Journal of the Chinese Institute of Engineers, Vol. 26, No. 3, pp. 279-288 (2003)

SUB-OPTIMAL FINITE DIMENSIONAL OBSERVER-BASED BOUNDARY CONTROLLER DESIGN FOR DISTRIBUTED PARAMETER SYSTEMS ON A PLANAR DOMAIN

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ABSTRACT

In this paper, a certain class of distributed parameter systems is considered. We propose a three-step design method for finding finite dimensional observer-based boundary feedback controllers. The first step is called the boundary-equation normalization, which transforms the boundary and system equations into a normal form. The second step is called the boundary input transformation, which integrates the boundary input equation into the system equation, and forms a type of distributed parameter system called the general boundary input system. The final step is to design the desired finite dimensional controller, based on the general boundary input system model. The design procedure utilizes the finite dimensional linear quadratic optimal control theory, so well-developed computation tools can be applied. Though the acquired controllers are only sub-optimal for the distributed parameter systems, an estimation of the performance degradation from that of the ideal case is derived for comparison purpose.

Key Words: sub-optimal controller, distributed parameter systems, boundary control, state observer.

I. INTRODUCTION

Compared with the bounded controller design problem for distributed parameter systems, the boundary controller design problem for distributed parameter systems is much more difficult to handle. In the bounded control problem, there are many similarities to the controller design problem for finite dimensional systems. Besides the unbounded nature of boundary controller design problems, the solution of boundary controller design problems also depends on the shapes of boundaries and the structure of control inputs. Thus it will be easier if we can transfer the boundary controller design problem into an equivalent bounded controller design problem. For a certain class of distributed parameter systems with special boundary conditions, this can be achieved (Lu and Fong, 2000; 2001; 2003). Such class of distributed parameter systems is said to be in a normal form.

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Given a distributed parameter system with control inputs on the boundary of its domain, first we check whether distributed parameter system can be transformed into a normal form, or whether boundary-equation normalization can be accomplished. If so, we show how the boundary inputs of the transformed normal form can be integrated into the system equation defined in certain interpolation space as bounded control inputs. Systems in such a mathematical model are called the general boundary control systems (Hinata, 1999; Inaba and Hinata, 1993), for which it is possible to design finite dimensional observer-based controllers.

In the stabilization problem of distributed parameter systems with bounded control, a main difference from that of finite dimensional systems is that we often need the controller to be finite dimensional. This is for practical engineering considerations. There are many methods for the stabilization of distributed parameter systems (Curtain, 1993). The method we adopted here is the so-called exact state-space design method, which has three major approaches. The first one is by Schumacher (1983) and Curtain (1984), who show the existence of finite dimensional controllers for a class of distributed parameter systems. Almost at the same time, another work by Sakawa (1983; 1984) gives a design scheme showing that under some conditions, a finite dimensional stabilization controller can be derived. In these two methods, Schumacher shows the existence of a finite dimensional controller, but does not guarantee the controller order to be small; while Sakawa proves that under some conditions, a finite dimensional controller of a pre-specified order may be obtained. The third approach is given by Bernstein and Hyland (1986), but the method needs to solve some operator-type Riccati equations, and has to face difficulties in numerical computations.

In our observer-based boundary feedback controller design, we use the work of Sakawa (1983) to show that under certain mild conditions, it is possible to design a finite dimensional stabilizing controller. However, in this paper, an explicit design procedure that gives finite dimensional sub-optimal controllers, is presented. Compared with the infinite dimensional linear quadratic optimal method that is needed to solve operator-type Riccati equations, the method proposed here only needs to solve finite dimensional Riccati equations. Though the controller is only sub-optimal for the overall system, we give a theorem that estimates the performance degradation from the ideal case with state feedback.

In Section II, we formulate the systems to be considered and introduce the boundary-equation normalization process. In Section III, we describe the boundary input transformation method. In Section IV, some of the notations and the result of Sakawa (1983), needed for the bounded controller design are presented. In Section V, we propose the design procedure that will give a sub-optimal finite dimensional controller, and analyze the performance of the controller. In Section VI an example is prepared to illustrate the design procedure, and the simulation results are shown. Finally in Section VII conclusions are made.

II. PLANT FORMULATION AND NORMALIZATION

Let $\boldsymbol{R}$ be the field of real numbers, $\boldsymbol{R}^{\ell}$ be the vector space of ℓ-tuple vectors with elements from $\boldsymbol{R}$, and $\boldsymbol{I}=\{1, 2, 3, 4\}$, $N=\{0, 1, 2, 3, \cdots\}$ be two index sets. The vectors in $\boldsymbol{R}^2$ are represented by $\boldsymbol{x}=[x_1\ \ x_2]^T$. Consider a domain $\boldsymbol{\Omega}\subset\boldsymbol{R}^2$ with the boundary $\partial\Omega=\cup_{i\in I}\boldsymbol{\Gamma}_i$, where

$$\boldsymbol{\Gamma}_1=\{[x_1\ \ x_2]^T|0\le x_1\le l_1,\ x_2=0\}$$

$$\boldsymbol{\Gamma}_2=\{[x_1\ \ x_2]^T|x_1=0,\ 0\le x_2\le l_2\}$$

$$\boldsymbol{\Gamma}_3=\{[x_1\ \ x_2]^T|0\le x_1\le l_1,\ x_2=l_2\}$$

$$\boldsymbol{\Gamma}_4=\{[x_1\ \ x_2]^T|x_1=l_1,\ 0\le x_2\le l_2\} \tag{1}$$

and positive l_1, $l_2\in\boldsymbol{R}$ are given. For $i\in\boldsymbol{I}$, denote the outward unit normal vector on the boundary $\boldsymbol{\Gamma}_i$ of $\boldsymbol{\Omega}$ by $\boldsymbol{n}_i$. In this paper, suppose the distributed parameter systems to be controlled are formulated as

$$\begin{cases} \dfrac{\partial z(t,\boldsymbol{x})}{\partial t}=[\dfrac{\partial z}{\partial x_1}\ \ \dfrac{\partial z}{\partial x_2}]\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22}\end{bmatrix}\begin{bmatrix}\dfrac{\partial z}{\partial x_1} \\ \dfrac{\partial z}{\partial x_2}\end{bmatrix} \\ b_i\dfrac{\partial z}{\partial \boldsymbol{n}_i}\bigg|_{\Gamma_i}=u_i(t),\ i\in\boldsymbol{I} \\ \boldsymbol{y}(t)=\begin{bmatrix}\displaystyle\int_{\Omega} c_1(\boldsymbol{x})z(t,\boldsymbol{x})d\boldsymbol{x} \\ \vdots \\ \displaystyle\int_{\Omega} c_p(\boldsymbol{x})z(t,\boldsymbol{x})d\boldsymbol{x}\end{bmatrix} \\ z(0,\boldsymbol{x})=z_0(\boldsymbol{x}) \end{cases} \tag{2}$$

where $z(t,\ \cdot)\in\boldsymbol{Z}=\boldsymbol{L}_2(\boldsymbol{\Omega})$, the space of the Lebesgue square integrable functions defined on Ω, is the state function of the time variable $t\in[0,\infty)$ and spatial variable $\boldsymbol{x}$, $a_{11}>0$, $a_{22}>0$, and $b_i\neq 0$ for $i\in I$ are real constants, $u_i(\cdot)$ for $i\in\boldsymbol{I}$ is the ith boundary control input function of $t\in[0,\infty)$ with the continuous second derivative, $\boldsymbol{y}(\cdot)$ is the output vector function with $c_i(\cdot)\in\boldsymbol{Z}$ for $i=1, \ldots, p$, and $z_0(\cdot)\in\boldsymbol{Z}$ is the initial condition of the system. Systems described by (2) are said to be in the normal form. Though in this paper it is assumed that the systems to be controlled are in the normal form, many distributed parameter systems may be converted into this form via linear or nonlinear state transformations. A special class of such systems is described by the equation

$$\begin{cases} \dfrac{\partial \tilde z(t,\boldsymbol{x})}{\partial t}=[\dfrac{\partial \tilde z}{\partial x_1}\ \ \dfrac{\partial \tilde z}{\partial x_2}]\begin{bmatrix} \tilde a_{11} & \tilde a_{12} \\ \tilde a_{21} & \tilde a_{22}\end{bmatrix}\begin{bmatrix}\dfrac{\partial \tilde z}{\partial x_1} \\ \dfrac{\partial \tilde z}{\partial x_2}\end{bmatrix} \\ \tilde b_i\dfrac{\partial \tilde z}{\partial \tilde{\boldsymbol{n}}_i}\bigg|_{\tilde\Gamma_i}=u_i(t),\ i\in\boldsymbol{I} \\ \tilde{\boldsymbol{y}}(t)=\begin{bmatrix}\displaystyle\int_{\tilde\Omega} \tilde c_1(\boldsymbol{x})\tilde z(t,\boldsymbol{x})d\boldsymbol{x} \\ \vdots \\ \displaystyle\int_{\tilde\Omega} \tilde c_p(\tilde{\boldsymbol{x}})\tilde z(t,\boldsymbol{x})d\boldsymbol{x}\end{bmatrix} \\ \tilde z(0,\boldsymbol{x})=\tilde z_0(0,\boldsymbol{x}) \end{cases} \tag{3}$$

defined on a domain $\tilde{\boldsymbol{\Omega}}$ with the boundary $\partial\tilde{\boldsymbol{\Omega}} = \cup_{i\in I}\tilde{\boldsymbol{\Gamma}}_i$, where $\tilde{\boldsymbol{\Gamma}}_i \in \boldsymbol{R}^2$ is a line segment in $\boldsymbol{R}^2$, and $\tilde{\boldsymbol{n}}_i$ is the outward unit normal vector on $\tilde{\boldsymbol{\Gamma}}_i$, $i\in\boldsymbol{I}$. If there exist a constant nonsingular matrix $\boldsymbol{T}\in\boldsymbol{R}^{2\times 2}$ and constants θ_1, $\theta_2\in\boldsymbol{R}$ such that

(i) the affine transformation $\boldsymbol{T}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ maps $\tilde{\boldsymbol{\Omega}}$ onto $\boldsymbol{\Omega}$, and

(ii) $\boldsymbol{T}\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix}\boldsymbol{T}^T = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ with certain positive constants a_{11}, a_{22}, then clearly such systems may be converted into the normal form. The conversion of a system description so that it fits the form formulated in (2) is called boundary-equation normalization.

To proceed, we need the following notations. Let the inner product on $\boldsymbol{Z}$ be defined as $<z_a, z_b>_Z = \int_{\boldsymbol{\Omega}} z_a(\boldsymbol{x})z_b(\boldsymbol{x})d\boldsymbol{x}$ for all z_a, $z_b\in\boldsymbol{Z}$. The space of all bounded linear operators from the space $\boldsymbol{X}$ to the space $\boldsymbol{Y}$ is denoted by $\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{Y})$. For any operator $\boldsymbol{A}$, $\boldsymbol{D}(\boldsymbol{A})$ represents its domain, $\mathcal{J}(\boldsymbol{A})$ its range, and $K(\boldsymbol{A})$ its kernel. Finally, the norms of matrices and operators are both symbolized by $\|\cdot\|$, and the context will decide the nature of the norms.

The system described by (2) may be expressed by a more compact state-space form

$$\begin{cases} \dfrac{dz(t)}{dt} = \tilde{\boldsymbol{A}}z(t) \\ \boldsymbol{H}z(t) = u(t) \\ y(t) = \boldsymbol{C}z(t) \\ z(0) = z_0 \end{cases} \tag{4}$$

where

$$\tilde{\boldsymbol{A}}z = a_{11}\frac{\partial^2 z(t,\boldsymbol{x})}{\partial x_1^2} + a_{22}\frac{\partial^2 z(t,\boldsymbol{x})}{\partial x_2^2} \tag{5}$$

$$\boldsymbol{H}z = \begin{bmatrix} b_1(-1)\dfrac{\partial z}{\partial x_2}\Big|_{x_2=0} \\ b_2\dfrac{\partial z}{\partial x_1}\Big|_{x_1=l_1} \\ b_3\dfrac{\partial z}{\partial x_2}\Big|_{x_2=l_2} \\ b_4(-1)\dfrac{\partial z}{\partial x_1}\Big|_{x_1=0} \end{bmatrix} \tag{6}$$

$$\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} \tag{7}$$

$$\boldsymbol{C}z(t) = \begin{bmatrix} \left\langle c_1(\boldsymbol{x}), z(t,\boldsymbol{x})\right\rangle_Z \\ \vdots \\ \left\langle c_p(\boldsymbol{x}), z(t,\boldsymbol{x})\right\rangle_Z \end{bmatrix} \tag{8}$$

Note that $\tilde{\boldsymbol{A}}:\mathcal{D}(\tilde{\boldsymbol{A}})\subset\boldsymbol{Z}\to\boldsymbol{Z}$ is an unbounded linear operator densely defined on $\boldsymbol{Z}$. We are interested in the state-space model (4) because if it owns the three properties listed in Definition 1 below, then it may lead to another model which facilitates the task of designing observer-based feedback controllers for the system described by (2).

III. BOUNDARY INPUT TRANSFORMATION

Definition 1: *(Inaba and Hinata, 1993) Distributed parameter systems described by the state-space model (4) are called general boundary input systems if*

(i) $\boldsymbol{A}=\tilde{\boldsymbol{A}}\big|_{\mathcal{D}(\tilde{A})\cap\mathcal{K}(H)}$: $D(\boldsymbol{A})\subset\boldsymbol{Z}\to\boldsymbol{Z}$ *generates an analytic semi-group* $\boldsymbol{T}(t)$ *on* $\boldsymbol{Z}$.

(ii) There exist an $\omega>\omega_0$*, the growth constant of* $\boldsymbol{T}(t)$*, and a* $\boldsymbol{G}(\omega)\in\boldsymbol{B}(\boldsymbol{R}^4, \boldsymbol{Z})$ *such that* $(\omega\boldsymbol{I}-\tilde{\boldsymbol{A}})\boldsymbol{G}(\omega)=0$ *and* $\boldsymbol{H}\boldsymbol{G}(\omega)\boldsymbol{u}=\boldsymbol{u}$ *for all* $\boldsymbol{u}=[u_1\ u_2\ u_3\ u_4]^T\in\boldsymbol{R}^4$.

(iii) There exists a real number $\alpha\in[0,1)$ *such that* $\mathcal{J}[\boldsymbol{G}(\omega)]\subset\mathcal{D}[(\omega\boldsymbol{I}-\boldsymbol{A})^\alpha]$.

The following Theorem shows that under certain mild technical assumptions, (4) is indeed a general boundary input system with (5)-(8).

Theorem 1: *(Lu and Fong, 2001) For* $\tilde{\boldsymbol{A}}$ *and* $\boldsymbol{H}$ *from (5) and (6), respectively, suppose* $\boldsymbol{A}=\tilde{\boldsymbol{A}}\big|_{\mathcal{D}(\tilde{A})\cap\mathcal{K}(H)}$ *as in Definition 1. Then the following are true.*

(i) The operator $\boldsymbol{A}$ *has real eigenvalues* λ_{n_1,n_2}, $(n_1, n_2)\in\boldsymbol{N}\times\boldsymbol{N}$*, of finite magnitudes. Moreover, the corresponding eigenvectors* ϕ_{n_1,n_2}, $(n_1, n_2)\in\boldsymbol{N}\times\boldsymbol{N}$*, form a complete basis in* $\boldsymbol{Z}$.

(ii) Systems described by (4) are general boundary input systems and there exist constants g^1_{0,n_2}, $g^2_{n_1,0}$, g^3_{0,n_2}, $g^4_{n_1,0}$, $(n_1, n_2)\in\boldsymbol{N}$*, such that* $\boldsymbol{G}(\omega)$ *can be expressed as*

$$[\boldsymbol{G}(\omega)\boldsymbol{u}](\boldsymbol{x})$$

$$=[\sum_{n_2\in N} g^1_{0,n_2}\phi_{0,n_2}(x_2)]u_1(t) + [\sum_{n_1\in N} g^2_{n_1,0}\phi_{n_1,0}(x_1)]u_2(t)$$

$$+[\sum_{n_2\in N} g^3_{0,n_2}\phi_{0,n_2}(x_2)]u_3(t) + [\sum_{n_1\in N} g^4_{n_1,0}\phi_{n_1,0}(x_1)]u_4(t)\,.$$

Proof: (i) With (5) and (6), it is easy to verify that the operator $\boldsymbol{A}=\tilde{\boldsymbol{A}}\big|_{\mathcal{D}(\tilde{A})\cap\mathcal{K}(H)}$ has only a point spectrum. Suppose (λ, ϕ) is an eigenvalue and eigenvector pair in the point spectrum of $\boldsymbol{A}$ and satisfies $(\boldsymbol{A}-\lambda I)\phi=0$. Then we have

$$\begin{cases} a_{11}\dfrac{\partial^2\phi}{\partial x_1^2}+a_{22}\dfrac{\partial^2\phi}{\partial x_2^2}-\lambda\phi=0 \\ -\dfrac{\partial\phi}{\partial x_2}\Big|_{x_2=0}=0 \\ \dfrac{\partial\phi}{\partial x_1}\Big|_{x_1=l_1}=0 \\ \dfrac{\partial\phi}{\partial x_2}\Big|_{x_2=l_2}=0 \\ -\dfrac{\partial\phi}{\partial x_1}\Big|_{x_1=0}=0 \end{cases} \tag{9}$$

for which the solutions are easily shown to be

$$\begin{cases} \phi_{0,0}(x_1, x_2)=1 \\ \phi_{0,n_2}(x_1, x_2)=\sqrt{2}\cos(\frac{n_2\pi}{l_2}x_2)\ ,\ n_1=0,\ n_2\geq 1 \\ \phi_{n_1,0}(x_1, x_2)=\sqrt{2}\cos(\frac{n_1\pi}{l_1}x_1)\ ,\ n_1\geq 1,\ n_2=0 \\ \phi_{n_1,n_2}(x_1, x_2)=\sqrt{2}\cos(\frac{n_1\pi}{l_1}x_1)\sqrt{2}\cos(\frac{n_2\pi}{l_2}x_2) \\ \quad n_1\geq 1,\ n_2\geq 1 \end{cases} \tag{10}$$

Therefore for every $(n_1, n_2)\in \boldsymbol{N}\times\boldsymbol{N}$, ϕ_{n_1, n_2} is an eigenvector of $\boldsymbol{A}$. It is straightforward to prove that the set $\{\phi_{n_1, n_2}|(n_1, n_2)\in N\times N\}$ forms a complete basis in $\boldsymbol{Z}$.

(ii) First of all, for every $z\in Z$, the operator $\boldsymbol{T}(t)$ defined by

$$\boldsymbol{T}(t)z=\sum_{(n_1, n_2)\in N\times N}\left\langle z, \phi_{n_1,n_2}\right\rangle_Z e^{\lambda_{n_1,n_2}t}\phi_{n_1,n_2} \tag{11}$$

is the analytic semigroup that $\boldsymbol{A}$ generates on $\boldsymbol{Z}$. Secondly, we want to find the appropriate ω and $\boldsymbol{G}(\omega)$. For any $\omega>\omega_0$, the growth constant of $\boldsymbol{T}(t)$, define the convenient symbol $f_u(\boldsymbol{x})=[\boldsymbol{G}(\omega)u](\boldsymbol{x})$. Then the equation $\tilde{\boldsymbol{A}}\boldsymbol{G}(\omega)u=\omega\boldsymbol{G}(\omega)u$ can be rewritten as $\tilde{A}f_u(\boldsymbol{x})=\omega f_u(\boldsymbol{x})$, which suggests we examine

$$\begin{aligned} f_u(\boldsymbol{x}) &= [\boldsymbol{G}(\omega)u](\boldsymbol{x}) \\ &= c_1(t)e^{\sqrt{\frac{\omega}{a_{11}}}x_1}+c_2(t)e^{-\sqrt{\frac{\omega}{a_{11}}}x_1} \\ &\quad +c_3(t)e^{\sqrt{\frac{\omega}{a_{22}}}x_2}+c_4(t)e^{-\sqrt{\frac{\omega}{a_{22}}}x_2} \end{aligned}$$

for some $c_i(t)$, $i\in \boldsymbol{I}$, independent of $\boldsymbol{x}$. Substituting the above expression into the other requirement that $\boldsymbol{G}(\omega)$ must satisfy, i.e., $\boldsymbol{HG}(\omega)\boldsymbol{u}=\boldsymbol{u}$, yields the conditions for the functions $c_i(t)$'s:

$$\begin{bmatrix} 1 & -1 \\ e^{\sqrt{\frac{\omega}{a_{22}}}l_2} & -e^{-\sqrt{\frac{\omega}{a_{22}}}l_2} \end{bmatrix}\begin{bmatrix} c_3(t) \\ c_4(t) \end{bmatrix}=\begin{bmatrix} \dfrac{1}{-\sqrt{\dfrac{\omega}{a_{22}}}}u_1(t) \\ \dfrac{1}{\sqrt{\dfrac{\omega}{a_{22}}}}u_3(t) \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} e^{\sqrt{\frac{\omega}{a_{11}}}l_1} & -e^{-\sqrt{\frac{\omega}{a_{11}}}l_1} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}=\begin{bmatrix} \dfrac{1}{\sqrt{\dfrac{\omega}{a_{11}}}}u_2(t) \\ \dfrac{1}{-\sqrt{\dfrac{\omega}{a_{11}}}}u_4(t) \end{bmatrix} \tag{13}$$

These two sets of linear equations in $c_i(t)$'s allow us to express $c_i(t)$'s in terms of $u_i(t)$. For the sake of notational brevity, we write

$$c_1(t)=\alpha_{12}u_2(t)+\alpha_{14}u_4(t)$$

$$c_2(t)=\alpha_{22}u_2(t)+\alpha_{24}u_4(t)$$

$$c_3(t)=\alpha_{31}u_1(t)+\alpha_{33}u_3(t)$$

$$c_4(t)=\alpha_{41}u_1(t)+\alpha_{43}u_3(t)$$

where α_{ij}'s are constants depending only on a_{11}, a_{22} and ω. Thus we have

$$\begin{aligned} [\boldsymbol{G}(\omega)u](\boldsymbol{x}) &= (\alpha_{31}e^{\sqrt{\frac{\omega}{a_{22}}}x_2}+\alpha_{41}e^{-\sqrt{\frac{\omega}{a_{22}}}x_2})u_1(t) \\ &\quad +(\alpha_{12}e^{\sqrt{\frac{\omega}{a_{11}}}x_1}+\alpha_{22}e^{-\sqrt{\frac{\omega}{a_{11}}}x_1})u_2(t) \\ &\quad +(\alpha_{33}e^{\sqrt{\frac{\omega}{a_{22}}}x_2}+\alpha_{43}e^{-\sqrt{\frac{\omega}{a_{22}}}x_2})u_3(t) \\ &\quad +(\alpha_{14}e^{\sqrt{\frac{\omega}{a_{11}}}x_1}+\alpha_{24}e^{-\sqrt{\frac{\omega}{a_{11}}}x_1})u_4(t) \end{aligned}$$

With this explicit expression of $[\boldsymbol{G}(\omega)u](\boldsymbol{x})$, we are able to expand it with respect to the basis $\{\phi_{n_1, n_2}\}$. The result is written as

$$\begin{aligned} &[\boldsymbol{G}(\omega)u](\boldsymbol{x}) \\ &=[\sum_{n_2\in N} g^1_{0,n_2}\phi_{0,n_2}(x_2)]u_1(t)+[\sum_{n_1\in N} g^2_{n_1,0}\phi_{n_1,0}(x_1)]u_2(t) \\ &\quad +[\sum_{n_2\in N} g^3_{0,n_2}\phi_{0,n_2}(x_2)]u_3(t)+[\sum_{n_1\in N} g^4_{n_1,0}\phi_{n_1,0}(x_1)]u_4(t) \end{aligned}$$

This is a full characterization of the operator $\boldsymbol{G}(\omega)$, with which the boundedness property can be easily derived. Finally, since $\boldsymbol{R}^4$ is a finite dimensional space and $\boldsymbol{G}(\omega)\in\boldsymbol{B}(\boldsymbol{R}^4,\boldsymbol{Z})$, $\mathcal{J}[\boldsymbol{G}(\omega)]$ is also finite dimensional, which means that at least $\alpha=0$ is such that $\mathcal{J}[\boldsymbol{G}(\omega)]\subset\mathcal{D}[(\omega I-\boldsymbol{A})^{\alpha}]=\boldsymbol{Z}$. This completes the proof. ■

After establishing that under the assumptions, (4) with (5)-(8) is a general boundary input system, we quote the following Theorem to obtain a model with a more familiar appearance. The symbols inherit those defined before.

Theorem 2: *(Hinata, 1999; Inaba and Hinata, 1993) For a general boundary input system (4), if we define $\boldsymbol{Z}_0=\boldsymbol{Z}$, $\boldsymbol{Z}_1=\mathcal{D}(\boldsymbol{A})$, the interpolation space $\boldsymbol{Z}_\alpha=\mathcal{D}[(\omega I-\boldsymbol{A})^{\alpha}]$, and operators $\boldsymbol{A}_\alpha:\mathcal{D}(\boldsymbol{A}_\alpha)=\boldsymbol{Z}_\alpha\subset\boldsymbol{Z}_{\alpha-1}\rightarrow\boldsymbol{Z}_{\alpha-1}$ and $\boldsymbol{B}_\alpha\in\mathcal{B}(\boldsymbol{R}^4,\boldsymbol{Z}_{\alpha-1})$, respectively, as*

$$\boldsymbol{A}_\alpha z=[(\omega I-\boldsymbol{A})^{\alpha-1}]^{-1}\boldsymbol{A}(\omega I-\boldsymbol{A})^{\alpha-1}z \tag{14}$$

$$\boldsymbol{B}_\alpha u=(\omega I-\boldsymbol{A}_\alpha)\boldsymbol{G}(\omega)u \tag{15}$$

then the system model (4) is equivalent to

$$\begin{cases}\dfrac{dz(t)}{dt}=\boldsymbol{A}_\alpha z(t)+\boldsymbol{B}_\alpha u(t)\\ y(t)=\boldsymbol{C}z(t)\\ z(0)=z_0\end{cases} \tag{16}$$

Note that since $\boldsymbol{C}$ is a bounded linear operator defined on $\boldsymbol{Z}$, we can get the same output y(t) as $\boldsymbol{C}Z(t)$ here.

The model (16) provides a description of the systems to be studied without explicit formulation of the boundary conditions, and facilitates the development of observer-based feedback controller design methods. In fact, if we continue to utilize the analysis results presented in Theorem 1, we even have series expansions of the operators $\boldsymbol{A}_\alpha$ and $\boldsymbol{B}_\alpha$.

Corollary 1: Let $\mu_{n_1,n_2}=\omega-\lambda_{n_1,n_2}$, then we have the following expansion

$$\boldsymbol{Z}_\alpha=\{z=\sum_{(n_1,n_2)\in N\times N}\zeta_{n_1,n_2}\phi_{n_1,n_2}\Big|$$

$$\sum_{(n_1,n_2)\in N\times N}(\mu^{\alpha}_{n_1,n_2}\zeta_{n_1,n_2})^2<\infty\} \tag{17}$$

$$\boldsymbol{A}_\alpha z=\sum_{(n_1,n_2)\in N\times N}\lambda_{n_1,n_2}\zeta_{n_1,n_2}\phi_{n_1,n_2}$$

$$\forall z=\sum_{(n_1,n_2)\in N\times N}\zeta_{n_1,n_2}\phi_{n_1,n_2}\in\boldsymbol{Z}_\alpha \tag{18}$$

$$\boldsymbol{B}_\alpha u=(\sum_{n_2\in N}\mu_{0,n_2}g^1_{0,n_2}\phi_{0,n_2})u_1(t)$$

$$+(\sum_{n_1\in N}\mu_{n_1,0}g^2_{n_1,0}\phi_{n_1,0})u_2(t)$$

$$+(\sum_{n_2\in N}\mu_{0,n_2}g^3_{0,n_2}\phi_{0,n_2})u_3(t)$$

$$+(\sum_{n_1\in N}\mu_{n_1,0}g^4_{n_1,0}\phi_{n_1,0})u_4(t)\,,\ \forall u\in\boldsymbol{R}^4 \tag{19}$$

To simplify the notation of $\boldsymbol{B}_\alpha u$, we define $\boldsymbol{\psi}^i_n$ and b^i, $i\in\boldsymbol{I}$, directly from Corollary 1 as

$$\boldsymbol{B}_\alpha u=(\sum_{n_2\in N}\boldsymbol{\psi}^1_{n_2}\phi_{0,n_2})u_1(t)+(\sum_{n_2\in N}\boldsymbol{\psi}^3_{n_2}\phi_{0,n_2})u_3(t)$$

$$+(\sum_{n_1\in N}\boldsymbol{\psi}^2_{n_1}\phi_{n_1,0})u_2(t)+(\sum_{n_1\in N}\boldsymbol{\psi}^4_{n_1}\phi_{n_1,0})u_4(t) \tag{20}$$

$$=b^1u_1(t)+b^2u_2(t)+b^3u_3(t)+b^4u_4(t) \tag{21}$$

At the end of this Section it is mentioned that due to part (i) of Theorem 1, the systems studied here are called spectral systems (Curtain and Zwart, 1995) when they are described by (16).

IV. OBSERVER-BASED FINITE-DIMENSIONAL CONTROLLERS

In the distributed parameter systems described by (16), note that $\boldsymbol{B}_\alpha$ and $\boldsymbol{C}$ are bounded operators. By Eq. (18), it is not difficult to check that eigenvalues of $\boldsymbol{A}_\alpha$ are also λ_{n_1,n_2}, $(n_1, n_2)\in\boldsymbol{N}\times\boldsymbol{N}$, which are all real and non-positive. Now re-label eigenvalues λ_{n_1,n_2} as distinct λ_i for i=1, 2, ... such that $\lambda_j<\lambda_k$ for $j>k$, and let the eigenfunctions of $\boldsymbol{A}_\alpha$ corresponding to the distinct eigenvalue λ_i, i=1, 2, $\cdots$, be denoted by τ_{ij}, $j=1,\cdots,m_i$. The set $\{\tau_{ij}|i=1, 2, \cdots$ and $j=1,\cdots,m_i.\}$ still forms a basis of $\boldsymbol{Z}_\alpha$. For a pair of positive integers $l<n$ chosen in advance, define the projection

$$\boldsymbol{P}_n z=\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left\langle z,\tau_{ij}\right\rangle_Z\tau_{ij}$$

$$\boldsymbol{Q}_n=I-\boldsymbol{P}_n$$

and the quantities $z_{ij}=\left\langle z,\tau_{ij}\right\rangle_Z$, $z_i^T(t)=[z_{i1}\ \cdots\ z_{im_i}]$, $\boldsymbol{\eta}_1^T=[z_1^T(t)\ \cdots\ z_l^T(t)]$, $\boldsymbol{\eta}_2^T=[z_{l+1}^T(t)\ \cdots\ z_n^T(t)]$, $\boldsymbol{A}_1=diag[\lambda_1 I_{m_1}\ \cdots\ \lambda_l I_{m_l}]$, $A_2=diag[\lambda_{l+1}I_{m_{l+1}}\ \cdots\ \lambda_n I_{m_n}]$, $\boldsymbol{b}_{ij}=[\left\langle b^1,\tau_{ij}\right\rangle_Z\ \cdots\ \left\langle b^4,\tau_{ij}\right\rangle_Z]$, $\tilde{\boldsymbol{B}}_i^T=[\boldsymbol{b}_{i1}^T\ \cdots\ \boldsymbol{b}_{im_i}^T]$, $\boldsymbol{B}_1^T=[\tilde{\boldsymbol{B}}_1^T(t)\ \cdots\ \tilde{\boldsymbol{B}}_l^T(t)]$, $\boldsymbol{B}_2^T=[\tilde{\boldsymbol{B}}_{l+1}^T(t)\ \cdots\ \tilde{\boldsymbol{B}}_n^T(t)]$, $c_{ij}^k=\left\langle c_k(x),\tau_{ij}\right\rangle_Z$,

$$\tilde{C}_i = \begin{bmatrix} c_{i1}^1 & \cdots & c_{im_i}^1 \\ \vdots & & \vdots \\ c_{i1}^p & \cdots & c_{im_i}^p \end{bmatrix}$$

$C_1=[\tilde{C}_1 \cdots \tilde{C}_l]$, $C_2=[\tilde{C}_{l+1} \cdots \tilde{C}_n]$, and $S_n\xi=[\langle Q_n c_1, \xi\rangle_Z \cdots \langle Q_n c_p, \xi\rangle_Z]^T$ for $\xi\in Q_n(Z_\alpha)$. To meet the subsequent need, it is further assumed that the pair (A_1, B_1) is controllable and the pair (A_1, C_1) is observable, which is equivalent to assuming that $rank(\tilde{B}_i)=m_i$ and $rank(\tilde{C}_i)=m_i$. With all these definitions and assumptions we can construct a finite dimensional subsystem

$$\begin{cases} \dot{\eta}_1 = A_1\eta_1 + B_1u(t) \\ \dot{\eta}_2 = A_2\eta_2 + B_2u(t) \\ y(t) = C_1\eta_1(t) + C_2\eta_2(t) + S_nQ_nz(t) \end{cases} \tag{22}$$

of the system (16) (Sakawa, 1983). Based on this subsystem it is possible to construct a finite-dimensional observer

$$\begin{cases} \dot{v}_1 = A_1v_1 + B_1u(t) + G_1(y - G_1v_1 - C_2v_2) \\ \dot{v}_2 = A_2v_2 + B_2u(t) \end{cases} \tag{23}$$

and to consider an observer-based state feedback controller

$$u=F_1v_1(t) \tag{24}$$

where F_1 and G_1 are constant matrices of appropriate dimensions, and need to be determined. The following Theorem, which is a summary of the results in (Sakawa, 1983), gives a set of conditions for designing the observer-based controller (23) and (24) to stabilize the system (16).

Theorem 3: *(Sakawa, 1983) Consider the system (16) controlled by the observer-based controller (23) and (24). Define the matrix and the operator*

$$A_{11} = \begin{bmatrix} A_1 & B_1F_1 \\ G_1C_1 & A_1 - G_1C_1 + B_1F_1 \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G_1S_n \\ 0 & Q_nB_\alpha F_1 & 0 \end{bmatrix} \tag{25}$$

respectively. Suppose l, n, F_1, and G_1 are chosen such that $\|e^{A_{11}t}\|\le Me^{\rho t}$ for all t with some $M\ge1$ and $\rho<0$, and $\lambda_{n+1}<\sigma(A_{11})<\rho+M\|\tilde{B}\|<\lambda_{l+1}<0$, where $\sigma(\cdot)$ denotes the largest real part of all eigenvalues of its argument matrix, then the overall closed-loop system is exponentially stable. Moreover, λ_{n+1} becomes more negative and $\|\tilde{B}\|$ becomes smaller when n increases.

V. SUB-OPTIMAL CONTROLLER DESIGN

In this Section, we give the following controller design procedure to determine the two matrices F_1 and G_1. First, we find F_1 based on linear quadratic optimal control. For the finite dimensional subsystem described by

$$\dot{\eta}_1 = A_1\eta_1 + B_1u(t) \tag{26}$$

consider the performance index

$$V^2 = \int_0^\infty (u^TRu + \eta_1^TQ\eta_1)dt \tag{27}$$

It is well known (Anderson and Moore, 1990) that given every pair of positive definite weighting matrices Q and R, a stabilizing state feedback law $u(t)=F_1^*\eta_1(t)$ may be found for the subsystem (26) by minimizing the performance index (27), where $F_1^*=-R^{-1}B_1^T\overline{P}$ and $\overline{P}$ satisfies the finite dimensional Riccati equation

$$\overline{P}A_1 + A_1^T\overline{P} - \overline{P}B_1R^{-1}B_1^T\overline{P} + Q = 0 \tag{28}$$

Furthermore, the minimal value of the performance index (27) corresponding to the control law $u(t)=F_1^*\eta_1(t)$ is

$$V^{*2} = \int_0^\infty \eta_1^T(Q + F_1^{*T}RF_1^*)\eta_1dt \tag{29}$$

The matrix G_1 can also be determined with this approach by considering the pair $\{A_1^T, -C_1^T\}$. To fit our purpose, the weighting matrices should be selected to produce F_1 and G_1 which satisfy the conditions of Theorem 3. This is always possible, since the eigenvalues of A_{11} are the union of those of $A_1+B_1F_1$ and $A_1-G_1C_1$, which may be adjusted by properly setting the weighting matrices to get the appropriate $\sigma(A_{11})$, ρ, and M. Besides, a larger integer n may be used when necessary.

However, in this paper we are not only concerned with the overall system stability, but also the overall performance. Notice that the minimal value (29) of the performance index corresponds to the state feedback $u(t)=F_1^*\eta_1(t)$, but the actual control signal here is the observer-based state feedback $u(t)=F_1^*v_1(t)$, which is much easier to implement than

$u(t)=F_1^*\eta_1(t)$. It is interesting to estimate that with the more practical control law, to what extent will the performance index degrade. Thus consider the same performance index

$$\tilde{V}^2=\int_0^\infty \eta_1^T(Q+F_1^{*T}RF_1^*)\eta_1 dt \tag{30}$$

where the state trajectory $\eta_1(\cdot)$ corresponds to $u(t)=F_1^*v_1(t)$ rather than $u(t)=F_1^*\eta_1(t)$. Below a Theorem provides a bound on the difference between V^* in (29) and $\tilde{V}$ in (30).

Theorem 4: *Consider the system (16) controlled by the observer-based controller (23) and (24) with initial conditions $v_1(0)=0$ and $v_2(0)=0$. Suppose the conditions in Theorem 3 are satisfied. Decompose the matrix $Q+F_1^{*T}RF_1^*$ as D^TD, where D is a symmetric matrix. Let $\overline{\gamma}=\lambda_{l+1}+\rho$*

$$\overline{A}_1=\begin{bmatrix} 0 & B_1F_1^* & 0 \\ 0 & 0 & G_1S_n \\ 0 & Q_nB_\alpha F_1^* & 0 \end{bmatrix} \tag{31}$$

$\overline{M}=M\|\overline{A}_1\|$, $k_0=\|G_1\|\|C_2\|\|z_0\|$, $p_0=\rho^2\|z_0\|^2[k_0(\overline{M}+\overline{\gamma}+\rho)(2\lambda_{l+1}-k_0)-\lambda_{l+1}\overline{M}(\overline{M}+1)]$, $q_0=2\lambda_{l+1}M\overline{M}(M+\overline{\gamma})(\overline{M}+\overline{\gamma})(2\rho+\overline{\gamma})$. Then we have $|V^*-\tilde{V}|<\|D\|\sqrt{\dfrac{p_0}{q_0}}$.

Proof: The states η_1, v_1, and Q_nz of the overall closed-loop system satisfy

$$\frac{d}{dt}\begin{bmatrix} \eta_1(t) \\ v_1(t)-\eta_1(t) \\ Q_nz(t) \end{bmatrix}$$

$$=\begin{bmatrix} A_1+B_1F_1^* & B_1F_1^* & 0 \\ 0 & A_1-G_1C_1 & G_1S_n \\ 0 & Q_nB_\alpha F_1^* & AQ_n \end{bmatrix}\begin{bmatrix} \eta_1(t) \\ v_1(t)-\eta_1(t) \\ Q_nz(t) \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ G_1C_2e^{A_2t}[\eta_2(0)-v_2(0)] \\ 0 \end{bmatrix}$$

or more compactly

$$\dot{\tilde{\omega}}(t)=(\overline{A}_0+\overline{A}_1)\tilde{\omega}(t)+\tilde{\psi}(t) \tag{32}$$

where

$$\tilde{\omega}=\begin{bmatrix} \eta_1 \\ v_1-\eta_1 \\ Q_nz \end{bmatrix}$$

$$\overline{A}_0=\begin{bmatrix} A_1+B_1F_1^* & 0 & 0 \\ 0 & A_1-G_1C_1 & 0 \\ 0 & 0 & AQ_n \end{bmatrix}$$

$$\tilde{\psi}(t)=\begin{bmatrix} 0 \\ G_1C_2e^{A_2t}[\eta_2(0)-v_2(0)] \\ 0 \end{bmatrix}$$

Consider the approximate system

$$\dot{\hat{\omega}}(t)=\overline{A}_0\hat{\omega}(t)+\tilde{\psi}(t) \tag{33}$$

and the space $\overline{Z}=R^{l_0}\oplus R^{l_0}\oplus Q_nZ$, where $\hat{\omega}^T=\hat{\eta}_1^T(\hat{v}_1-\hat{\eta}_1)^T(Q_n\hat{z})^T]^T$ and $l_0=\sum_{i=1}^{l}m_i$. For $w=[\omega_1^T\ \omega_2^T\ \omega_3^T]^T\in\overline{Z}$, define the norm on $\overline{Z}$ as $\|\omega\|_{\overline{Z}}=\sqrt{\|\omega_1\|_{R^{l_0}}^2+\|\omega_2\|_{R^{l_0}}^2+\|\omega_3\|_Z^2}$. Also define $\overline{D}=diag[D\ 0\ 0]\in\mathcal{B}(\overline{Z},\overline{Z})$. Thus we have

$$V^{*2}=\int_0^\infty \hat{\omega}^T(t)\overline{D}^T\overline{D}\hat{\omega}(t)dt=\int_0^\infty \|\overline{D}\hat{\omega}(t)\|_{\overline{Z}}^2 dt$$

$$=\int_0^\infty \|D\hat{\eta}_1(t)\|_{R^{l_0}}^2 dt \tag{34}$$

Observe that

$$\tilde{V}^2=\int_0^\infty \|\overline{D}\tilde{\omega}(t)\|_{\overline{Z}}^2 dt \tag{35}$$

is the performance index of the closed-loop system. Introducing the norm for square integrable signals defined on $\overline{Z}$ as $\|\cdot\|_{L_2(\overline{Z})}=\{\int_0^\infty\|\cdot\|_{\overline{Z}}^2dt\}^{1/2}$ and taking advantage of its associated inequality $\left|\|\omega_1\|_{L_2(\overline{Z})}-\|\omega_2\|_{L_2(\overline{Z})}\right|\le\|\omega_1-\omega_2\|_{L_2(\overline{Z})}$, we get

$$|V^*-\tilde{V}|=\left|\|\overline{D}\hat{\omega}\|_{L_2(\overline{Z})}-\|\overline{D}\tilde{\omega}\|_{L_2(\overline{Z})}\right|$$

$$\le\|\overline{D}\hat{\omega}-\overline{D}\tilde{\omega}\|_{L_2(\overline{Z})}$$

$$=\{\int_0^\infty\|\overline{D}(\hat{\omega}-\tilde{\omega})\|_{\overline{Z}}^2dt\}^{1/2} \tag{36}$$

Now, let $T_0(t)$ and $T_1(t)$ be the semigroups generated by $\overline{A}_0$ and $\overline{A}_0+\overline{A}_1$, respectively. The solutions of Eqs. (32) and (33) can be represented, respectively, as

$$\hat{\omega}(t)=T_0(t)\omega(0)+\int_0^t T_0(t-\tau)\tilde{\psi}(\tau)d\tau$$

$$\tilde{\omega}(t)=T_1(t)\omega(0)+\int_0^t T_1(t-\tau)\tilde{\psi}(\tau)d\tau$$

Then we have the inequality

$$\left\|(\hat{\omega}-\tilde{\omega})(t)\right\|_{\overline{Z}}\leq\left\|(T_0-T_1)(t)\right\|_{B(\overline{Z})}\left\|\omega(0)\right\|_{\overline{Z}}+\int_0^t\left\|(T_0-T_1)(t-\tau)\right\|_{B(\overline{Z})}\left\|\tilde{\psi}(\tau)\right\|_{\overline{Z}}d\tau \tag{37}$$

where $B(\overline{Z})=B(\overline{Z},\overline{Z})$. By the conditions of Theorem 3 and the boundedness of $\overline{A}_1$, $\left\|T_0(t)\right\|_{B(\overline{Z})}\leq Me^{\rho t}$ and $\left\|(T_0-T_1)(t)\right\|_{B(\overline{Z})}\leq Me^{\rho t}(e^{M\|\overline{A}_1\|_{\overline{Z}}t}-1)$. Moreover, since $\left\|e^{A_2t}\right\|\leq e^{\lambda_{l+1}t}$ and $v_1(0)=0$, $v_2(0)=0$, we have $\left\|\tilde{\psi}(t)\right\|_{\overline{Z}}\leq\left\|G_1\right\|\left\|C_2\right\|e^{\lambda_{l+1}t}\left\|z_0\right\|$. Substituting these upper bounds into (37), we obtain the following inequality

$$\left\|(\hat{\omega}-\tilde{\omega})(t)\right\|_{\overline{Z}}$$

$$\leq Me^{\rho t}(e^{M\|\overline{A}_1\|_{\overline{Z}}t}-1)\left\|z_0\right\|+\int_0^t Me^{\rho t}(e^{M\|\overline{A}_1\|_{\overline{Z}}(t-\tau)}-1)\cdot\left\|G_1\right\|\left\|C_2\right\|e^{\lambda_{l+1}(t-\tau)}\left\|z_0\right\|d\tau$$

Then by a lengthy but direct computation, we finally have

$$\int_0^\infty\left\|(\tilde{\omega}-\hat{\omega})\right\|_{\overline{Z}}^2dt\leq\frac{p_0}{q_0} \tag{38}$$

which gives

$$\left|V^*-\tilde{V}\right|<\left\|D\right\|\sqrt{\frac{p_0}{q_0}} \tag{39}$$

and finishes the proof. ■

VI. AN EXAMPLE

Suppose $l_1=1$ and $l_2=1$, i.e., $\Omega=\{[x_1\ x_2]^T|0\leq x_1\leq 1, 0\leq x_2\leq 1\}$. Consider the distributed parameter system described by the following mathematical model

$$\begin{cases}\dfrac{\partial z(t,x)}{\partial t}=\dfrac{\partial^2}{\partial x_1^2}z(t,x)+\dfrac{\partial^2}{\partial x_2^2}z(t,x)\\ -\dfrac{\partial}{\partial x_2}z(t,x)\big|_{x_2=0}=u_1(t)\\ \dfrac{\partial}{\partial x_1}z(t,x)\big|_{x_1=1}=u_2(t)\\ \dfrac{\partial}{\partial x_2}z(t,x)\big|_{x_2=1}=0\\ -\dfrac{\partial}{\partial x_1}z(t,x)\big|_{x_1=0}=0\\ z(0,x)=2,\ \forall x\in\Omega\\ y(t)=\displaystyle\int_\Omega 0.1\sin(x_1+x_2)z(t,x)dx_1dx_2\end{cases} \tag{40}$$

For this system, ϕ_{n_1,n_2} in Theorem 1 may be expressed as

$$\begin{cases}\phi_{0,0}(x_1,x_2)=1\\ \phi_{0,n_2}(x_1,x_2)=\sqrt{2}\cos(n_2\pi x_2)\ ,\ n_2\geq 1\\ \phi_{n_1,0}(x_1,x_2)=\sqrt{2}\cos(n_1\pi x_1)\ ,\ n_1\geq 1\\ \phi_{n_1,n_2}(x_1,x_2)=\sqrt{2}\cos(n_1\pi x_1)\sqrt{2}\cos(n_2\pi x_2)\\ \quad n_1\geq 1, n_2\geq 1\end{cases} \tag{41}$$

Note that in the above expressions the amplitude of ϕ_{n_1,n_2} is scaled for convenience, but it is easy to check that $A\phi_{n_1,n_2}=\lambda_{n_1,n_2}\phi_{n_1,n_2}$, where $\lambda_{n_1,n_2}=-(n_1^2+n_2^2)\pi^2$. Also, for this system $\omega_0=0$. We choose $\omega=1$, and G is thus

$$[G(1)u](x)$$

$$=\frac{e^{x_2}+e^{2-x_2}}{e^2-1}u_1(t)+\frac{e^{1+x_1}+e^{1-x_1}}{e^2-1}u_2(t)$$

$$=(\sum_{n_2\in N}g^1_{0,n_2}\phi_{0,n_2})u_1(t)+(\sum_{n_1\in N}g^2_{n_1,0}\phi_{n_1,0})u_2(t)$$

where

$$g^1_{0,n_2}=\begin{cases}1 & n_2=0\\ \dfrac{\sqrt{2}}{1+n_2^2\pi^2} & n_2\geq 1\end{cases} \tag{42}$$

$$g^2_{n_1,0}=\begin{cases}1 & n_1=0\\ \dfrac{(-1)^{n_1}\sqrt{2}}{1+n_1^2\pi^2} & n_1\geq 1\end{cases} \tag{43}$$

Now re-label eigenvalues λ_{n_1,n_2} as distinct λ_i for $i=1, 2, \ldots$ such that $\lambda_j<\lambda_k$ for $j>k$. Here the first three distinct eigenvalues are $\lambda_1=0$, $\lambda_2=-\pi^2$, and $\lambda_3=-2\pi^2$. The eigenfunction corresponds to λ_1 is $\tau_{11}=1$, those to λ_2 are $\tau_{21}=\sqrt{2}\cos(\pi x_1)$ and $\tau_{22}=\sqrt{2}\cos(\pi x_2)$, and the one

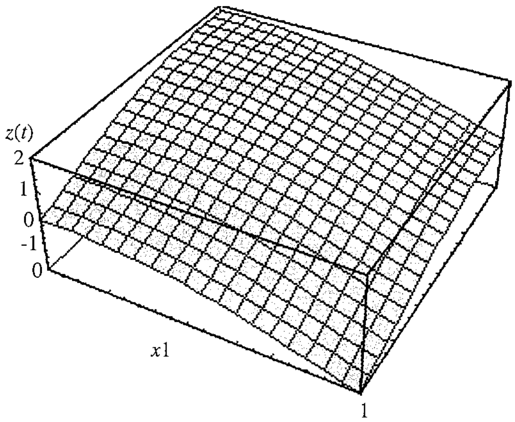

Fig. 1 Controlled state response at t=0.1

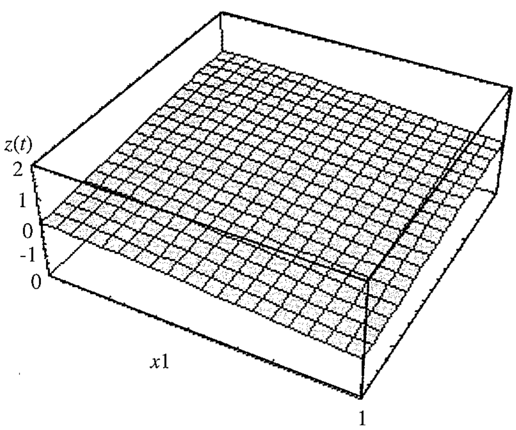

Fig. 2 Controlled state response at t=0.3

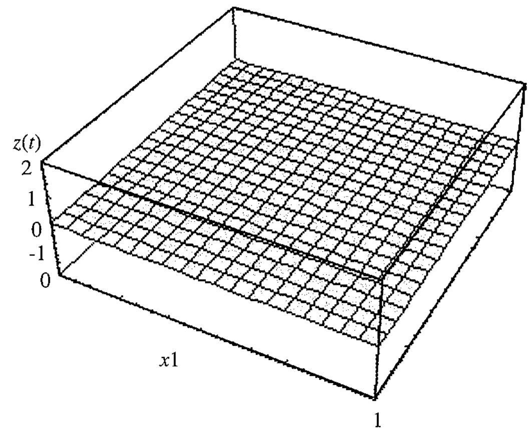

Fig. 3 Controlled state response at t=0.6

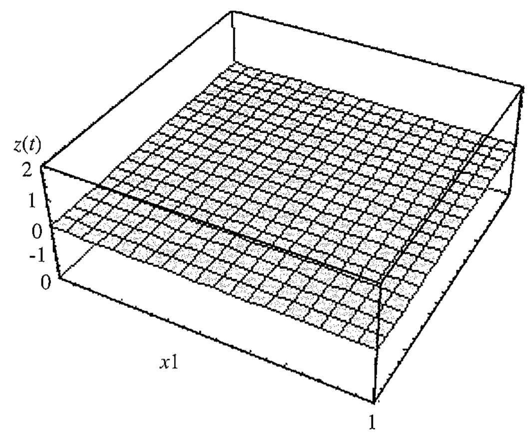

Fig. 4 Controlled state response at t=1.0

to λ_3 is $\tau_{31}=\sqrt{2}\cos(\pi x_1)\sqrt{2}\cos(\pi x_2)$, which imply m_1=1, m_2=2, and m_3=1. Our controller design starts with the choice of l=1, n=2,

$$\boldsymbol{A}_1=[0],\ \boldsymbol{B}_1=[1\ 1],\ \boldsymbol{C}_1=[0.7736] \tag{44}$$

and

$$\boldsymbol{A}_2=\begin{bmatrix}-\pi^2 & 0\\ 0 & -\pi^2\end{bmatrix},\ \boldsymbol{B}_2=\begin{bmatrix}0 & -\sqrt{2}\\ \sqrt{2} & 0\end{bmatrix},$$

$$\boldsymbol{C}_2=[-0.145\ \ -0.145] \tag{45}$$

After some trials, the weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ are selected to be

$$\boldsymbol{Q}=[12] \text{ and } \boldsymbol{R}=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix} \tag{46}$$

which gives

$$\boldsymbol{F}_1=\begin{bmatrix}-7.746\\ -7.746\end{bmatrix} \tag{47}$$

The observer gain is similarly set to $\boldsymbol{G}_1$=20. Thus, we have $\sigma(\boldsymbol{A}_{11})=-15.472$, $\|e^{A_{11}}\|\le 5.3e^{-13t}=Me^{\rho t}$ and $\|\tilde{\boldsymbol{B}}\|\le 0.567$. At this point, we check the conditions of Theorem 3 and get

$$-2\pi^2=\lambda_3<\sigma(\boldsymbol{A}_{11})<\rho<\rho+M\|\tilde{\boldsymbol{B}}\|<\lambda_2=-\pi^2 \tag{48}$$

Therefore, the observer-based controller

$$\begin{cases}\dot{v}_1=[1\ 1]u+20(y-0.7736v_1-[-0.145\ \ -0.145]v_2)\\ \dot{v}_2=\begin{bmatrix}-\pi^2 & 0\\ 0 & -\pi^2\end{bmatrix}v_2+\begin{bmatrix}0 & -\sqrt{2}\\ \sqrt{2} & 0\end{bmatrix}u\\ u=\begin{bmatrix}-7.746\\ -7.746\end{bmatrix}v_1\end{cases} \tag{49}$$

makes the overall closed-loop system exponentially stable. Furthermore, since $\|\overline{\boldsymbol{A}}_1\|\le 15.49$, by Theorem 4 we can get the bound $|V^*-\tilde{V}|<0.49$ with $V^*=3.0984$. Fig. 1 to Fig. 4 show the simulation result of the controlled state function $z(t, x)$ at t=0.1, 0.3, 0.6, and 1.0, respectively. Clearly the closed-loop system is exponentially stabilized by the boundary control.

VII. CONCLUSIONS

A systematic method is proposed for finding a sub-optimal boundary controller for a certain class of distributed parameter systems on a planar domain. The controller is finite dimensional and observer-based, so it is easy to implement. An example with simulation results is given. With the first part of the proposed systematic method, it is believed that many other finite dimensional control laws can also be adapted for distributed parameter systems.

ACKNOWLEDGMENTS

This work is supported by the National Science Council of the Republic of China under Grant NSC 88-2213-E-002-089.

NOMENCLATURE

$B(\boldsymbol{X}, \boldsymbol{Y})$	The space of all bounded linear operators from the space X to the space Y
$\mathcal{D}(\boldsymbol{A})$	The domain of an operator A
I	The identity matrix or identity operator
I	The set of $\{1, 2, 3, 4\}$
$\mathcal{J}(\boldsymbol{A})$	The range of an operator A
$\mathcal{K}(\boldsymbol{A})$	The kernel of an operator A
$\boldsymbol{N}$	The set of nonnegative integers
n_i	The outward unit normal vector on Γ_i
$\boldsymbol{R}$	The set of real numbers
$\boldsymbol{R}^\ell$	The vector space of ℓ-tuple vectors with elements from R
$\boldsymbol{T}(\cdot)$	The analytic semigroup generated by the operator A
u_i	The ith control input
V^2	A performance index
$\boldsymbol{x}$	A vector in R^2 with the coordinate vector $[x_1\ x_2]^T$
$\boldsymbol{Z}$	$L_2(\Omega)$, the Lebesgue square integrable functions defined on the domain Ω
$\boldsymbol{Z}_\alpha$	The interpolation space $\mathcal{D}[(\omega I-A)^\alpha]$
$z(\cdot, \cdot)$	The state function of the system to be controlled
$\partial\boldsymbol{\Omega}$	The boundary of the domain Ω
$\boldsymbol{\Gamma}_i$	The ith boundary line of the domain Ω
$\boldsymbol{\Omega}$	A planar domain on R^2
ω_0	The growth constant of $T(\cdot)$
$\langle \cdot, \cdot \rangle_Z$	The inner product function defined on Z
$\|\cdot\|$	Norms of matrices and operators

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Manuscript Received: Mar.25, 2002
Revision Received: Oct. 18, 2002
and Accepted: Nov. 11, 2002