# CLOSED-FORM DESIGN OF ALLPASS FRACTIONAL DELAY FILTERS

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## ABSTRACT

In this paper, we propose a novel allpass (AP) fractional delay (FD) filter whose denominator polynomial is obtained by truncating the power series of a certain function. This function is derived from the frequency response of the denominator whose magnitude response is related to the desired phase response through the Hilbert transform since the denominator of a stable AP filter is of minimum phase. The target function and corresponding power series are calculated analytically and expressed in closed form. The closed-form expressions facilitate the analysis of stability. According to the properties for the coefficients of the denominator polynomial, we show that the proposed AP filter is stable for positive delay. Numerical examples indicate that the phase delays of the proposed filters are flat around $\omega = 0$.

## 1. INTRODUCTION

Digital implementation of fractional delay (FD) occurs in many applications such as sound synthesis and timing adjustment in digital receivers. Therefore, design of digital FD filters is important and has been widely studied and reported in the literature[1]. Several closed-form FD filters are investigated because there exist efficient and tunable structures for implementation. FIR FD filters with closed-form coefficients can be derived by windowing their ideal impulse responses [1, 2], by solving the Vandermonde system [3], or by expanding a certain function to power series [4]. The Vandermonde FIR FD filters which are identical to the series expansion have maximally flat (MF) frequency responses [4]. The MF FIR FD filters can be implemented in module [4] or in tunable structures such as the Farrow structure [1].

Allpass filters are a natural choice to design the FD filters since the AP filters have unity magnitude responses within the whole frequency band structurally. However, it is necessary for AP filters to check its stability which is guaranteed for the FIR filters. The methods of AP filters design can be surely applied to the FD filters [5]. Like the FIR MF FD filters, closed-form AP FD filters whose coefficients are obtained by solving the Vandermonde system have MF phase delay [6]. There exist tunable structures for the MF AP FD filters [7].

In this paper, we proposed a new AP FD filter with closed-form coefficients by series expansion. The phase delay of the denominator is calculated from the desired overall delay and accordingly the magnitude response can be derived through the Hilbert transform based on the fact that the denominator is of minimum phase. The ideal transfer function of the denominator can be derived by the magnitude and the phase responses. The denominator polynomials of the AP filters are obtained by truncating the power series of the ideal transfer function. Stability checked by a theorem about the bound of the zeros of a polynomial is guaranteed for positive delay.

## 2. FREQUENCY RESPONSE OF DENOMINATOR

The transfer function of an $N$th-order real-coefficient AP filter is represented by

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \ldots + a_0 z^{-N}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{z^{-N}A(z^{-1})}{A(z)} \tag{1}$$

where the numerator is the mirror-image polynomial of the denominator. Although we usually let $a_0 = 1$ for AP filters to prevent from the null solution, in this paper we do not make such an assumption to facilitate our derivation and discussion. The phase response of the AP filter can be expressed by

$$\arg[H(e^{j\omega})] = -N\omega - 2\arg[A(e^{j\omega})] \tag{2}$$

where $\arg[A(e^{j\omega})]$ is the phase response of the denominator. Given the desired frequency response $P(\omega)$, we want to find a set of coefficients $a_n$'s so that $\arg[H(e^{j\omega})] \approx P(\omega)$, or equivalently,

$$\arg[A(e^{j\omega})] \approx -\frac{1}{2}[N\omega + P(\omega)]. \tag{3}$$

Suppose the desired phase response for the $N$th-order AP FD filter is

$$P(\omega) = -(N + d)\omega \tag{4}$$

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where $d$, restricted by $-1 < d < 1$ in this paper, is the fractional part of the delay. Then, Eq.(3) gives the desired phase response of the denominator

$$\arg[A(e^{j\omega})] \approx \frac{1}{2}d\omega. \quad (5)$$

Because the denominator $A(z)$ is of minimum phase, we assume that its magnitude response $|A(e^{j\omega})|$ approximates a minimum phase system with phase response expressed by Eq.(5). Therefore, the magnitude $|A(e^{j\omega})|$ is related to the phase response $\arg[A(e^{j\omega})]$ by the discrete Hilbert transformation [2]. Specifically speaking,

$$\ln|A(e^{j\omega})| = \mathcal{C} + \frac{1}{2\pi}\mathcal{P}\int_{-\pi}^{\pi} \arg[A(e^{j\theta})]\cot\left(\frac{\omega-\theta}{2}\right)d\theta \quad (6)$$

where the symbol $\mathcal{P}$ denotes the Cauchy principal value of the integral. The constant $\mathcal{C}$ in Eq.(6) is calculated by

$$\mathcal{C} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|A(e^{j\omega})|d\omega. \quad (7)$$

Since $\mathcal{C}$ represents a scaling factor to $|A(e^{j\omega})|$, it will be canceled out in case of AP filter. Hence we let $\mathcal{C} = 0$ without loss of generality.

Substituting Eq.(5) into Eq.(6), we may express $|A(e^{j\omega})|$ in closed form. By means of the Leibniz's theorem for differentiation of an integral [8, p.11], it is easy to calculate the integral of Eq.(6).

**Property 1** *The magnitude response of a minimum phase system with phase response $\frac{1}{2}d\omega$ is*

$$|A(e^{j\omega})| = (2 + 2\cos\omega)^{-d/2}. \quad (8)$$

*Proof*: Let

$$f(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\cot\left(\frac{\omega-\theta}{2}\right)d\theta. \quad (9)$$

We obtain that $\ln|A(e^{j\omega})| = \frac{1}{2}d \times f(\omega)$ after substituting Eq.(5) into Eq.(6). By the Leibniz's theorem, we have

$$\frac{d}{d\omega}f(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\left[\frac{\partial}{\partial\omega}\cot\left(\frac{\omega-\theta}{2}\right)\right]d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\left[-\frac{1}{2}\csc^2\left(\frac{\omega-\theta}{2}\right)\right]d\theta = \tan\frac{1}{2}\omega.$$

Therefore,

$$f(\omega) = \mathcal{K} + \int \tan\frac{1}{2}\omega d\omega = \mathcal{K} - \ln\cos^2\frac{1}{2}\omega \quad (10)$$

where $\mathcal{K}$ is a constant. To determine $\mathcal{K}$, we let $\omega = 0$, equate Eq.(9) with (10), and obtain that

$$\mathcal{K} = f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\cot\left(-\frac{1}{2}\theta\right)d\theta = -\ln 4.$$

Therefore, $\ln|A(e^{j\omega})| = -\frac{1}{2}d\ln[4\cos^2(\frac{1}{2}\omega)]$ and the magnitude response can be obtained and expressed as Eq.(8).

Since the magnitude and the phase responses of the denominator are obtained, we may express its transfer function in closed form. This transfer function could be regarded as the ideal transfer function of the AP FD filter. Substituting $z + z^{-1}$ for $2\cos\omega$ in Eq.(8) and expressing the phase response as $z^{d/2}$, we obtain the following property.

**Property 2** *The ideal transfer function of the denominator for an $N$th-order AP FD filter with total delay of $N + d$ is*

$$A_{id}(z) = (1 + z^{-1})^{-d}. \quad (11)$$

*Remark.* In fact, the denominators of Thiran's MF AP FD filters converge to $A_{id}(z)$ as $N$ approaches infinity. By expressing the coefficients of Thiran's MF AP FD filters as

$$a_m = \frac{(-1)^m (N+d)_m}{(2N+1+d)_m}\binom{N}{m}, \quad (12)$$

we have

$$\lim_{N\to\infty} a_m = \binom{-d}{m} \quad (13)$$

where $\binom{-d}{m}$ is the coefficients of the Taylor's series of $A_{id}(z)$ expanded at $z^{-1} = 0$, and $(x)_n$ is the Pochhammer's symbol defined by $(x)_0 = 1$ and $(x)_n = x \times (x+1) \times \cdots \times (x+n-1)$.

Based on the magnitude response expressed by Eq.(8) or the ideal transfer function of Eq.(11), we may find the coefficients of the denominator $A(z)$ by well-developed methods for FIR filter design. In this paper, we will expand Eq.(11) into its power series to obtain the closed-form coefficients directly.

**Property 3** *The transfer function of the denominator of an $N$th-order AP FD filter can be expressed by*

$$A(z) = \sum_{n=0}^{N} \frac{(d)_n}{n!}\left(\frac{1-z^{-1}}{2}\right)^n. \quad (14)$$

*Proof*: This result can be obtained by expanding the ideal transfer function around $z = 1$. Expressing $A_{id}(z)$ in Eq.(11) as $2^{-d}\left[1 + (z^{-1} - 1)/2\right]^{-d}$, expanding it as the binomial series, and truncating the resulting series up to the first $N+1$ terms, we obtain Eq.(14) except the scaling factor of $2^{-d}$. The factor can be dropped since it will be cancelled out in the AP transfer function.

According to Property 3, it is easy to obtain the filter coefficients $a_n$'s in closed form by expanding the right side of Eq.(14) and then collecting terms of the same power. After some algebraic manipulations, the coefficients in Eq.(1)can be expressed as

$$a_n = \frac{(-1)^n}{n!\,2^n}\sum_{k=0}^{N-n} \frac{(d)_{k+n}}{k!\,2^k} \quad (15)$$

for $n = 0, 1, \ldots, N$.

The coefficients of the $N$th-order AP FD filter is now obtained in closed form. Although we derive the coefficients of the filter according to the assumption of minimum phase system for the denominator, this assumption does not guarantee stability of $A(z)$ in Eq.(14) because of truncation of series. In next section, we will discuss the stability of the proposed AP filter.

## 3. STABILITY PROBLEM

To test the stability of the proposed AP filter, we may apply the Schur-Cohn criterion or the more efficient Jury-Marden criterion [9]. Nevertheless, it is difficult to evaluate the Schur determinants because the summation in Eq.(14) can not be simplified furthermore. It is also complicated to establish the Jury-Marden arrays for the same reason. Therefore, we have to find another way to test the stability. In this paper, we will apply the Eneström-Kakeya theorem [10, 11] stated as the following:

**Theorem 1** *Let* $p(x) = \sum_{n=0}^{N} a_n x^{N-n}$, $N \geq 1$, *be a polynomial with* $a_n > 0$ *for* $0 \leq n \leq N$. *Let* $r_n = a_{n+1}/a_n$ *for* $0 \leq n < N$. *Then all the zeros of* $p(x)$ *are contained in the annulus*

$$\min_n r_n \leq |x| \leq \max_n r_n.$$

By the closed-form expression of coefficients in Eq.(14), we can show that the coefficients are decreasing in modulus. In fact, we have the following property.

**Property 4** *The denominator* $A(z)$ *can be represented as*

$$A(z) = h_0 - h_1 z^{-1} + h_2 z^{-2} - \cdots + (-1)^N h_N z^{-N} \quad (16)$$

*for* $0 < d < 1$ *where* $h_0 > h_1 > h_2 > \cdots > h_N > 0$. *On the other hand, for* $-1 < d < 0$, *we have*

$$A(z) = g_0 + g_1 z^{-1} - g_2 z^{-2} + \cdots + (-1)^{N-1} g_N z^{-N} \quad (17)$$

*where* $g_0 > g_1 > g_2 > \cdots > g_N > 0$.

*Proof*: We show the property only for $0 < d < 1$. The proof for the case of $-1 < d < 0$ is similar. It is obvious that $h_n = (-1)^n a_n > 0$ for $n = 0, 1, \ldots, N$. Besides, we have

$$\begin{aligned} h_{n+1} - h_n &= -\frac{(d)_n}{n!\, 2^n} + \sum_{k=1}^{N-n} \frac{(k-n-1)(d)_{k+n}}{(n+1)!\, k!\, 2^{k+n}} \\ &< -\frac{(d)_n}{n!\, 2^n} + \sum_{k=1}^{\infty} \frac{(k-n-1)(d)_{k+n}}{(n+1)!\, k!\, 2^{k+n}} \\ &= \frac{2^{n+d}(d-1)_{n+1}}{(n+1)!} < 0. \end{aligned}$$

By Property 4, we can show that the proposed AP filter is stable for $0 < d < 1$. Let $\hat{A}(z) = A(-z)$. Because the coefficients of $\hat{A}(z)$ are positive and strictly monotone decreasing, the zeros of $\hat{A}(z)$ lie in $|z| < 1$ by Theorem 1. Therefore the zeros $A(z)$ also lie in $|z| < 1$. We conclude that the AP FD filter with denominator $A(z)$ is stable.

We can not conclude that the AP filter is stable for $-1 < d < 1$ by Property 4 and Theorem 1. However, by numerically computing the zeros of $A(z)$, the largest moduli of zeros are less than unity within the range of interest. Fig. 1 shows the plot of largest moduli for $N = 5, 15, 25, 35, 45$ and 55 and $-1 < d < 0$. The pole of largest modulus for $N = 55$ and $d = -0.99$ is the negative real pole of $-0.99963284345625$ whose modulus is near but less than unity.

## 4. DESIGN RESULTS

As an illustration of the proposed filters, Fig. 2 and 3 show the design results of the 10th-order AP FD filters. Fig. 2 is the plot of the phase delays versus normalized frequency for $d = -0.8, -0.6, \ldots, 0.6, 0.8$. The phase delays are flat around $\omega = 0$ within a wide bandwidth. Fig. 3 shows the poles for $d = -0.9$ and 0.9. It is obvious that the poles of both cases are inside the unit circle. However, for $d = -0.9$, there is a real negative pole near the unit circle.

## 5. CONCLUSIONS

A new AP FD filter expressed in closed form is proposed in this paper. The denominator polynomial of the AP filter is obtained by truncating the power series of the function of $(1 + z^{-1})^{-d}$. We derive the function by the frequency response of the denominator where its magnitude response is related to the phase response through the Hilbert transform since the denominator of a stable AP filter is of minimum phase. To analyze the stability, we apply a theorem about the bound for the zeros of a polynomial to the denominator. The closed-form expressions facilitate analysis of stability. Without taking the Schur-Cohn criterion, we show that the proposed filter is stable for $0 < d < 1$. The stability of filters with negative $d$ is demonstrated by numerical computing the poles. Within the range of interest, we show that all the poles are inside unit circle. Design examples indicate that the phase delays are flat around DC.

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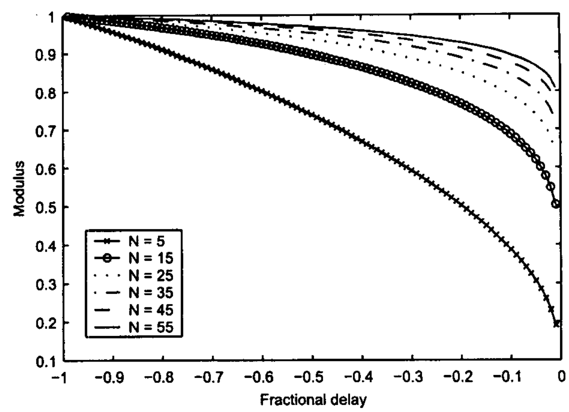


**Fig. 1**. The plot of the largest poles in modulus for delay in $-1 < d < 0$.

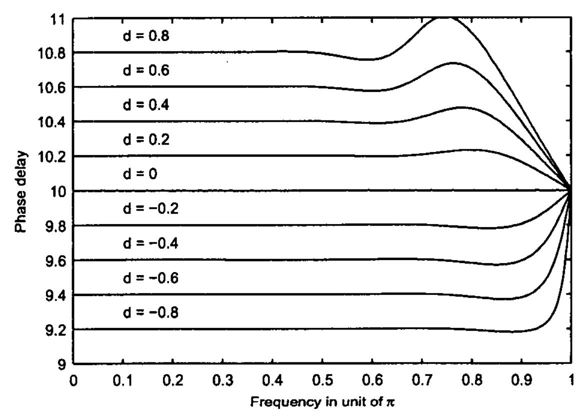


**Fig. 2**. The plot of the phase delays for $N = 10$.

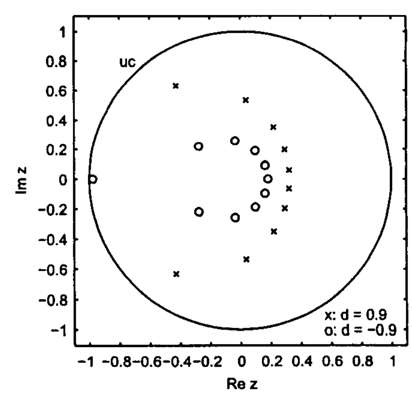


**Fig. 3**. The plot of the pole locations for $N = 10$. "uc" stands for the unit circle.