Generalized Autoregressive Spectral Estimation

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ABSTRACT

An autoregressive spectral estimation method is developed to reduce the noise effect in prediction coefficient estimation. This method solves the prediction coefficients from a generalized Yule-Walker equation which is formed by the data and its generalized autocorrelation sequence. This method provides several control parameters for the spectral estimator to combat the unmodeled additive noise in the linear least square sense. Through the efficient use of information by this method, data size will be directly helpful in noise suppression.

INTRODUCTION

Autoregressive power spectral density estimation (AR PSD) is a well known high resolution spectral analysis tecnique[1]. One primary limitation to the AR spectral estimator is its sensitivity to white noise. Various approaches have been proposed to combat the noise problem, e.g.,[2-6]. In AR PSD estimation, the linear prediction coefficients (LPC) must be estimated from the Yuler-Walker equation first. Since the autoregressive property holds over different data domains, the LPC can be estimated simultaneous over these domains to reduce its sensitivity to noise. The data domains include data itself, autocorrelation of data and the generalized autocorrelation domains which are to be defined latter. The method of estimating LPC based on data and its generalized autocorrelation sequence is named generalized autoregressive (GAR) method here. The GAR method has two major advantages : a) No change to the AR model of the underlying process. b) lower threshold effect of noise.

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Comparisons of this algorithm to the covariance method will be given latter. Its relative performance to other techniques can be inferred from existing literatures[1].

GENERALIZED AUTOREGRESSION

An AR process $x(n)$ can be represented by the following difference equation.

$$x(n) = \sum_{i=1}^{p} a_i x(n-i) + u(n) \qquad (1)$$
$$n = p, \cdots, N$$

where a_i's are the LPC and $u(n)$ is the driven force.

It is well known that the autocorrelation sequence (ACS) $r_1(k)$ of $x(n)$ obeys the same difference equation also:

$$r_1(k) = \sum_{i=1}^{p} a_i r_1(k-i) \qquad (2)$$
$$k = p, \cdots, p+L_1$$

where L_1 is the effective length of $r_1(k)$. $r_m(k)$ is defined as the $m-th$ generation of autocorrelation sequence hereafter. Treating $r_1(k)$ as a realization of the random process $R_1(\cdot)$, one may perform the autorcorrelation operation on $r_1(k)$ to produce the second generation of ACS $r_2(l)$, and $r_2(l)$ obeys the same difference equation as

$$r_2(l) = \sum_{i=1}^{p} a_i r_2(l-i) \qquad (3)$$
$$l = p, \cdots, p+L_2$$

A generalized Yule-Walker (GYW) equation can be formed by (1),(2) and (3). The resulting equation is

$$(\mathbf{D}^H\mathbf{D}) \cdot \mathbf{a} = \mathbf{D}^H \mathbf{d} \qquad (4)$$

where $\mathbf{D}$ is the generalized data matrix defined as

$$\begin{bmatrix} x(N-1) & \cdots & x(N-p) \\ x(N-2) & \cdots & x(N-p+1) \\ \vdots & & \vdots \\ x(p) & \cdots & x(1) \\ r_1(p+L_1-1) & \cdots & r_1(L_1) \\ r_1(p+L_1-2) & \cdots & r_1(L_1-1) \\ \vdots & & \vdots \\ r_1(p-2) & \cdots & r_1(0) \\ r_2(p+L_2-1) & \cdots & r_2(L_2) \\ r_2(p+L_2-2) & \cdots & r_2(L_2-1) \\ \vdots & & \vdots \\ r_2(p-2) & \cdots & r_2(0) \end{bmatrix},$$

$$\begin{aligned} \mathbf{a} = & \quad (a(1), \cdots, a(p))^T, \\ \mathbf{d} = & \quad (\mathbf{x}, \mathbf{r_1}, \mathbf{r_2})^T, \\ \mathbf{x} = & \quad (x(N), \cdots, x(p+1)), \\ \mathbf{r_1} = & \quad (r_1(p+L_1-1), \cdots r_1(p-1)) \text{ and} \\ \mathbf{r_2} = & \quad (r2(p+L_2-1), \cdots r_2(p-1)). \end{aligned}$$

The AR parameters in $\mathbf{a}$ can be estimated by solving the least square solution of the GYW equation.

This concept can be applied further to include more generations of ACS into the GYW equation to reduce its sensitivity to noise. Longer data will allow longer L_m and more generations of ACS to use in the GYW equation. An additional choice is to add the backward prediction equations into the GYW equation as the covariance method does. In fact, the selection of number of generations of ACS to be used depends not only on the data length but also on the SNR, which will be discussed in next section.

After the LPC is solved, the theorectical AR PSD is

$$P_{xx}(f) = \frac{\sigma_u^2}{|1 - \sum_{k=1}^{p} a_k \exp(j2\pi f)|^2} \qquad (5)$$

where σ_u^2 is the variance (power) of the driven force $u(n)$ in equation (1). Its value can be estimated from the theory of maximum likelihood estimation.

THE GENERALIZED AR METHOD

Although GAR is simple in formulation as above, its implementation to real data corrupted by noise will be complicated by several decision problems. However, comparing to standard AR methods[1], those decisions open the door for the GAR PSD estimator to control the effect of noise.

In real data processing, the random process $x(n)$ is sampled with error and therefore estimating generations of ACF from x(n) is in error also. For the GAR with one generation, the required data are

$$\begin{aligned} y(n) &= x(n) + \Delta x(n) \\ &= \sum_{i=1}^{P} a_i x(n-i) + \Delta x(n) \qquad (6) \end{aligned}$$

and

$$\begin{aligned} r_{yy}(k) &= r_{xx}(k) + \Delta r_{xx}(k) \\ &= \sum_{i=1}^{p} a_i r_{xx}(k-i) + \Delta r_{xx}(k) \qquad (7) \end{aligned}$$

where $y(n)$ is the sample of $x(n)$ with error $\Delta x(n)$ which includes white noise, $r_{yy}(k)$ is the estimate of $r_{xx}(k)$, and $\Delta r_{xx}(k)$ is the estimate error of $r_{xx}(k)$ due to $\Delta x(n)$ and effect of finite data length. The LPC is estimated as the least square error (LSE) solution of eq. (6) and (7). To find an unbiased LSE solution, it is required that $\Delta x(n)$ and $\Delta r_{xx}(k)$ have zero mean and equal variance of $\sigma_{\Delta x}^2$ and $\sigma_{\Delta r}^2$. In general, Δx will be zero mean but $E[\Delta r_{xx}] = 0$ is questionable. Thus the GAR solution could be biased. The other concern of equal variance will required us to utilize weighted least square method by introducing a weight W to make

$$Var[\Delta x] = Var[W \Delta r]. \qquad (8)$$

Thus eq. (7) becomes

$$\begin{aligned} r'_{yy}(k) &= W r_{yy}(k) \\ &= W r_{xx}(k) + W \Delta r_{xx}(k) \\ &= \sum_{i=1}^{p} a_i W r_{xx}(k-i) + W \Delta r_{xx}(k) \, (9) \end{aligned}$$

and the LSE solution is solved from eq. (6) and (9).

In fact, for high SNR, standard AR methods provide the optimum estimate of $\mathbf{a}_i$, generalization will cause bias to the solution. For low SNR, the phenomenon is reversed, i.e. the generalized solution will be more accurate. From the above properties, we must select a best number of generations, this is in genaral a complicated decision problem. For the case of short data with low SNR, it is sure that GAR method will always use the maximum number of generations to combat noise.

There is another decision problem in the selection of the effective length L_m of ACS's used in GYW equation. An emperical choice is $\frac{L_m}{3} \leq L_{m+1} \leq \frac{L_m}{2}$[7], with $L_0 = N$. Since L_m is in general greater than the order of the AR model, the use of $r_m(k)$, with $k > p$, in the GYW will make GAR method possesses the same advantages as those methods using high order Yule-Walker equations[4]. This property combined with the number of generations used in the GYW equation make the GAR method be able to work at low SNR by using longer data. Using more data to combat noise is always a reliable way in the statistical sense.

COMPUTER SIMULATION

The performance of GAR is shown by sinusoidal spectrum estimation first. The signal consists of two complex sinusoids with equal power, at frequencies of 0.30 and 0.35, and N=40. Fig.1~3 show the overlayed spectrums estimated by GAR for three generations, i.e., $m = 0, 1, 2$, when $p = 4$ and $SNR = 6dB$. Note that zero-th generation is exactly the covariance method.

To compare the performance of GAR method with the modified covariance method[1], their capability in probability of resolution[8] are given for different SNR. In the implementation of GAR, an algorithm to estimate the optimum weights for each generation of ACF is required. In this simulation a suboptimum way to implement GAR is used. The algorithm set all weights equal to unit and check the prediction error at successive generations. When the prediction error increases abruptly at certain generation, the algorithm stops and the solution of its previous generation is accepted.

The signals are considered as been resolved if they appear as two zeros of $A(z) = 1 - \sum_{i=1}^{P} a_i z^{-i}$ at their right frequencies and are closest to the unit circle. Fig.4 shows the probability of resolution for the modified covariance method and GAR method when $p = 8$ and SNR is varied from 0 to 25 dB. The improvement in SNR is about 5 dB.

The performance of GAR for general AR process has been verified, but due to the limitation of space, it is not given here.

CONCLUSION

A generalized autoregressive method is proposed for the estimation of model parameters of an AR process imbedded in noise. Some preliminary study does show its capability in noise suppression. This capability comes from efficient use of the information across different data domains. Several control parameters appearing in GAR formulation require further study for their optimum values.

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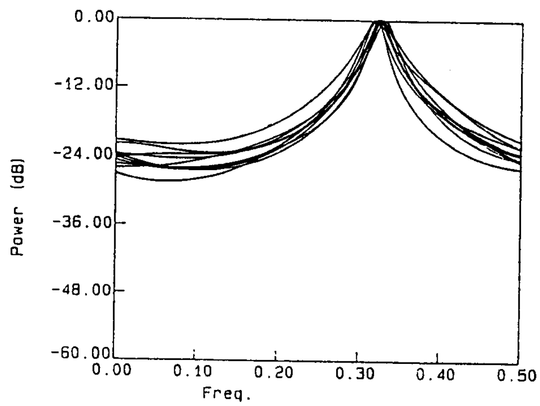

Fig.1 Overlayed spectrum of two equal power, closely spaced sinusoids using the covariance method.

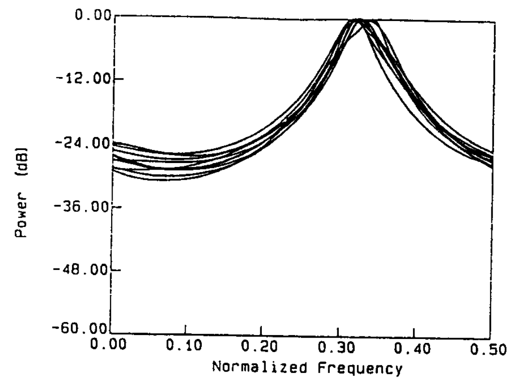

Fig.2 Overlayed spectrum for the same signal as Fig.1, but using GAR with one generation.

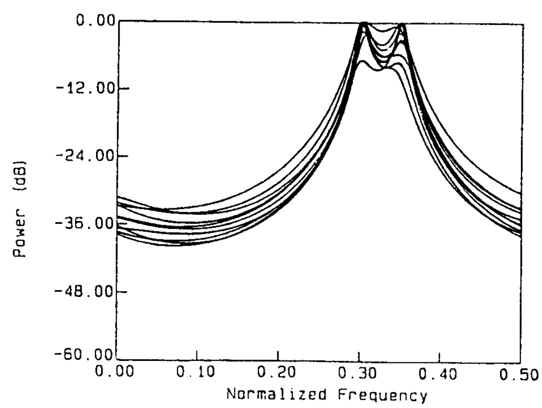

Fig.3 Overlayed spectrum for the same signal as Fig.1, but using GAR with two generation.

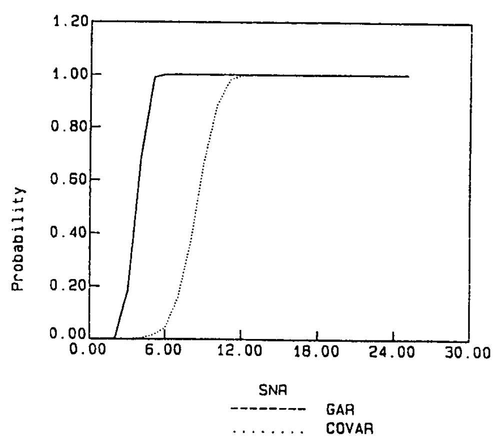

Fig.4 Comparison of probability of resolution for two equal power sinusoids with different SNR.