An Improved Approach for Importance Sampling Simulation in Multilevel Digital Transmission Systems

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Abstract

Computer simulation is currently the most powerful approach in evaluation of the error rate performance of a digital transmission system. Classical Monte Carlo (MC) method is the fundamental method, but requires a very large sample size. The importance sampling (IS) technique has been proposed to significantly reduce the required sample size for estimating the bit error rate (BER) of binary systems. In this paper an improved approach for this problem is presented, in which the BER and symbol error rate (SER) of binary as well as multilevel systems are discussed for both classical MC and various IS techniques. Numerical simulation results are further provided to support the discussion.

1 Introduction

Evaluation of the error rate performance of a digital transmission system is the most basic problem in communication system design. The concerned error rate can be bit error rate (BER), or symbol error rate (SER). For systems which are not too complex, analysis is always helpful to estimate the error rate performance. Yet simulation is the most legitimate method for obtaining the accurate estimate, which can then be used to verify the analysis. For systems which are too complex to analyze, simulation is in fact the only approach to obtain the detailed performance.

The fundamental method of estimating the error rate from a simulation is the classical Monte Carlo (MC) method [1], which consists of basically counting errors. The more advanced techniques for simulation include the "variance-reduction" methods [1], in which importance sampling (IS) is one of the most attractive methods to be used in simulating digital transmission systems. In the last decade, substantial progress has been achieved in the simulation of BER of binary systems using IS techniques [2, 3, 4].

In this paper, we extend the formulation for both classical MC and IS techniques to include BER and SER of multilevel systems based on symbol by symbol observation. A new IS technique and analysis of BER and SER based on the extended formulation are also presented. In the following, we first briefly summarize the previous work on BER of binary systems in Section 2 for development purposes. The extended formulation is then presented in Section 3. In Section 4.1 the conventional IS technique for BER of binary systems is shown to be applicable in simulating BER and SER of M-ary systems, and a new improved IS technique is presented in Section 4.2. Numerical results for all the discussed techniqus are provided in Section 5. Finally the conclusion is given in Section 6.

2 Summary of Previous Work

We shall very briefly summarize in this section the previous work on transmission performance simulation using vector channel model for further developments in this paper. A discrete-time vector channel model [5, 6] for binary PAM is depicted in Figure 1. Let $\{X_k\}$ be a sequence of random symbols such that the outcome $x_k \in X = \{x | x = \pm 1\}$, and let $g = \{g_0, g_1, \ldots, g_\nu\}$ be the normalized channel impulse response such that $\sum_{i=0}^{\nu} g_i^2 = 1$, then the channel output sequence is $\{Z_k\} = \{X_k\} * g + \{N_k\}$, where '$*$' stands for convolution and N_k is additive white Gaussian noise (AWGN) with variance σ^2. The system is intersymbol-interference (ISI) free if $\nu = 0$. The system has small to moderate ISI if $\nu \geq 1$ and g_0 dominates over all $g_i, 1 \leq i \leq \nu$, where ν is called the ISI *span*. The appropriate detector for this system is simply a threshold detector with the threshold set at V_T. The detector decides $\hat{X}_k = -1$ if $Z_k < V_T$ and $\hat{X}_k = 1$ otherwise, where $\hat{X}_k$ is the estimate of X_k at the detector output.

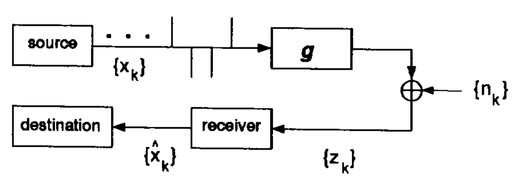

Figure 1: Vector channel model for PAM and QAM transmission systems.

If we assume $P_{e0} = P_{e1}$, where $P_{ei}, i = 0, 1$ are the error rate when 0 and 1 are transmitted, then

$$P_e = P_{e0} = \int_{-\infty}^{\infty} h_0(z_k) f_{Z_k}(z_k)\, dz_k = E[h_0(z_k)] \quad (1)$$

where P_e is the BER, $f_{Z_k}(z_k)$ is the probability density

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function (p.d.f.) of Z_k, and

$$h_0(z_k) = \begin{cases} 1, & z_k > V_T \\ 0, & z_k < V_T. \end{cases} \quad (2)$$

The classical MC estimator of P_e is

$$\hat{P}_e = \frac{1}{N}\sum_{k=0}^{N-1} h_0(z_k) \quad (3)$$

where N is the number of simulation samples in classical MC simulation. Let $\sigma^2_{\rm MC}$ be the variance of the classical MC estimator $\hat{P}_e$, Gaussian approximation will show that $(\hat{P}_e - 1.96\sigma_{\rm MC}, \hat{P}_e + 1.96\sigma_{\rm MC})$ is a 95 percent confidence interval for P_e, and it requires approximately $10/P_e$ samples to achieve an estimator variance $\sigma^2_{\rm MC} = P_e^2/10$ and thus an interval $(\hat{P}_e - 0.62P_e, \hat{P}_e + 0.62P_e)$ of 95 percent confidence [2]. For a typical error probability of, say, 10^{-6}, classical MC requires 10^7 samples per simulation run, which is impractical.

The IS technique is a variance-reduction method that can obtain a good estimate of P_e with dramatically less samples. It simply consists of introducing another p.d.f. $f^*_{Z_k}(z_k)$ which is preferable for sampling purposes. That is, we rewrite Equation (1) as

$$P_e = \int_{-\infty}^{\infty} h_0(z_k) w^*_{Z_k}(z_k) f^*_{Z_k}(z_k)\,dz_k = E[h_0(z_k) w^*_{Z_k}(z_k)] \quad (4)$$

where $w^*_{Z_k}(z_k) = f_{Z_k}(z_k)/f^*_{Z_k}(z_k)$ is the likelihood ratio, or "weight" of $f_{Z_k}(z_k)$ with respect to $f^*_{Z_k}(z_k)$, and P_e is now estimated by

$$\hat{P}^*_e = \frac{1}{N^*}\sum_{k=0}^{N^*-1} h_0(z_k) w^*_{Z_k}(z_k) \quad (5)$$

where N^* is the number of simulation sample in IS simulation. If $\hat{P}^*_e$ is also approximated by Gaussian distribution, let $\sigma^2_{\rm IS}$ be the variance of the IS estimator $\hat{P}^*_e$, then if $\sigma^2_{\rm IS} = \sigma^2_{\rm MC} = P_e^2/10$, the confidence interval will be $(\hat{P}_e - 0.62P_e, \hat{P}_e + 0.62P_e)$, which is identical to that of the classical MC with the same estimator variance. An efficient IS technique should achieve $\sigma^2_{\rm IS} = P_e^2/10$ with N^* as small as possible, and the efficiency of an IS technique is measured as $r_{\rm MC/IS} = N/N^*$ with $\sigma^2_{\rm IS} = \sigma^2_{\rm MC}$.

Two IS techniques for BER of binary systems have been proven to be very successful, one is the conventional IS (CIS) proposed by Shanmugan [2], the other is the improved IS (IIS) proposed by Lu and Yao [4]. Instead of directly generating the new p.d.f. $f^*_{Z_k}(z_k)$, both techniques generate a new noise p.d.f. $f^*_{N_k}(n_k)$ to indirectly change the p.d.f. of Z_k. The CIS technique [2] is to generate the zero mean Gaussian noise with a new variance $\sigma^2_* > \sigma^2$, i.e.,

$$f^*_{N_k}(n_k) = \frac{1}{\sqrt{2\pi}\sigma_*}\exp\left(-\frac{n_k^2}{2\sigma_*^2}\right). \quad (6)$$

This techniques is very robust, and will be refered to as CIS in all the following discussions. The IIS technique [4] consists of translating the noise p.d.f. by a constant c, i.e.,

$$f^{**}_{N_k}(n_k) = f_{N_k}(n_k - c). \quad (7)$$

This technique is even more efficient than CIS, and will be refered to as IIS in all the following discussions.

3 Reformulation

As we summarized in Section 2, only BER of binary systems is seriously considered in the previous literature. However, many communication systems used today employ multilevel modulations. Hence we shall reformulate the problem for a more generalized analysis.

3.1 Vector Channel Model

The vector channel model for M-ary PAM is basically the same as that of binary PAM depicted in Figure 1, except that the outcome $x_k \in X = \{x \mid x = \pm1, \pm3, \ldots, \pm(M-1)\}$. The appropriate detector here is now the symbol by symbol dectector. Let $\mathcal{D}(x)$ be the decision region of $x \in X$, the detector decides that $\hat{X}_k = x$ if $Z_k \in \mathcal{D}(x)$. The decision regions for $M = 2$ and 4 are plotted in Figure 2 (a) and (b). Note that the minimum Euclidean distance (ED) of these systems is $2g_0$.

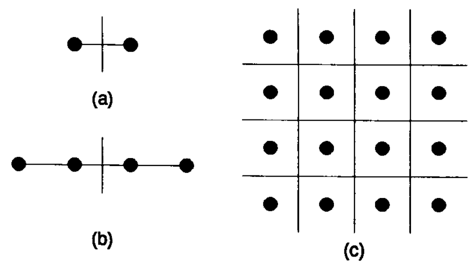

Figure 2: Decision regions for various PAM and QAM constellations.

The performance measures we consider in this paper are both BER and SER for all M. The SER is defined as

$$P_s = P[\hat{X}_k \neq X_k] \quad (8)$$

where $\hat{X}_k$ is the estimated symbol at the detector output. The classical MC simulation is simply counting the relative frequency of the symbol error. Define the symbol error indicator function

$$I_E(x_k, \hat{x}_k) = \begin{cases} 1 & \hat{x}_k \neq x_k \\ 0 & \hat{x}_k = x_k \end{cases} \quad (9)$$

and then we have

$$P_s = E[I_E(x_k, \hat{x}_k)] \quad (10)$$

the expectation of the indicator function.

For binary systems, i.e., the case of $M = 2$, SER is the same as BER. For the case of $M = 2^m, m \geq 2$, BER can be obtained by identifying the number of erroneous bits in the erroneous symbols. This can be expressed by the decomposition of the symbol error indicator

$$I_E(x_k, \hat{x}_k) = \sum_{x \neq x'} I_{x,x'}(x_k, \hat{x}_k) \qquad (11)$$

where

$$I_{x,x'}(x_k, \hat{x}_k) = \begin{cases} 1 & (x_k, \hat{x}_k) = (x, x') \\ 0 & \text{otherwise} \end{cases} \qquad (12)$$

is the indicator function of the symbol $x \in X$ being transmitted but the symbol $x' \in X$ being detected, and $(x_k, \hat{x}_k)$ is the 2-tuple of the actually transmitted and estimated symbols, and '$\times$' stands for Cartesian product. Let $b(\cdot)$ be the binary representation of a symbol, and $d_H(b(x), b(x'))$ be the Hamming distance between the binary representation of x and x', $(x, x') \in X \times X$. Then the BER is

$$P_b = E\left[\sum_{x \neq x'} \frac{d_H(b(x), b(x'))}{m} I_{x,x'}(x_k, \hat{x}_k)\right] \qquad (13)$$

where m is the number of bits of the binary representation.

For QAM, the above model for PAM still applies, except that the real symbol X_k is replaced by the complex symbol X_k^c because two orthogonal carriers are used, where $X_k^c = X_k^i + jX_k^q$, $X_k^c \in X^c$, where X^c is a two-dimensional constellation, the real-valued output Z_k replaced by the complex-valued output Z_k^c, and the real-valued noise N_k by the complex-valued noise N_k^c. The estimated symbol $\hat{X}_k$ is also replaced by the complex symbol $\hat{X}_k^c$. The two-dimensional constellation 16-QAM its decision regions are plotted in Figure 2 (c). For notational convenience, we will use the notation of the PAM systems to refer to both the PAM and QAM systems in the following discussion.

3.2 Classical MC Simulation

With a slight abuse of notation, we will use the density function in the generic sense for both the "probability density function" of the continuous variable and the "probability mass function" of the discrete variable. Let

$$\boldsymbol{X}_k = \{X_{k-\nu}, X_{k-\nu+1}, \ldots, X_k\} \qquad (14)$$

be the vector of consecutive random symbols of length $\nu + 1$ ending at time k, the SER can be expressed as

$$P_s = E[I_E(\boldsymbol{x}_k, n_k)] = E[I_E(x_k, \hat{x}_k)]. \qquad (15)$$

The classical MC estimator of the SER is then the sample mean

$$\hat{P}_s = \frac{1}{N} \sum_{k=0}^{N-1} I_E(x_k, \hat{x}_k) \qquad (16)$$

which is simply the relative frequency of symbol errors. Likewise, the classical MC estimator of the BER is

$$\hat{P}_b = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{x \neq x'} \frac{d_H(b(x), b(x'))}{m} I_{x,x'}(x_k, \hat{x}_k). \qquad (17)$$

Since the BER estimator is based on the decomposition of the SER estimator, we will focus on the SER estimator in the following.

3.3 IS Simulation

To apply IS technique, we generate the noise N_k by a density $f^*_{N_k}(n_k)$ different from $f_{N_k}(n_k)$. Let

$$w^*_{N_k}(n_k) = f_{N_k}(n_k)/f^*_{N_k}(n_k) \qquad (18)$$

be the "weight", or "likelihood ratio" of $f_{N_k}(n_k)$ with respect to $f^*_{N_k}(n_k)$. Then $I_E(x_k, \hat{x}_k)w^*_{N_k}(n_k)$ is an unbiased indicator of P_s under the new density $f^*_{N_k}(n_k)$, since

$$E(I_E(x_k, \hat{x}_k)w^*_{N_k}(n_k)) = P_s. \qquad (19)$$

Likewise,

$$\sum_{x \neq x'} \frac{d_H(b(x), b(x'))}{m} I_{x,x'}(x_k, \hat{x}_k) w^*_{N_k}(n_k)$$

is an unbiased indicator of P_b. The IS estimator of P_s is then

$$\hat{P}^*_s = \frac{1}{N^*} \sum_{k=0}^{N^*-1} I_E(x_k, \hat{x}_k) w^*_{N_k}(n_k) \qquad (20)$$

and the IS estimator of P_b is

$$\hat{P}^*_b = \frac{1}{N^*} \sum_{k=0}^{N^*-1} \sum_{x \neq x'} \frac{d_H(b(x), b(x'))}{m} I_{x,x'}(x_k, \hat{x}_k) w^*_{N_k}(n_k). \qquad (21)$$

Again, because the IS BER estimator is also based on the IS SER estimator, a good IS technique for SER will be a good IS technique for BER. We will thus concentrate on the IS techniques for SER in the following.

4 Two IS Techniques

Because of the decision regions for M-ary systems in Figure 2 and the fact that all symbols are assumed to be approximately equally probable, the new noise density function used in IS should be symmetric with respect to the vertical axis to minimize the estimator variance. Thus, only the CIS method can also be used here for M-ary systems, while the IIS method can not be directly used because it is not symmetric. However, as described below we can modify IIS to obtain a new technique to be used here. In the following both these techniques will be discussed first assuming an ideal ISI free channel for analytical tractability.

4.1 The Conventional Importance Sampling (CIS) Technique

Case A: Binary PAM

This has been analyzed in the work of Lu and Yao [4] with results summarized below: the optimal value of σ_* is

$$\sigma_{*,\text{opt}} \approx \sqrt{1 + \sigma^2 + \frac{9}{16}\sigma^4} \qquad (22)$$

and the efficiency of this technique as compared to classical MC is

$$r_{MC/CIS} \approx 2\sigma \exp(\frac{1}{2\sigma^2} - \frac{1}{2}) \tag{23}$$

which implies a dramatic decrease in the required simulation samples when $\sigma^2_{CIS,opt} = \sigma^2_{MC}$.

Case B: Multilevel PAM and QAM

For multilevel PAM with $M = 2^m, m \geq 2$, if all the symbols are equally probable, we have a very similar result

$$r_{MC/CIS} \approx \frac{\sigma}{1 - 2^{-m}} \exp(\frac{1}{2\sigma^2} - \frac{1}{2}) \tag{24}$$

and the optimum value of σ_* is also given by Equation (22).

For 4QAM we have

$$\sigma_{*,opt} \approx \sqrt{\frac{1}{2} + \frac{5}{4}\sigma^2 + \frac{49}{32}\sigma^4} \tag{25}$$

and

$$r_{MC/CIS} \approx \frac{\sigma^2 d}{\sigma_*} \exp(\frac{1}{2\sigma^2} - \frac{1}{2}). \tag{26}$$

For other QAM constellations, if the minimum ED is 2, the optimum value of σ_* is also approximately given by Equation (25), and the efficiency depends on the particular constellation used, and lies between one and two times that of 4QAM.

4.2 A New Improved Importance Sampling (NIIS) Technique

To achieve an IS technique as efficient as possible, we should have the weight $f_{N_k}(n_k)/f^*_{N_k}(n_k)$ as close to constant as possible in the important region[1], i.e., the region for $I_E(x_k, \hat{x}_k) = 1$. But at the same time, $f^*_{N_k}(n_k)$ must be a density function that is easy to generate, and the resulting technique must be robust. These are obviously contradicting constraints. The CIS technique discussed in Section 4.1 is robust, and the noise density function is easy to generate, but the weight

$$f_{N_k}(n_k)/f^*_{N_k}(n_k) = \frac{\sigma^*}{\sigma} \exp(\frac{-n_k^2}{\sigma^2 2 d_1^{-2}}) \tag{27}$$

is not so close to constant as it decays exponentially with the n_k^2, where

$$d_1 = \sqrt{1 - \frac{\sigma^2}{\sigma_*^2}}. \tag{28}$$

To improve on this, we propose here to use a new noise density

$$f^{***}_{N_k}(n_k) = \frac{1}{2} f_{N_k}(n_k + c) + \frac{1}{2} f_{N_k}(n_k - c) \tag{29}$$

where c is a parameter to be optimized. As plotted in Figure 3, the density is derived from the Gaussian density simply by a symmetric shift, hence is very easy to generate and the resulting technique will be robust. For $c > 2\sigma$ and $n_k > 1$, the weight for this density

$$f_{N_k}(n_k)/f^{***}_{N_k}(n_k) \approx 2 \exp(\frac{-2c|n_k| + c^2}{2\sigma}) \tag{30}$$

decays exponentially with only $|n_k|$, which is closer to constant. This method will be refered to as the new improved importance sampling (NIIS) technique in all the following discussions.

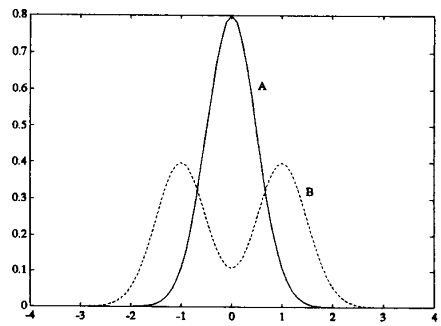

Figure 3: Density function of the Gaussian noise $f_{N_k}(n_k)$ (curve A) and the proposed noise $f^{***}_{N_k}(n_k)$ (curve B).

Following the same approach as in the preceding subsection, the variance σ^2_{NIIS} of the NIIS technique can be easily evaluated.

Case A: Binary System

For binary system,

$$\sigma^2(I_E(x_k, \hat{x}_k) w^{***}_{N_k}(n_k)) \approx 2 \exp(\frac{c^2}{\sigma^2}) Q(\frac{1+c}{\sigma}) - Q^2(\frac{1}{\sigma}) \tag{31}$$

and thus

$$\sigma^2_{NIIS} \approx \frac{1}{N^{***}} \left[2 \exp(\frac{c^2}{\sigma^2}) Q(\frac{1+c}{\sigma}) - Q^2(\frac{1}{\sigma}) \right]. \tag{32}$$

Using the approximation $Q(t) \approx \frac{1}{\sqrt{2\pi}t} \exp(\frac{-t^2}{2})$ to obtain the optimum values

$$c_{opt} \approx \sqrt{1 + \sigma^2} \tag{33}$$

and the efficiency of this technique

$$r_{MC/NIIS} \approx \frac{\exp(\frac{1}{2\sigma^2})}{1 - \sigma/\sqrt{2\pi}} \tag{34}$$

which is better than Equation (23) for $1/\sigma^2 \gg 1$.

Case B: Multilevel PAM and QAM

For multilevel PAM with $M = 2^m, m \geq 2$, if all the symbols are equally probable, we have a very similar result

$$r_{MC/NIIS} \approx \frac{\exp(\frac{1}{2\sigma^2})}{(2 - 2^{1-m})(1 - \sigma/\sqrt{2\pi})} \tag{35}$$

and the optimum value of c is also given by Equation (33).

For multilevel QAM it is difficult to find a close form expression for the NIIS estimator variance. But numerical intergration has been performed to find the approximate optimum value of c,

$$c_{\text{opt}} \approx \sqrt{\frac{1}{4} + \sigma^2}. \tag{36}$$

However, numerical integration also showed that for multilevel QAM this NIIS technique is not as good as CIS. Hence for multilevel QAM CIS should be used. Yet it is possible that the combination of CIS and NNIS could be efficient for multilevel QAM, i.e., one could probably use the CIS technique on the i-channel, while the NIIS technique on the q-channel. The analysis for such a case is still open.

5 Numerical Results

In the numerical simulation to be discussed here, all random variables are derived from the uniform $(0,1)$ distribution with suitable transformations. The uniform $(0,1)$ distribution is generated by a maximum length linear congrunential pseudo-random generator defined by $f(z) = 16807z \bmod 2^{31} - 1$ [7]. The Gaussian distribution is derived using the Kinderman-Ramage algorithm [8]. The distribution specified by Equation (29) is derived by adding cB to a Gaussian variable, where c is a constant and B is a Bernoulli trial with equally probable outcomes ± 1.

First we shall consider the ISI free 8PAM. Tables 1 (a) list the simulation results of BER using the CIS technique, whereas Tables 1 (b) list the results of BER using the NIIS technique proposed in this paper at various SNR (in dB). In Table 1, we use 10 simulation runs to obtain the average value and standard deviation (devi in Tables) of the estimators. It can be found by comparing Tables 1 (a) and (b) that CIS for PAM is less efficient than NIIS for PAM, since CIS for 8PAM achieves approximately the same estimator deviation as NIIS for 8PAM using 3 to 4 times more samples. This result is in good agreement with the analysis in Section 4.2.

Next we consider ISI free 4QAM. As discussed in Section 4.2, CIS is better than NIIS for QAM, hence we list the results of CIS only for 4QAM in Table 2 for BER. The number of simulation runs are also 10. It can be found by comparing Table 2 with Table 1 (a) that CIS for QAM is less efficient than CIS for PAM, since CIS for 4QAM achieves approximately the same estimator deviation as CIS for 8PAM using 5 to 6 times more samples. This result is in good agreement with the analysis in Section 4.1.

Table 1: (a) BER simulation results for 8PAM using the CIS technique.

σ	SNR	N^*	$\hat{P}_b^*$	devi	P_b
0.2702	24.6	300	6.97E-5	1.64E-5	6.23E-5
0.2344	25.8	550	5.63E-6	1.20E-6	5.80E-6
0.2104	26.8	1000	5.93E-7	7.54E-8	5.85E-7
0.1925	27.5	1800	6.06E-8	9.17E-9	5.98E-8

Table 1: (b) BER simulation results for 8PAM using the NIIS technique.

σ	SNR	N^{***}	$\hat{P}_b^{***}$	devi	P_b
0.2702	24.6	100	5.83E-5	1.71E-5	6.23E-5
0.2344	25.8	180	6.07E-6	1.11E-6	5.80E-6
0.2104	26.8	350	5.78E-7	8.95E-8	5.85E-7
0.1925	27.5	600	6.07E-8	6.19E-9	5.98E-8

Table 2: BER simulation results for 4QAM using the CIS technique.

σ	SNR	N^*	$\hat{P}_b^*$	devi	P_b
0.2702	14.4	1800	1.07E-4	1.99E-5	1.07E-4
0.2344	15.6	3000	1.09E-5	1.74E-6	9.94E-6
0.2104	16.6	5000	9.86E-7	9.09E-8	1.00E-6
0.1925	17.3	7000	9.96E-8	8.00E-9	1.03E-7

6 Conclusion

We have generalized the classical MC and IS methods to include BER and SER of multilevel PAM and QAM. Several efficient IS techniques, including one newly proposed, have been discussed.

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