Proceedings of the 37th IEEE
Conference on Decision & Control
Tampa, Florida USA • December 1998

Stability Radius of Linear Normal Distributed Parameter Systems with Multiple Directional Perturbations [1]

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Abstract

In this note, the stability robustness problem of linear time-invariant normal distributed parameter systems with multiple bounded or relative bounded directional perturbations is considered. The Lyapunov stability criterion is used to derive the system stability radius, i.e., the extent of perturbation within which the system can keep stability.

1. Introduction

In this paper we consider the robust stability problem of a class of distributed parameter systems (DPS) [1, 2]. Just like the commonly discussed finite dimensional systems, DPS also have the stability and stability robustness problems. However, due to the intricacy of underlying mathematics, it is generally more difficult to study these problems for DPS. Various methods, such as [7], that can be successfully used in finite dimensional systems seem not directly applicable in DPS. In the literature, many authors have managed to study the stability robustness problem of DPS with unstructured bounded perturbation [3], structural perturbation [6], and time-varying perturbation [4].

Here we discuss via the Lyapunov stability approach a case of DPS with multiple structural perturbations, called the directional perturbations, which are operators each multiplied by an unknown constant. It is shown that with this approach the bounded and relative bounded operators can be treated together, and bounds on the unknown constants can be found to ensure the system stability. More specifically, the bound of each unknown constant can be derived separately for each perturbation operator.

2. Problem Formulation

Let Z be a Hilbert Space with the inner product function $\langle\cdot,\cdot\rangle$, and $A_0 : D(A_0) \subset Z \to Z$ be a closed, linear unbounded operator densely defined on Z, with $D(A_0)$ denoting the domain of A_0. Assume that A_0 is the infinitesimal generator of a C_0-Semigroup $T_0(t)$. Thus, the mild solution of the system:

$$\begin{cases} \frac{d}{dt}z(t) = A_0 z(t) \\ z(0) = z_0, \quad z_0 \in Z \end{cases} \tag{1}$$

can be written as $z(t) = T_0(t)z_0$ [1, 2]. We say that A_0 or $T_0(t)$ is exponentially stable if there exist M and $\omega > 0$ such that $\|T_0(t)\| \leq Me^{-\omega t}$.

For an exponentially stable A_0, consider the following system with multiple directional perturbations:

$$\begin{cases} \frac{d}{dt}z(t) = (A_0 + \sum_{i=1}^{N_0} k_i A_i)z(t) \\ z(0) = z_0, \quad z_0 \in Z \end{cases} \tag{2}$$

where for each $i = 1,\ldots,N_0$, A_i is a known perturbation operator, which can be bounded or unbounded. If we let $k = (k_1, ..., k_{N_0}) \in \mathcal{R}^{N_0}$, then we wish to find an upper bound $\bar{k}$ of $\|k\|_2 = \sqrt{\sum_{i=1}^{N_0} k_i^2}$ such that when $\|k\|_2 < \bar{k}$, the system described by (2) is still exponentially stable.

3. Main Result

Theorem 1 *Suppose A_0 is a normal operator [2] generating the C_0-semigroup $T_0(t)$, then the followings are true:*

(1) $(A_0 + A_0^*) : D(A_0) \subset Z \to Z$ is the infinitesimal generator of $T_0^*(t)T_0(t)$, and $A_0 + A_0^*$ is exponentially stable provided A_0 is.

(2) If A_0 is exponentially stable, and $P_0 z = \int_0^\infty T_0^*(t)T_0(t)z\,dt$, then $P_0 z \in D(A_0)$, $P_0 z \in D(A_0^*)$, and $(A_0^* + A_0)P_0 z = -z$, where $A_0 P_0$ and $A_0^* P_0$ both belong to $\mathcal{L}(Z)$, the space of bounded linear operators defined on Z.

Proof: Omitted for the sake of brevity.

For the perturbation operators A_i's, we define the set of relative bounded perturbations of A_0.

Definition 1 *[5] Let $A_0 : D(A_0) \subset Z \to Z$ be an unbounded operator defined on Z. The set of relative bounded perturbation operators with respect to A_0 is defined as $\mathcal{P}_u(A_0) = \{A : D(A) \subset Z \to Z \mid D(A_0) \subset D(A),\ \exists\, \alpha, \beta \geq 0$ such that $\forall z \in D(A_0), \|Az\| \leq \alpha\|A_0 z\| + \beta\|z\|\}$.*

From the definition of $\mathcal{P}_u(A_0)$, we note that $\mathcal{L}(Z) \subset \mathcal{P}_u(A_0)$. A rich amount of examples of relative bounded operators can be found in [5]. It is noted that a relative bounded operator can be unbounded by itself.

[1] This research is supported by the National Science Council of the Republic of China under Grants NSC 86-2213-E-002-018

Theorem 2 *Let $A_0 : D(A_0) \subset Z \rightarrow Z$ be an exponentially stable normal operator, and the C_0-semigroup $T_0(t)$ generated by A_0 satisfy $\|T_0(t)\| \leq Me^{-\omega t}$, where $M, \omega > 0$ are constants. Assume that there are perturbations A_i, $i = 1, \ldots, N_0$, satisfying $A_i^* \in \mathcal{P}_u(A_0^*)$, and $(A_0 + \sum_{i=1}^{N_0} k_i A_i) : D(A_0) \subset Z \rightarrow Z$ generates a C_0-semigroup $T_{N_0}(t)$. Let*

$$\overline{k}_i = \inf_{\|z\|=1} \frac{(1-\epsilon)}{|\langle (P_0 A_i + A_i^* P_0) z, z\rangle|} \tag{3}$$

where $0 < \epsilon < 1$, and $\overline{k} = (\sum_{i=1}^{N_0} (\overline{k}_i)^{-2})^{-\frac{1}{2}}$. If $\|k\|_2 \leq \overline{k}$, then the perturbed system (2) is still exponentially stable.

Proof: Omitted for the sake of brevity.

In this Theorem, the bound $\overline{k}$ is based on $\overline{k}_i$'s, for which the formula involves solving infimum over all z with $\|z\| = 1$. This is not always easy to do for given A_0 and A_i's. To provide simplified but convenient conditions, we give the following two Corollaries.

Corollary 1 *Under the assumption of Theorem 2. If an estimation of the relative bounded coefficients α_i and β_i such that $\|A_i^* z\| \leq \alpha_i \|A_0^* z\| + \beta_i \|z\|$ exists, then a lower bound of $\overline{k}_i$ is*

$$\overline{k}_i \geq \frac{(1-\epsilon)}{(1+R_0)\alpha_i + \frac{M}{2\omega}\beta_i} \tag{4}$$

where $R_0 = \|A_0 P_0\|$ and $0 < \epsilon < 1$.

Proof: Omitted for the sake of brevity.

Corollary 2 *Under the assumption of Theorem 2. If A_0 is further assumed to be self-adjoint, i.e., $A_0^* = A_0$, and a pair of relative bounded coefficients α_i and β_i is known, then another lower bound of $\overline{k}_i$ is*

$$\overline{k}_i \geq \frac{(1-\epsilon)}{\alpha_i + \frac{M}{\omega}\beta_i} \tag{5}$$

where $0 < \epsilon < 1$.

Proof: Omitted for the sake of brevity.

4. Example

The following is an example about diffusion equations. Consider the following system defined on $Z = L_2(0,1)$:

$$\begin{cases} \frac{\partial z}{\partial t} = (\frac{\partial^2}{\partial x^2} + \gamma I)z + k_1(\frac{\partial z}{\partial x}) + k_2 v(x) \int_0^1 h(x) z(t,x) dx \\ z(t,0) = z(t,1) = 0 \\ z(0,x) = z_0 \end{cases}$$

where $h(x) \geq 0$ with $\|h(x)\| = 1$, $\|v(x)\| = 1$, and $\gamma < 0$. Put this system into the framework of our discussion, we have $A_0 z = (\frac{\partial^2}{\partial x^2} + \gamma I)z$ with $D(A_0) = \{z \in Z \mid z \text{ and } \frac{\partial z}{\partial x} \text{ absolutely continuous}, \frac{\partial^2 z}{\partial x^2} \in Z, z(t,0) = z(t,1) = 0\}$. It is easy to verify that $A_0^* = A_0$, i.e., A_0 is self-adjoint, and since $\gamma < 0$, the semigroup $T_0(t)$ satisfies $\|T_0(t)\| \leq e^{\gamma t}$. The perturbation operators are $A_1 z = \frac{\partial}{\partial x} z$, with $D(A_1) = \{z \in Z \mid z \text{ absolutely continuous}, \frac{\partial z}{\partial x} \in Z, z(t,0) = z(t,1) = 0\}$ and $A_2 z = v(x) \int_0^1 h(x) z(t,x) dx\cdot$.

First we check that $A_1 \in \mathcal{P}_u(A_0^*)$ with $\alpha_1 = \frac{1}{n-1}$ and $\beta_1 = \frac{2n(n+1)}{(n-1)}$, where n is any positive integer larger than unity [5]. Also $A_1^* = -A_1$ and $D(A_1^*) = D(A_1)$. Therefore we have

$$\|A_1^* z\| \leq \frac{1}{n-1}\|A_0^* z\| + \frac{2n(n+1)}{(n-1)}\|z\|,$$

Thus by Corollary 2, a lower bound of $\overline{k}_1$ can be obtained as

$$\overline{k}_1 \geq (1-\epsilon)\frac{|\gamma|(n-1)}{|\gamma| + 2n(n+1)}$$

For $\epsilon = 0.1$, we give the resulting values of $\overline{k}_1$, $\overline{k}_2$, and $\overline{k}$ in the following table:

γ, n	$\overline{k}_1$	$\overline{k}_2$	$\overline{k}$
$\gamma = -1, n = 3$	0.072	0.9	0.0717
$\gamma = -2, n = 3$	0.138	1.8	0.1376
$\gamma = -10, n = 4$	0.54	9	0.539

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