflection coefficient of Γ_k is connected at the k-th port, the relationship between $S_{ij}^{(k)}$ of this reduced (n-1)-port network and S_{ij} of the n-port network to be solved is given as

$$S_{ij}^{(k)} = S_{ij} + \frac{S_{ik}S_{kj}\Gamma_k}{1 - S_{kk}\Gamma_k}. \quad (1)$$

In (1) the port numbering is the same for the n-port network and the (n-1)-port network.

As given in [4], the matrix equation to relate the diagonal elements of n-port scattering matrix and the elements of reduced (n-1)-port scattering matrices is

$$[R_{n\times n}][S_d] = [S_r], \quad (2)$$

where

$$[S_d] = [\Gamma_1 S_{11} \quad \Gamma_2 S_{22} \quad \cdots \quad \Gamma_n S_{nn}]^t, \quad (3)$$

$$[S_r] = [\Gamma_1 S_{11}^{(2)} - \Gamma_2 S_{22}^{(1)} \quad \Gamma_2 S_{22}^{(3)} - \Gamma_3 S_{33}^{(2)} \quad \cdots \quad \Gamma_1 S_{11}^{(n)} - \Gamma_n S_{nn}^{(1)}]^t, \quad (4)$$

and

$$[R_{n\times n}] = \begin{bmatrix} 1-\Gamma_2 S_{22}^{(1)} & -\left(1-\Gamma_1 S_{11}^{(2)}\right) & 0 & \cdots & 0 & 0 \\ 0 & 1-\Gamma_3 S_{33}^{(2)} & -\left(1-\Gamma_2 S_{22}^{(3)}\right) & \cdots & 0 & 0 \\ 0 & 0 & 1-\Gamma_4 S_{44}^{(3)} & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\Gamma_n S_{nn}^{(n-1)} & -\left(1-\Gamma_{n-1} S_{n-1\ n-1}^{(n)}\right) \\ 1-\Gamma_n S_{nn}^{(1)} & 0 & 0 & \cdots & 0 & -\left(1-\Gamma_1 S_{11}^{(n)}\right) \end{bmatrix}$$

(5)

for $n \geq 3$. In (2), $[R_{n\times n}]$ and $[S_r]$ are matrices related to the reduced (n-1)-port scattering parameters and the reflection coefficients of terminators used, whereas $[S_d]$ contains the diagonal elements to be solved.

However, the determinant of $[R_{n\times n}]$ can be proved to be zero. This means the elements of S_{ii} in (3) can not be solved explicitly, but can be expressed in a polynomial form in terms of one element, such as S_{11}. In addition, we can prove that only two of the terminators used to reduce the measured ports are required to be known instead of n as in [4]. In the following derivation of the formula-

0-7803-5135-5/99/$10.00 © 1999 IEEE

tion for solving off-diagonal elements, S_{11} will be solved. One can then solve all the off-diagonal and diagonal elements, hence reconstruct the n-port scattering matrix.

For the off-diagonal elements, one can obtain the expression as

$$S_{ji}^{(k)}S_{ij}+S_{ij}^{(k)}S_{ji}=S_{ji}^{(k)}S_{ij}^{(k)}+\frac{S_{ii}^{(j)}}{\Gamma_j}-S_{ii}^{(k)}S_{jj}^{(k)}+\left(S_{jj}^{(k)}-\frac{1}{\Gamma_j}\right)S_{ii}+\left(S_{ii}^{(k)}-S_{ii}^{(j)}\right)S_{jj}. \quad (6)$$

Since all the diagonal elements are expressed in terms of S_{11} (or designated as t in the following for simplicity). The right hand side of (6) is then a polynomial of t with the order of one. For $n\geq 4$, if a different port h is taken, (6) becomes

$$S_{ji}^{(h)}S_{ij}+S_{ij}^{(h)}S_{ji}=S_{ji}^{(h)}S_{ij}^{(h)}+\frac{S_{ii}^{(j)}}{\Gamma_j}-S_{ii}^{(h)}S_{jj}^{(h)}+\left(S_{jj}^{(h)}-\frac{1}{\Gamma_j}\right)S_{ii}+\left(S_{ii}^{(h)}-S_{ii}^{(j)}\right)S_{jj}. \quad (7)$$

From (6) and (7), S_{ij} and S_{ji} are polynomials of t with the order of one.

One can now substitute S_{ii}, S_{jj}, S_{ij} and S_{ji} into

$$(S_{ii}^{(j)}-S_{ii})(\frac{1}{\Gamma_j}-S_{jj})=S_{ij}S_{ji}, \quad (8)$$

which is a second-order polynomial equation of t. The correct value of t can be determined by substituting the resulting values of S_{ij} into

$$\Delta=S_{ik}^{(j)}-S_{ik}-\frac{S_{ij}S_{jk}\Gamma_j}{1-S_{jj}\Gamma_j}, \quad (9)$$

to yield a very small value of Δ based on (1). Therefore, all the scattering parameters of an n-port network can be calculated from the reduced (n-1)-port scattering matrices. One can repeat this port reduction process to reduce the measured ports to a minimal order of three as given in [4].

For a reciprocal network,

$$S_{ij}=S_{ji} \text{ and } S_{ij}^{(k)}=S_{ji}^{(k)}. \quad (10)$$

The correct value of t can be solved in a similar manner as that in the non-reciprocal case. However, one can find that the formulation is valid till n is 3. This means that the minimal order of measured ports can be further reduced to be two. Therefore, one can use the derived PRM formulation to acquire the scattering matrix of a reciprocal n-port network from the measurements using a two-port network analyzer.

III. EXPERIMENTAL RESULTS AND VERIFICATION

The experiments on using the developed PRM contain two parts: experiment 1 for reciprocal network and experiment 2 for non-reciprocal network.

In experiment 1, the DUT (or 'R' network) is a microstrip circuit on a 50 *mil* RT/duroid 6006 substrate as shown in Fig. 1(a). The measurement setup is shown in Fig. 2 with two terminators for each port. Terminator 1 is for PRM process, and terminator 2 is for the measurement verification. Table 1 illustrates the port arrangement at different order of PRM process. The first column is the port of a four-port DUT. The ports of intermediate three-port scattering matrices required to reconstruct the four-port scattering matrix are listed in the second column. The type of terminator connected is also given. For example, the first element 123_1 in column 2 represents a three-port scattering matrix of ports 1, 2 and 3 with terminator 1 connected at port 4. The ports of two-port scattering matrices required to reconstruct each three-port scattering matrix are given in the third column. The actual measured ports are given in the last column. There are a total of six sets of two-port scattering matrices to be measured. Typical elements, S_{11}, S_{21} and S_{14}, of the reconstructed four-port scattering matrix of 'R' network are shown in Fig. 3.

In experiment 2, the four-port DUT used in experiment 1 is connected with an isolator (Narda IOS-4080) at port 2 to become a non-reciprocal network (or 'NR' network) as shown in Fig. 1(b). Since it is lack of three-port vector network analyzer, the four sets of three-port scattering matrix corresponded to the second column in Table 1 are calculated using the experiment 1 results of four-port scattering matrix and the measured two-port scattering matrix of isolator. Results of the reconstructed scattering parameters of four-port 'NR' network are shown in Fig. 4.

The reconstructed four-port scattering matrix is verified with the measured two-port scattering matrices by connecting two terminator 2's (50 Ω loads) at the other two ports as illustrated in Fig. 2. In addition, these two-port scattering matrices are calculated using the four-port scattering matrix of 'NR' network with the measured reflection coefficients of 50 Ω loads. Typical measured and calculated results (denoted as MS_{ij} and CS_{ij}) are given in Fig. 4, which shows MS_{ij} and CS_{ij} are almost identical. This shows the accuracy of the reconstructed S_{ij} of 'NR' network. The measured port pair for MS_{ij} is given on the top center of each figure. Some discrepancies can be observed between S_{ij} and MS_{ij} in Fig. 4. This means that the imperfect terminators may degrade measured results by directly using a two-port network analyzer.

IV. REFERENCES

[1] J. C. Tippet and R. A. Speciale, "A rigorous technique for measuring the scattering matrix of a multiport device with a two-port network analyzer," *IEEE Trans. Microwave Theory and Tech.*, vol. MTT-30, no. 5, pp. 661-666, May 1982.

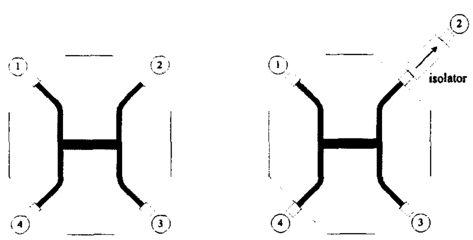
[2] H. Dropkin, "Comments on 'A rigorous technique for measuring the scattering matrix of a multiport device with a two-port network analyzer'," *IEEE Trans. Microwave Theory and Tech.*, vol. MTT-31, no. 1, pp. 79-81, Jan. 1983.

[3] D. Woods, "Multiport-network analysis by matrix renormalization employing voltage-wave S-parameters with complex normalization," *Proc. of Inst. Elec. Eng.*, vol. 124, no. 3, pp. 198-204, Mar. 1977.

[4] W. Lin and C. Ruan, "Measurement and calibration of a universal six-port network analyzer," *IEEE Trans. Microwave Theory and Tech.*, vol. MTT-37, no. 4, pp. 734-742, Apr. 1989.

Ports of resulting four-port S-matrix	Ports of intermediate three-port S-matrix	Ports of measured two-port S-matrix	Ports of actually measured two-port S-matrix
1234	123_1	12_11	12_11
		13_11	13_11
		23_11	23_11
	124_1	12_11	---
		14_11	14_11
		24_11	24_11
	134_1	13_11	---
		14_11	---
		34_11	34_11
	234_1	23_11	---
		24_11	---
		34_11	---

Table 1. Port description of the scattering matrix in PRM process.



(a) (b)

Fig. 1 Circuit layouts of (a) a reciprocal four-port network (or 'R' network) and (b) a non-reciprocal four-port network (or 'NR' network).

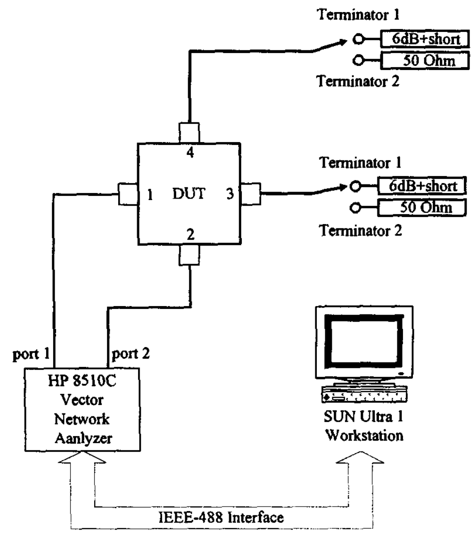


Fig. 2 Measurement arrangement of a four-port network (DUT) with its ports 1 and 2 connected to HP8510C. The actual measured ports are based on Table 1.

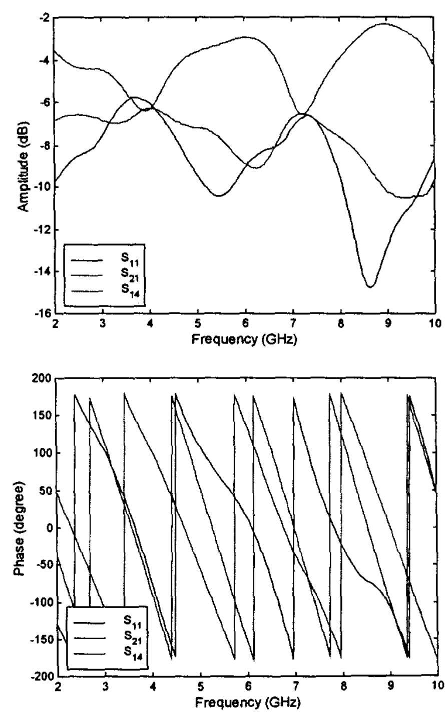
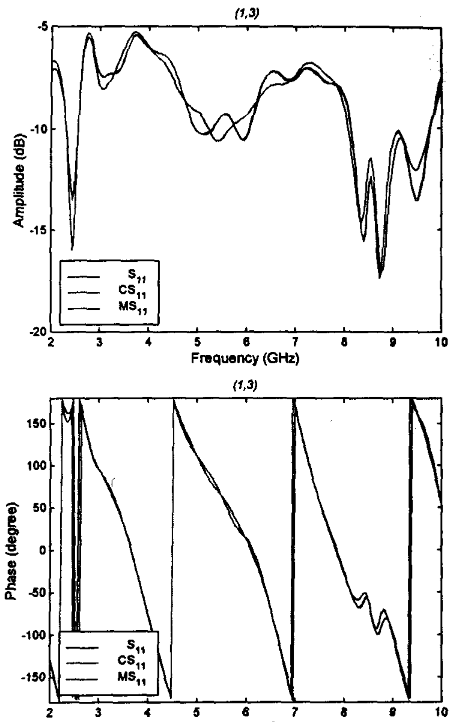
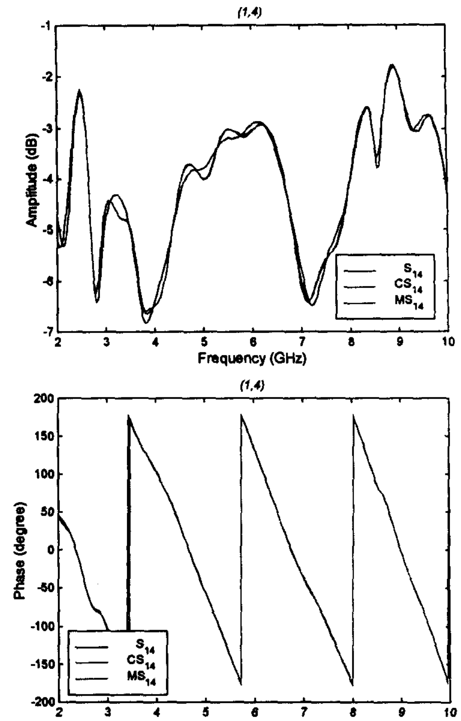


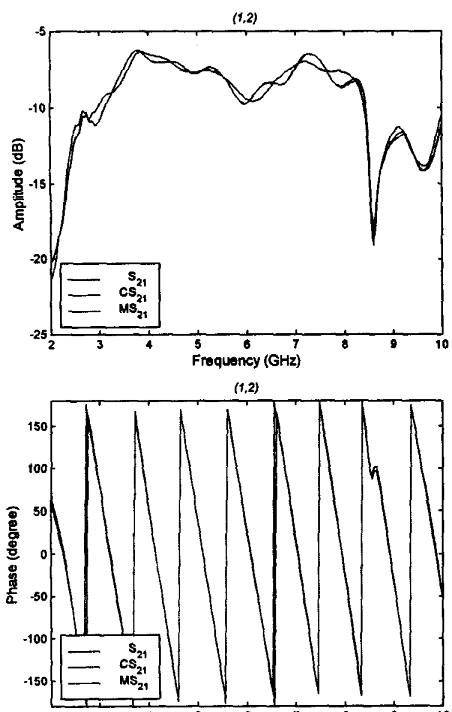
Fig. 3 Results of input (S_{11}), coupled path (S_{21}) and direct path (S_{14}) characteristics of 'R' network using the PRM process.



(a)



(c)



(b)

Fig. 4 Comparison of (a) input (S_{11}), (b) coupled path (S_{21}) and (c) direct path (S_{14}) characteristics of 'NR' network. S_{ij} is the scattering parameter reconctructed by PRM. MS_{ij} is the measured two-port scattering matrix and CS_{ij} is the calculated two-port scattering matrix.