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Set-Membership Identification for Continuous-Time Systems with Nonparametric Uncertainties and Disturbances

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Abstract

An on-line set-membership identification problem is formally formulated for linear time-invariant continuous-time systems which have bounded disturbances as well as nonparametric uncertainties. Based on the formulation, an efficient ellipsoidal-bounding algorithm is proposed to estimate the parameter set of the system from the available input-output timed data.

1 Introduction

"Set-membership identification" is referred to as a class of techniques for estimating parameters of a system under *a priori* information that constrains the solution to certain set. Many theoretical and applied results have been developed in this context, e.g., [1] consider the systems with bounded disturbances, [4] consider the systems with nonparametric uncertainties, and [2] [3] consider the systems with both nonparametric uncertainties and disturbances. However, all the results mentioned above are purely for discrete-time systems.

In this paper, we formally formulate an on-line set-membership identification problem for linear time-invariant single-input single-output continuous-time systems with nonparametric uncertainties and disturbances. The goals of this problem are to on-line find a parameter set estimate which always contains the parameter set of the true system, and to reduce monotonically the estimation error with respect to time. To solve this problem, we use the available input-output timed data to delineate a *posterior* parameter set which accounts for *a priori* knowledge of the nonparametric uncertainty and the disturbance. Then, we derive an ellipsoidal set to bound the intersection of the *prior* parameter set and the *posterior* parameter set. Finally, we propose an efficient algorithm to minimize the volume of the ellipsoidal-bounding set.

Here, we establish some notations and definitions that will be used throughout the paper. Let $\mathbf{R}_0$ and $\mathbf{R}^q$ denote the set of nonnegative real numbers and q dimensional real vectors, respectively. Let $|x|_1 := \sum_i |x_i|$, $|x|_2 := (\sum_i x_i^2)^{1/2}$, and $|x|_\infty := \max_i |x_i|$ denote the 1-norm, 2-norm, and ∞-norm, respectively, in $\mathbf{R}^q$ space. For any $t \in \mathbf{R}_0$ and any signal $x : \mathbf{R}_0 \to \mathbf{R}^q$, we define some norms on x as

$$\|x\|_1 := \int_0^\infty |x(\tau)|_1 \, d\tau \qquad \|x\|_{1,t} := \int_0^t |x(\tau)|_1 \, d\tau$$
$$\|x\|_2 := \left(\int_0^\infty |x(\tau)|_2^2 \, d\tau\right)^{1/2} \qquad \|x\|_{2,t} := \left(\int_0^t |x(\tau)|_2^2 \, d\tau\right)^{1/2}$$
$$\|x\|_\infty := \sup_{\tau\in\mathbf{R}_0} |x(\tau)|_\infty \qquad \|x\|_{\infty,t} := \sup_{\tau\in[0,t]} |x(\tau)|_\infty$$

Let Δ be a linear time-invariant continuous-time system with proper stable transfer matrix $\Delta(s)$. Let h_Δ be the impulse response of Δ and define induced L^2 norm on Δ as

$$\|\Delta\| := \sup_{\|x\|_2 \neq 0} \frac{\|\Delta x\|_2}{\|x\|_2}$$

Let M be a square matrix, then we denote the determinant of M by $\det(M)$, the symmetric positive definiteness of M by $M > 0$, and the symmetric positive semidefiniteness of M by $M \geq 0$.

2 Problem Formulation

The problem is to on-line identify a model set of a system G suitable for (adaptive) robust control design by using the available input-output timed data $\{u(\tau), y(\tau)\}_{\tau=0}^t$, $t \in \mathbf{R}_0$, which are generated by a single-input single-output system described by $y = Gu + d$, where u is an applied input, y is the observed output, and d is the disturbance.

The model set $\mathcal{M}$ is defined as follows:

$$\mathcal{M} := \{y = G_{\theta,\Delta}u + d : G_{\theta,\Delta} \in \mathcal{G},\ d \in \mathcal{D}\}$$

where

$$\mathcal{G} := \{G_{\theta,\Delta} : \theta \in S_0,\ \|\Delta\| \leq 1\}$$
$$\mathcal{D} := \{d = \Delta_V W_V v : \|\Delta_V\| \leq 1,\ \|v\|_\infty \leq 1\}$$

S_0 is known and is referred to as the *prior parameter set*, Δ_V is an uncertain proper stable system, and W_V is a known proper stable system. The transfer function of $G_{\theta,\Delta}$ is characterized by a coprime factor form as follows:

$$G_{\theta,\Delta}(s) = \left(\frac{D_\theta(s)}{\Lambda(s)} + \Delta_D(s)W_D(s)\right)^{-1}\left(\frac{N_\theta(s)}{\Lambda(s)} + \Delta_N(s)W_N(s)\right)$$

where

$$D_\theta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$
$$N_\theta(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0$$
$$\theta = [a_{n-1}, \ldots, a_0, b_m, \ldots, b_0]^T$$
$$\Delta(s) = \begin{bmatrix} \Delta_N(s) & \Delta_D(s) \end{bmatrix},\ W(s) = \begin{bmatrix} W_N(s) & 0 \\ 0 & W_D(s) \end{bmatrix}$$

with $\Lambda(s)$ being a Hurwitz polynomial with degree $\deg(\Lambda) \geq n \geq m$. Δ is an uncertain proper stable system and W is a known proper stable system.

Throughout this paper the following assumption is made.

(A1) $G \in \mathcal{G}$ and $d \in \mathcal{D}$.

$G \in \mathcal{G}$ means that there exists an $G_{\theta^*,\Delta^*} \in \mathcal{G}$ such that $G = G_{\theta^*,\Delta^*}$. We call this specific θ^* a *valid parameter* for representing the system. Because the valid parameter for a system may not be unique, it may seem ambiguous as to which valid parameter should be identified. Thus, the problem formulated here is to identify the *valid parameter set* $\Theta^* := \{\theta^* : G = G_{\theta^*,\Delta^*} \in \mathcal{G}\}$ To proceed, we have to impose some explicit assumptions concerning Θ^* and S_0.

A2: Let $\theta^* = \left[a_{n-1}^*, \ldots, a_0^*, b_m^*, \ldots, b_0^*\right]^T$. For each $k = 0, \ldots, n-1$, the sign of a_k^* is known and is the same for all $\theta^* \in \Theta^*$. Without loss of generality, the sign a_k^* is assumed positive.

A3: $S_0 := \{\theta : (\theta - \hat{\theta}(0))^T P^{-1}(0)(\theta - \hat{\theta}(0)) \leq 1\}$, where $q = m + n + 1$, $\hat{\theta}(0)$ and matrix $P(0)$ are known, and $P(0) > 0$.

To Summarize: The on-line set-membership identification problem is precisely formulated as follows.

Given: (i) $\Lambda(s)$, $W(s)$, m, n, $P(0)$, and $\hat{\theta}(0)$ for which it is known that $G \in \mathcal{G}$, (ii) W_V for which it is known that $d \in \mathcal{D}$, and (iii) $\{u(\tau), y(\tau)\}_{\tau=0}^t$, $t \in \mathbf{R}_0$.

Find: an on-line algorithm $A_t|_{t=0}^\infty$ which maps the given information into an ellipsoid in $\mathbf{R}^q$, namely,

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$$S_t := \{\theta : (\theta - \hat{\theta}(t))^T P^{-1}(t)(\theta - \hat{\theta}(t)) \le 1\}$$

which satisfies

$$\Theta^* \subset S_{t_j} \text{ and } L(S_{t_j}) \le L(S_{t_i}),\ \forall\, t_j \ge t_i \ge 0 \qquad (1)$$

where L denotes Lebesgue measure on $\mathbf{R}^q$. In other words, the ellipsoid S_t contains the valid parameter set Θ^* and the estimation error $L(S_t \setminus \Theta^*)$ is monotonically decreasing.

3 Main Results

We first use the available input-output timed data to delineate a *posterior* parameter set which accounts for *a priori* knowledge of the nonparametric uncertainty and the disturbance as follows.

Lemma 1 *Under assumptions (A1) and (A2), for any $t \in \mathbf{R}_0$, we have $\Theta^* \subset \Theta_t$ where*

$$\Theta_t := \left\{\theta : \|\tilde{y} - \phi^T\theta\|_{2,t}^2 \le 2\theta^T \int_0^t \bar{\psi}(\tau)\bar{\psi}^T(\tau)d\tau\ \theta + 2\beta^2(t)\right\}$$

$$\phi := [-F_{n-1}y, \ldots, -F_0 y, F_m u, \ldots, F_0 u]^T$$

$$\bar{\psi}(\tau) := [\|h_{F_{n-1}} w_V\|_{1,\tau}, \ldots, \|h_{F_0} w_V\|_{1,\tau}, 0, \ldots, 0]^T$$

$$\beta(t) = \|x\|_{2,t} + \left(\int_0^t \|h_{W_D} w_V\|_{1,\tau}^2 d\tau\right)^{1/2} + \left(\int_0^t \|h_{F_n} w_V\|_{1,\tau}^2 d\tau\right)^{1/2}$$

$$x := [W_N u, -W_D y]^T, \qquad \tilde{y} := F_n y$$

and the transfer function of system F_k is defined by $F_k(s) := s^k/\Lambda(s)$, for all $k = 0, \ldots, n$.

Then, we derive an ellipsoidal set to bound the intersection of the *posterior* parameter set Θ_{t_i} and the *prior* parameter set $S_{t_{i-1}}$ which is defined in the following Lemma.

Lemma 2 *Under assumptions (A1) and (A2), if $P(t_{i-1}) > 0$,*

$$\Theta^* \subset S_{t_{i-1}} := \{\theta : (\theta - \hat{\theta}(t_{i-1}))^T P^{-1}(t_{i-1})(\theta - \hat{\theta}(t_{i-1})) \le 1\}$$

$$M(t_i) := \int_0^{t_i} (\phi(\tau)\phi^T(\tau) - 2\bar{\psi}(\tau)\bar{\psi}^T(\tau))d\tau \ge 0 \qquad (2)$$

then, for any $\alpha \in \mathbf{R}_0$, $\Theta^* \subset \{S_{t_{i-1}} \cap \Theta_{t_i}\} \subset \Theta^\alpha_{t_{i-1},t_i}$, where

$$\Theta^\alpha_{t_{i-1},t_i} = \{\theta : (\theta - \hat{\theta}(t_i))^T \Gamma^{-1}(t_i)(\theta - \hat{\theta}(t_i)) \le \gamma(t_i)\}$$

$$\Gamma(t_i) = [P^{-1}(t_{i-1}) + \alpha \int_0^{t_i} \phi(\tau)\phi^T(\tau)d\tau - 2\alpha \int_0^{t_i} \bar{\psi}(\tau)\bar{\psi}^T(\tau)d\tau]^{-1} > 0$$

$$\hat{\theta}(t_i) = \hat{\theta}(t_{i-1}) + \alpha\Gamma(t_i)z(t_i) \qquad (3)$$

$$\gamma(t_i) = 1 + \eta\alpha + z^T(t_i)\Gamma(t_i)z(t_i)\alpha^2 \ge 0$$

$$\eta(t_i) = 2\beta^2(t_i) - \|\tilde{y} - \phi^T\hat{\theta}(t_{i-1})\|_{2,t}^2 + 2\|\bar{\psi}^T\hat{\theta}(t_{i-1})\|_{2,t}^2$$

$$z(t_i) = \int_0^{t_i} (\phi(\tau)(\tilde{y}(\tau) - \phi^T(\tau)\hat{\theta}(t_{i-1})) + 2\bar{\psi}(\tau)\bar{\psi}^T(\tau)\hat{\theta}(t_{i-1}))d\tau$$

The next issue is how to choose α. We propose that α should be chosen to minimize the volume of the ellipsoid $\Theta^\alpha_{t_{i-1},t_i}$ or, equivalently, $\det(\gamma(t_i)\Gamma(t_i))$. To find the minimizer α_i, we need to decompose real matrix $M(t_i)$ into dyadic form as follows:

$$M(t_i) = \sum_{k=1}^{\ell} \varphi_k(t_i)\varphi_k^T(t_i) \qquad (4)$$

where $\ell = \text{rank}(M(t_i))$ and $\varphi_k(t_i) = \lambda_k^{1/2}(t_i)e_k(t_i)$, where $e_k(t_i)$ is an eigenvector associated with a positive eigenvalue $\lambda_k(t_i)$ of $M(t_i)$ for all $k = 1, \ldots, \ell$.

Theorem 1 *Under the statement of Lemma 2, then*

$$det(\gamma(t_i)\Gamma(t_i)) = f(\alpha)\ det(P(t_{i-1}))$$

where $f(\alpha)$ is a rational function of α with finite order and can be calculated by the following steps:
Initialization: *$q = m + n + 1$, $\mu_0 = 1$, and $A_0 = P(t_{i-1})$.*
Step 1: Decompose $M(t_i)$ into the form in (4).
Recursion: *For $k = 1, \ldots, \ell$.*
Step 2: $\mu_k = (1 + \alpha\varphi_k^T(t_i)A_{k-1}\varphi_k(t_i))\mu_{k-1}$
Step 3: $A_k = A_{k-1} - (\alpha A_{k-1}\varphi_k(t_i)\varphi_k^T(t_i)A_{k-1})/(1 + \alpha\varphi_k^T(t_i)A_{k-1}\varphi_k(t_i))$
End Recursion: *If $k < \ell$, increment k and return to Step 2.*
Step 4: $\gamma(t_i) = 1 + \eta\alpha + z^T(t_i)A_\ell z(t_i)\alpha^2$
Step 5: $f(\alpha) = \gamma^q(t_i)/\mu_\ell$
Thus the value of α, say α_i, which minimizes $det(\gamma(t_i)\Gamma(t_i))$, can be analytically found as $\alpha_i = arg\ \min_{\alpha\in\mathcal{A}} f(\alpha)$, where $\mathcal{A} = \{0\} \cup \{\alpha > 0 : \frac{d}{d\alpha}f(\alpha) = 0\}$.

Algorithm: *(Set-Membership Identification, $A_t|_{t=0}^{\infty}$)*
In this algorithm, the required signals $\phi, \bar{\psi}_2, \eta, z$ are constructed and/or calculated on-line, and they are sampled at some time sequence $\{t_k\}$ in order to calculate an estimate S_t of the valid parameter set at real time t. The estimate S_t is calculated as follows:
Initialization: $t_0 = 0$, $\hat{\theta}(t_0) = \hat{\theta}(0)$, and $P(t_0) = P(0)$.
Recursion: For $i = 1, 2, 3, \ldots$

Step 1: Sample signals $\phi, \bar{\psi}, \eta, z$ at time t_i.

Step 2: If (2) is not satisfied, let $P(t_i) = P(t_{i-1})$, $\hat{\theta}(t_i) = \hat{\theta}(t_{i-1})$, and go to Step 7, else, continue.

Step 3: Find a minimizer α_i as described in Theorem 1 and let $\alpha = \alpha_i$.

Step 4: Update $\hat{\theta}(t_i)$ according to (3).

Step 5: If $\gamma(t_i) \le 0$, go to Step 9, else, continue.

Step 6: Update $P(t_i)$ according to $P(t_i) = \gamma(t_i)\Gamma(t_i)$

Step 7: Let $t_{i+1} = t$ and

$$S_t = \{\theta : (\theta - \hat{\theta}(t_i))^T P^{-1}(t_i)(\theta - \hat{\theta}(t_i)) \le 1\},\ \forall\, t \in [t_i, t_{i+1})$$

Step 8: Increment i and return to Step 1.

Step 9: If $\gamma(t_i) = 0$, let $S_t = \{\hat{\theta}(t_i)\}$ for all $t \in [t_i, \infty)$ and then stop. Otherwise give a massage that assumptions (A1), (A2), or (A3) are not consistent with the true system.

Theorem 2 *Under assumptions (A1), (A2), and (A3), the estimate S_t calculated by the algorithm $A_t|_{t=0}^{\infty}$ satisfies (1).*

Remark: The parameter sets we have developed so far assume a coprime factor form of nonparametric uncertainty with bounded induced L^2 norm. This is not a necessary restriction as the sets could also have been developed for other nonparametric uncertainty forms and other norms, e.g., the additive form, the multiplicative form, the induced L^∞ norm, the H^2 norm, or the H^∞ norm.

4 Conclusion

In this paper, we formulate and solve from an ellipsoidal-bounding standpoint an on-line set-membership identification problem for continuous-time systems which have nonparametric uncertainties and disturbances. A concrete on-line algorithm is specified for this problem. In spite of the nonparametric uncertainty and disturbance, this on-line algorithm guarantees that the valid parameter set for representing the true system is always contained in the parameter set estimate and the estimation error is monotonically decreasing with respect to time.

References

[1] E. Fogel and Y.F. Huang, "On the Value of Information in System Identification—Bounded Noise Case", *Automatica*, No. 2, 1982.

[2] R.L. Kosut, M.K. Lau, and S. Boyd, "Set-Membership Identification of Systems with Parametric and Nonparametric Uncertainty", *IEEE Trans. AC*, No. 7, 1992.

[3] B. Wahlberg and L. Ljung, "Hard Frequency-Domain Model Error Bounds from Least-Squares Like Identification Techniques", *IEEE Trans. AC*, No. 7, 1992.

[4] R.C. Younce and C.E. Rohrs, "Identification with Nonparametric Uncertainty", *IEEE Trans. AC*, No. 6, 1992.