ROBUST STATE FEEDBACK CONTROL THROUGH ACTUATORS WITH GENERALIZED SECTOR NONLINEARITIES AND SATURATION

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ABSTRACT

In control systems, actuators often have nonlinear characteristics that can not be neglected. For linear systems driven by actuators satisfying the generalized sector condition, a robust state feedback controller synthesis method is proposed to achieve the ultimate boundedness control. The method is based on the linear matrix inequality approach and is easy to apply. As an important special case of the generalized sector condition, the saturation characteristic of actuators is discussed separately, and non-conservative results are obtained.

KeyWords: Actuator nonlinearity, LMI, robust stability, saturation.

I. INTRODUCTION

Most actuators in real control systems are subject to some nonlinearities due to technological factors or physical constraints. Though many actuators are manufactured so as to have pretty good linear characteristics over their main operation ranges, nonlinearities, such as deadzone and saturation, inevitably exist. It is just a matter of degree. For certain applications, especially those that involve large amounts of power, if these nonlinearities are not properly accounted for, they will cause the overall performance to deteriorate, damage the system, or result in instability. Consequently, for many decades, control problems with nonlinear actuators have attracted considerable interest, and no less recently [8,11,12,14, 15]. In the celebrated Lur'e problem [16], a general class of nonlinearities are described by means of the so-called sector condition, and many analytic or graphic stability criteria are derived. However, some common nonlinearities, such as deadzone and hysteresis, which are often seen in hydraulic or electro-magnetic devices, do not satisfy the sector condition. Therefore, we will propose a generalized sector condition to cover more nonlinear characteristics of actuators and discuss the corresponding control problem.

In addition, we will discuss a specific problem involving actuators with the standard saturation characteristic. It is probably the most frequently studied problem when actuator nonlinearities are the issue because almost all actuators have this characteristic. To deal with this problem, one very important approach is to use the set invariance concept [3,20]. The main idea is to establish some positively invariant set in the state space. This approach can be further divided into approaches that use ellipsoidal and polyhedral invariant sets, which lead to various solution methods, depending on the adopted mathematical tools, such as linear programming [1,2,18, 19], convex optimization [6,11], eigenstructure assignment [5], and polynomial formulation [10]. For this input saturation problem, the goal is usually to find a stabilizing controller, or to find a large stability region in the state space, given a stabilizing controller and actuator saturation data [6,12,17]. Here, we will attempt to achieve both goals by utilizing the results derived for the general nonlinearity problem.

Besides actuator nonlinearities, the other feature of the control problem studied in this paper is plant uncertainty. We will adopt the polytopic linear differential inclusion (PLDI) description [4] for the plant and study the robust state feedback stabilization problem. The exact problem formulation will be given in Section 2. A sufficient condition for the existence of an ultimate boundedness controller in the presence of actuators satisfying the generalized sector condition will be developed in Section 3. The sufficient condition will be given in terms of a linear matrix inequality (LMI) so that it will be easy to search for feasible controllers, which will eventually regulate the system trajectories inside an ellipsoidal ultimate boundedness set with the major semi-axis shorter than a prescribed length. In Section 4,

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we shall apply the results given in Section 3 to deal with control problems with saturating actuators and look for state feedback gain and a stability region simultaneously. Here, all the results involve LMI or bilinear matrix inequality (BMI) and will be accompanied by examples to show how the proposed methods can be applied. Finally, some conclusions will be drawn in Section 5.

Before we start, we will define some notations first. For any two matrices X, $Y \in \mathcal{R}^{n\times n}$, $X \geq Y$ means that X, Y are symmetric, and that $X - Y$ is positive semi-definite. Similar notations will apply to symmetric positive/negative definite matrices. If $X > 0$, then $\lambda_{max}(X)$ denotes its largest eigenvalue. The notation $\mathrm{diag}(X_1, \ldots, X_\ell)$ stands for the block diagonal matrix with diagonal blocks $X_1, \ldots, X_\ell$. The transpose of a real matrix X is denoted by X^T. I_m is the $m \times m$ identity matrix, and e_i is the ith column of I_m. In a symmetric block matrix, for simplicity, the symbol * represents the submatrices that lie above the diagonal. Finally, the notation $\|\cdot\|$ denotes the 2-norm of the argument vector or matrix, and $\mathrm{Co}\{\cdot\}$ represents the convex hull of the set in the argument.

II. PROBLEM FORMULATION

Consider the uncertain system described by the mathematical model

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(0) = x_0, \\ \tilde{u}(t) &= Kx(t), \\ u(t) &= [\psi_1(t, \tilde{u}_1(t)) \cdots \psi_m(t, \tilde{u}_m(t))]^T, \end{aligned} \tag{1}$$

where

$$[A(t)\ B(t)] \in \mathrm{Co}\{[A_1\ B_1], \cdots, [A_l\ B_l]\}. \tag{2}$$

The first equation in (1) represents a plant descibed by PLDI with the state vector $x(t) \in \mathcal{R}^n$ and the input vector $u(t) = [u_1(t) \ldots u_m(t)]^T \in \mathcal{R}^m$. It is assumed that the pairs $\{A_i, B_i\}$, $i = 1, 2, \ldots, l$, are controllable. The plant is to be stabilized by the state feedback controller of the second equation in (1), where $\tilde{u}(t) = [\tilde{u}_1(t) \ldots \tilde{u}_m(t)]^T \in \mathcal{R}^m$ is the control signal vector, and $K = [k_1^T \ldots k_m^T]^T \in \mathcal{R}^{m\times n}$ is the state feedback gain matrix. The third equation in (1) represents m actuators which convert the control signals to inputs. The jth actuator is assumed to have the nonlinear characteristic $\psi_j(\cdot, \cdot) : [0, \infty) \times \mathcal{R} \to \mathcal{R}$, which belongs to the generalized sector $[\alpha_j, \beta_j]$ with bias $[\delta_{1j}, \delta_{2j}]$, as shown in Fig. 1. Note that the actuator nonlinear characteristic is not confined in the first and third quadrants, and does not necessarily pass the origin of the characteristic plane. This enables us to accommodate many of actuator nonlinearities, especially those which are multi-valued, such as backlash, and do not belong to the standard sector condition [16]. Mathematically, this

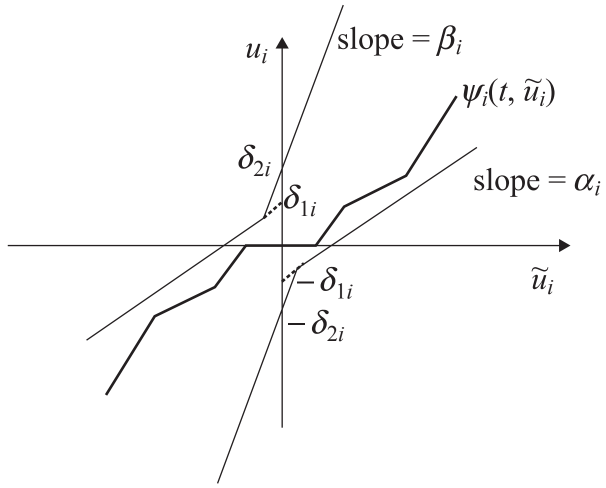

Fig. 1. Generalized sector nonlinearity.

is equivalent to saying that at any instant t, the signals $u_j(t)$ and $k_j^T x(t)$ satisfy one of the following conditions:

$$[u_j(t) - \alpha_j k_j x(t) + \delta_{1j}][u_j(t) - \beta_j k_j x(t) - \delta_{2j}] \leq 0 \tag{3}$$

or

$$[u_j(t) - \alpha_j k_j x(t) - \delta_{1j}][u_j(t) - \beta_j k_j x(t) + \delta_{2j}] \leq 0 \tag{4}$$

for $j = 1, \ldots, m$.

For the above system (1), we will discuss in Section 3 how to synthesize state feedback controllers to achieve the ultimate boundedness control. The following definition gives a precise description of the concept of ultimate boundedness.

Definition 1. [13] The solutions of

$$\dot{x}(t) = f[t, x(t)], \quad x(t_0) = x_0, \tag{5}$$

are ultimately bounded (with bound β) if there exists a $\beta > 0$ and if corresponding to any $\alpha > 0$ and $t_0 \geq 0$, there exists a $t_1(\alpha) > 0$ such that $\|x_0\| < \alpha$ implies that $\|x(t)\| < \beta$ for all $t \geq t_0 + t_1$.

In this paper, the control purpose is made more specific. Given a number $\gamma_o > 0$, it is desired that the state trajectories of the system (1) eventually enter and stay within some ellipsoidal set $\mathcal{E}_c = \{x \in \mathcal{R}^n \mid x^T P x \leq c\}$, where $P > 0$, $c > 0$, and the major semi-axis of $\mathcal{E}_c$ is no longer than γ_o. The problem is to derive conditions which enable us to find state feedback controllers such that the system (1) behaves as desired. In Section 4, we will focus on actuators with the standard saturation characteristic (to be defined later). In this case, the control purpose is to make the closed-loop system asymptotically stable. However, due to saturation, not all the trajectories can be brought back to the origin of the state space. Hence, the problem is to find a state feedback controller which forces the state trajectories that start from the largest possible ellipsoidal set $\mathcal{P} = \{x \in \mathcal{R}^n \mid x^T P x \le 1\}$ of initial conditions to converge. Here, the concept of a positively invariant set will be called upon, and a definition for it is given below.

Definition 2. [3,20] The set $\Omega \subset \mathcal{R}^n$ is said to be positively invariant for the system (5) if for all $x(t_0) \in \Omega$, the solution $x(t) \in \Omega$ for all $t > t_0$.

III. GENERALIZED SECTOR NONLINEARITIES

We will first present an LMI-based condition for the existence of state feedback controllers which solve the problem defined in Section 2.

Theorem 1. Consider the uncertain system (1) driven by nonlinear actuators which satisfy the generalized sector conditions depicted in Fig. 1. Let $S_{min} = \mathrm{diag}(\alpha_1, \ldots, \alpha_m)$, $S_{max} = \mathrm{diag}(\beta_1, \ldots, \beta_m)$, $\delta_1 = [\delta_{11} \ldots \delta_{1m}]^T$, and $\delta_2 = [\delta_{21} \ldots \delta_{2m}]^T$. For a given $\gamma_o > 0$, if the LMIs

$$\begin{bmatrix} \Pi_i & * & * & * \\ \frac{1}{2}(\delta_2 - \delta_1)^T B_i^T & -\hat{\epsilon} & * & * \\ \frac{1}{2}(S_{max} - S_{min})Y & \frac{1}{2}(\delta_2 + \delta_1) & -T^{-1} & * \\ Q & 0 & 0 & -\tau_{m+1}^{-1} I_n \end{bmatrix} < 0,$$

$$i = 1, 2, \ldots, l, \tag{6}$$

$$\tau_{m+1}^{-1} + \lambda \le 2\gamma_o, \tag{7}$$

and

$$\lambda I_n \ge Q \ge \hat{\epsilon} I_n, \quad T^{-1} > 0, \quad \hat{\epsilon} > 0, \quad \tau_{m+1}^{-1} > 0, \tag{8}$$

where

$$\begin{aligned} \Pi_i = {} & QA_i^T + A_i Q + \frac{1}{2} B_i (S_{max} + S_{min}) Y \\ & + \frac{1}{2} Y^T (S_{max} + S_{min})^T B_i^T + B_i T^{-1} B_i^T, \end{aligned} \tag{9}$$

have the feasible solutions $Q \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{m\times n}$, $T^{-1} \in \mathcal{R}^{m\times m}$, τ_{m+1}^{-1}, λ, and $\hat{\epsilon}$, then the static state feedback control $\tilde{u}(t) = Kx(t)$ with $K = YQ^{-1}$ causes the trajectories of the system (1) to converge to and stay within the ellipsoid set $\mathcal{E}_c = \{x \in \mathcal{R}^n \mid x^T P x \le c\}$, where $P = Q^{-1}$, $c = \tau_{m+1}^{-1}$, and γ_o is an upper bound of the major semi-axis length of $\mathcal{E}_c$.

Proof. For $\mathcal{E}_c = \{x \in \mathcal{R}^n \mid x^T P x \le c\}$ to be an ultimate boundedness region of the controlled system (1), it is sufficient to ensure that the time derivative of the Lyapunov function candidate $\upsilon(x) = x^T P x$ is negative along all the state trajectories of (1) outside $\mathcal{E}_c$ for all the considered generalized sector nonlinearities. Here, we require that $d\upsilon[x(t)]/dt$ be negative outside a ball $\mathcal{B}_\epsilon = \{x \in \mathcal{R}^n \mid x^T x \le \epsilon\}$ which is contained by the set $\mathcal{E}_c$ for all the considered generalized sector nonlinearities. This is implied by the S-procedure [4], or by the existence of positive τ_j and $\tilde{\tau}_j$, $j = 1, \ldots, m + 1$, such that

$$\begin{aligned} & x^T[A^T(t)P + PA(t)]x + x^T PB(t)u + u^T B^T(t)Px \\ & -\sum_{j=1}^{m} \tau_j (u_j - \alpha_j k_j x + \delta_{1j})(u_j - \beta_j k_j x - \delta_{2j}) \\ & -\tau_{m+1}(\epsilon - x^T x) < 0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} & x^T[A^T(t)P + PA(t)]x + x^T PB(t)u + u^T B^T(t)Px \\ & -\sum_{j=1}^{m} \tilde{\tau}_j (u_j - \alpha_j k_j x - \delta_{1j})(u_j - \beta_j k_j x - \delta_{2j}) \\ & -\tilde{\tau}_{m+1}(\epsilon - x^T x) < 0 \end{aligned} \tag{11}$$

for all nonzero x, u, and $[A(t)\ B(t)]$ satisfying (2). Let us discuss (10) first. Defining $\hat{\epsilon} = \tau_{m+1}\epsilon > 0$ and $T = \mathrm{diag}(\tau_1, \ldots, \tau_m)$, we can rewrite (10) as

$$\begin{aligned} & x^T[A^T(t)P + PA(t)]x + x^T PB(t)u + u^T B^T(t)Px \\ & -u^T Tu + u^T TS_{min} Kx + u^T TS_{max} Kx - x^T K^T S_{min} TS_{max} Kx \\ & +\delta_1^T TS_{max} Kx + u^T T\delta_2 - \delta_1^T Tu - \delta_2^T TS_{min} Kx \\ & +\delta_1^T T\delta_2 - \hat{\epsilon} + \tau_{m+1} x^T x < 0, \end{aligned} \tag{12}$$

which must hold for all $[A(t)\ B(t)] \in Co\{[A_1\ B_1], \ldots, [A_l\ B_l]\}$. In matrix language, this is equivalent to

$$\begin{bmatrix} A_i^T P + PA_i + \tau_{m+1} I_n - K^T S_{min} TS_{max} K & * & * \\ \frac{1}{2}(\delta_1^T TS_{max} - \delta_2^T TS_{min})K & \delta_1^T T\delta_2 - \hat{\epsilon} & * \\ B_i^T P + \frac{1}{2} T(S_{max} + S_{min})K & \frac{1}{2} T(\delta_2 - \delta_1) & -T \end{bmatrix} < 0 \tag{13}$$

for $i = 1, 2, \ldots, l$. By the Schur-complement [4], (13) is equivalent to

$$\begin{bmatrix} \begin{pmatrix} A_i^T P + PA_i + \tau_{m+1} I_n \\ -K^T S_{min} T S_{max} K + PB_i T^{-1} B_i^T P \\ +\frac{1}{2} K^T (S_{min}+S_{max})^T B_i^T P \\ +\frac{1}{2} PB_i (S_{min}+S_{max}) K \\ +\frac{1}{4} K (S_{min}+S_{max})^T T (S_{min}+S_{max}) K \end{pmatrix} & * \\ \begin{pmatrix} \frac{1}{2}(\delta_1^T T S_{max} - \delta_2^T T S_{min}) K \\ +\frac{1}{2}(\delta_2-\delta_1)^T B_i^T P \\ +\frac{1}{4}(\delta_2-\delta_1)^T T (S_{min}+S_{max}) K \end{pmatrix} & \begin{pmatrix} \delta_1^T T \delta_2 - \hat{\epsilon} \\ +\frac{1}{4}(\delta_2-\delta_1)^T (\delta_2-\delta_1) \end{pmatrix} \end{bmatrix} < 0. \tag{14}$$

Multiplying diag(P^{-1}, 1) from the left and right hand sides to (14) and letting $Q = P^{-1} > 0$, $Y = KQ$, we obtain

$$\begin{bmatrix} \begin{pmatrix} QA_i^T + A_i Q + \tau_{m+1} Q^2 + B_i T^{-1} B_i^T \\ +\frac{1}{2} Y^T (S_{min}+S_{max})^T B_i^T \\ +\frac{1}{2} B_i (S_{min}+S_{max}) Y \\ +\frac{1}{4} Y^T (S_{max}-S_{min})^T T (S_{max}-S_{min}) Y \end{pmatrix} & * \\ \begin{pmatrix} \frac{1}{2}(\delta_2-\delta_1)^T B_i^T \\ +\frac{1}{4}(\delta_2+\delta_1)^T T (S_{max}-S_{min}) Y \end{pmatrix} & -\hat{\epsilon} + \frac{1}{4}(\delta_2+\delta_1)^T T (\delta_2+\delta_1) \end{bmatrix} < 0. \tag{15}$$

By applying the Schur-complement to (15) twice, once with the pivot term $\tau_{m+1}Q^2$ and once with the pivot term $\frac{1}{4} Y^T(S_{max} - S_{min})^T T(S_{max} - S_{min})Y$, the main inequality (6) can be obtained. The condition $\mathcal{B}_\epsilon = \{x \in \mathcal{R}^n \mid x^T x \le \epsilon\} \subseteq \mathcal{E}_c = \{x \in \mathcal{R}^n \mid x^T P x \le c\}$ is satisfied by setting $c = \tau_{m+1}^{-1}$ and requiring, accordingly, $Q \ge \hat{\epsilon} I_n > 0$. In addition, the major semi-axis length of $\mathcal{E}_c$ is equal to $\sqrt{\tau_{m+1}^{-1} \cdot \lambda_{max}(Q)}$, which will be no greater than γ_o provided that (7) and the first inequality of (8) hold since $\sqrt{\tau_{m+1}^{-1} \cdot \lambda_{max}(Q)} \le \frac{1}{2}[\tau_{m+1}^{-1} + \lambda_{max}(Q)]$.

As for the inequality (11), parallel derivation leads to the following equivalent condition:

$$\begin{bmatrix} \tilde{\Pi}_i & * & * & * \\ \frac{1}{2}(-\delta_2+\delta_1)^T B_i^T & -\tilde{\epsilon} & * & * \\ \frac{1}{2}(S_{max}-S_{min})Y & \frac{1}{2}(-\delta_2-\delta_1) & -\tilde{T}^{-1} & * \\ Q & 0 & 0 & -\tilde{\tau}_{m+1}^{-1} I_n \end{bmatrix} < 0,$$

$i = 1, 2, \ldots, l,$ (16)

where $\tilde{\epsilon} = \tilde{\tau}_{m+1}\epsilon$ and

$$\tilde{\Pi}_i = QA_i^T + A_i Q + \frac{1}{2} B_i (S_{max}+S_{min}) Y + \frac{1}{2} Y^T (S_{max}+S_{min})^T B_i^T + B_i \tilde{T}^{-1} B_i^T.$$

However, multiplying diag(I_n, −1, I_m, 1) from the left and right hand sides to (16) to get

$$\begin{bmatrix} \tilde{\Pi}_i & * & * & * \\ \frac{1}{2}(\delta_2+\delta_1)^T B_i^T & -\tilde{\epsilon} & * & * \\ \frac{1}{2}(S_{max}-S_{min})Y & \frac{1}{2}(\delta_2+\delta_1) & -\tilde{T}^{-1} & * \\ Q & 0 & 0 & -\tilde{\tau}_{m+1}^{-1} I_n \end{bmatrix} < 0,$$

$i = 1, 2, \ldots, l,$

quickly tells us that the conditions derived from (11) are redundant, and that we only need conditions (6), (7), and (8). ■

Remark 1. The expressions (6), (7), and (8) form an LMI feasibility problem in the variables Q, Y, T^{-1}, $\hat{\epsilon}$, λ, and τ_{m+1}^{-1}, which can be solved with the help of [9] to determine a state feedback controller $\tilde{u} = Kx$ for the ultimate boundedness control. Since the feasibility problem may have more than one solution, solutions corresponding to low controller gains are usually desirable. Thus, we form the following convex optimization problem:

$$\begin{aligned} &\text{minmize} \quad \|Y\| \\ &\text{subject to} \quad (6), (7), \text{and } (8), \end{aligned} \tag{17}$$

where the two-norm of Y is chosen as the objective function to indirectly minimize the controller gain $K = YQ^{-1}$. Though the solution may not give the smallest gain matrix, our numerical experience indicates that excessively large gain matrices are indeed avoided.

Remark 2. When $\delta_{1j} = \delta_{2j} = 0$, the generalized sector condition described by (3) and (4) reduces to the standard sector condition [16] formulated by $(u_j - \alpha_j k_j x)(u_j - \beta_j k_j x) \le 0$. In this case, the control problem discussed previously turns into that of finding state feedback controllers to ensure asymptotic stability instead of ultimate boundedness of the closed-loop system trajectories. Results of this special case stated below will be applied to study the actuator saturation problem in the next section.

Corollary 1. Consider the polytopic uncertain system (1) driven by nonlinear actuators which satisfy the sector conditions

$$(u_j - \alpha_j k_j x)(u_j - \beta_j k_j x) \le 0, \quad j = 1, \ldots, m.$$

If there exist a positive definite matrix $Q \in \mathcal{R}^{n\times n}$, a positive definite matrix $T^{-1} \in \mathcal{R}^{m\times m}$, and a matrix $Y \in \mathcal{R}^{m\times n}$ satisfying

$$\begin{bmatrix} \Pi_i & * \\ \frac{1}{2}(S_{max} - S_{min})Y & -T^{-1} \end{bmatrix} < 0, \quad i = 1, 2, \ldots, l, \quad (18)$$

then the state feedback control $\tilde{u}(t) = Kx(t)$ with $K = YQ^{-1}$ will asymptotically stabilize the closed-loop system (1). Note that S_{max}, S_{min}, and Π_i are defined in **Theorem 1**.

Proof. It is easy to mimic the proof of **Theorem 1** to show that under the assumed conditions, we have

$$\begin{aligned}\frac{d[x^T(t)Px(t)]}{dt} &= x^T[A^T(t)P + PA(t)]x \\ &\quad + x^T PB(t)u + u^T B^T(t)Px \\ &< \sum_{j=1}^{m} \tau_j (u_j - \alpha_j k_j x)(u_j - \beta_j k_j x) \le 0,\end{aligned}$$

where $P = Q^{-1}$, for all nonzero x, u, and $[A(t)\ B(t)]$ satisfying (2). Here, we can see that in addition to the asymptotic stability as concluded above, it is also true that the ellipsoidal set $\mathcal{P} = \{x \in \mathcal{R}^n \mid x^T Px \le 1\}$ is a positively invariant set [3,20] of the closed-loop system. ■

We will give one example to illustrate the proposed ultimate boundedness controller synthesis approach.

Example 1. Consider the uncertain system (1) with two states, one input, and

$$A(t) = \begin{bmatrix} 0 & 0.5 \\ 0.8 & p(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} -0.5 \\ -0.2p(t) \end{bmatrix}, \quad p(t) \in [-1, 1],$$

$$\alpha_1 = 0.8, \quad \beta_1 = 1.2, \quad \delta_{11} = 0.2, \quad \delta_{21} = 0.3.$$

For $\gamma_o = 0.5$, we can solve (17) to obtain

$$K = [3.75 \quad 6.83], \quad Q = \begin{bmatrix} 0.223 & -0.074 \\ -0.074 & 0.09 \end{bmatrix},$$

$$Y = [0.33 \quad 0.34],$$

$$\tau_1^{-1} = 1.22, \quad \tau_2^{-1} = 0.741, \quad \hat{\epsilon} = 0.057.$$

In Fig. 2, the solid ellipse is the ultimate boundedness ellipsoidal set $\mathcal{E}_c$ we derived. The actuator is assumed to have the backlash characteristic with a deadband width of 0.25 for the purpose of simulation. Clearly, starting from the initial condition $x_o = [-3\ -4]^T$, the state trajectory (dashed line) of the closed-loop system subject to the uncertain parameter $p(t) = \sin(\frac{2\pi}{10})$ enters and stays within $\mathcal{E}_c$ eventually.

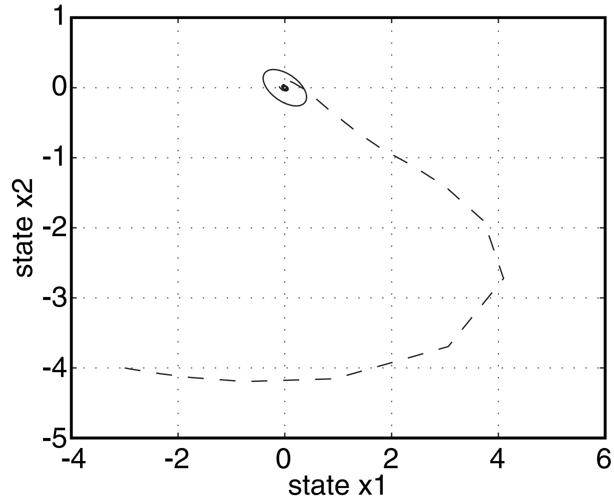

Fig. 2. Results of Example 1.

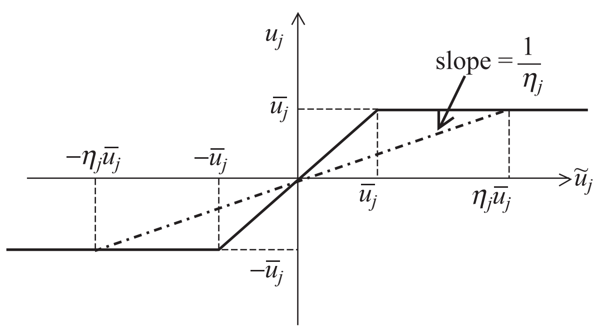

Fig. 3. Standard saturation characteristic.

IV. ACTUATORS WITH THE STANDARD SATURATION CHARACTERISTIC

In this section, we will focus on the polytopic uncertain system (1) driven by actuators that have the standard saturation characteristic; i.e., for $j = 1, \ldots, m$, and all $t \ge 0$,

$$\psi_j(t,\zeta) = \begin{cases} \bar{u}_j, & \zeta \ge \bar{u}_j, \\ \zeta, & -\bar{u}_j < \zeta < \bar{u}_j, \\ -\bar{u}_j, & \zeta \le -\bar{u}_j. \end{cases} \quad (19)$$

This characteristic is depicted in Fig. 3 using thick, solid lines.

At first sight, this particular problem can be solved by using **Corollary 1** in Section 3 with $S_{min} = 0$ and $S_{max} = I_m$. However, the range of applicable cases will be limited, as (18) actually is equivalent to

$$\begin{bmatrix} A_i^T P + PA_i & PB_i + \frac{1}{2}K^T T \\ B_i^T P + \frac{1}{2}TK & -T \end{bmatrix} < 0,$$

which is feasible only if $A_i^T P + PA_i < 0$ for some $P > 0$, implying that all A_i, $i = 1, 2, \ldots, l$, must be Hurwitz. This reflects the fact that when unstable plants and saturating actuators are involved, in general, it is impossible to stabilize the entire state space. Thus, one must change the control objective. The linear part of the standard saturation characteristic suggests that any stabilizing state feedback controller for the linear plant in (1) can still stabilize some state trajectories of the closed-loop system with the saturating actuators, provided that the initial condition x_o is close enough to the origin of the state space. Thus, the goal may become that of finding state feedback controllers which can handle the largest possible set of initial conditions, from which all the state trajectories will be brought back to the origin asymptotically. As a starting point, we apply **Corollary 1** of Section 3 with $S_{min} = S_{max} = I_m$ in (18) but add the constraint that the set $\mathcal{P} = \{x \in \mathcal{R}^n \mid x^T Px \le 1\}$ is inscribed by the set $\{x \in \mathcal{R}^n \mid |k_j x| \le \bar{u}_j, j = 1, \ldots, m\}$, which can be formulated by the LMI constraints

$$\begin{bmatrix} u_j^2 & e_j^T Y \\ Y^T e_j & Q \end{bmatrix} \ge 0, \quad j = 1, \ldots, m. \qquad (20)$$

Clearly, the idea is to have $|\tilde{u}_j| = |k_j x| \le \bar{u}_j, j = 1, \ldots, m$, so that no actuators will saturate. Also, since $\mathcal{P}$ is a positively invariant set, it can serve as the set of stabilized initial conditions we are looking for. However, this still causes an unnecessary restriction since to bring the system states back to the origin, the actuators need not be unsaturated at all times.

Suppose the jth actuator is allowed to saturate at most to the level $\eta_j \ge 1$, which means that $|\tilde{u}_j(t)|$ must not exceed $\eta_j \bar{u}_j$ for all $t \ge 0$. In this situation, the nonlinearity of the jth actuator belongs to the sector $[1/\eta_j, 1]$, as can be seen in Fig. 3. Hence, the method proposed in Section 3 can be applied again, with $S_{min} = \mathrm{diag}(1/\eta_1, 1/\eta_2, \ldots, 1/\eta_m)$ and $S_{max} = I_m$ in (18). In the extra constraints (20), the term $\bar{u}_j^2$ should be replaced with $\eta_j^2 \bar{u}_j^2$. The next issue is the question of whether η_j's are known beforehand. Usually, for technical or safety reasons, we have an upper bound $\bar{\eta}_j$ for each η_j but do not know how to pre-select a set of η_j's so that the resultant stabilized region $\mathcal{P}$ is as large as possible. This motivates us to examine the related LMIs more closely. Now (18) is

$$\begin{bmatrix} \Pi_i & \frac{1}{2}Y^T (I_m - S_{min})^T \\ \frac{1}{2}(I_m - S_{min})Y & -T^{-1} \end{bmatrix} < 0, \quad i = 1, 2, \ldots, l, \qquad (21)$$

where

$$\Pi_i = QA_i^T + A_i Q + \frac{1}{2}B_i (I_m + S_{min})Y + \frac{1}{2}Y^T (I_m + S_{min})^T B_i^T + B_i T^{-1} B_i^T$$

and $S_{min} = \mathrm{diag}(1/\eta_1, 1/\eta_2, \ldots, 1/\eta_m)$. The other set of LMIs is

$$\begin{bmatrix} \eta_j^2 \bar{u}_j^2 & e_j^T Y \\ Y^T e_j & Q \end{bmatrix} \ge 0, \quad j = 1, \ldots, m,$$

or equivalently

$$\begin{bmatrix} \bar{u}_j^2 & e_j^T Y / \eta_j \\ Y^T e_j / \eta_j & Q \end{bmatrix} \ge 0, \quad j = 1, \ldots, m. \qquad (22)$$

Note that (21) and (22) are LMIs with respect to the variables $\{Q, Y, T^{-1}\}$ as well as with respect to $\{Q, S_{min}, T^{-1}\}$. Thus, they are BMIs with respect to the variables $\{Q, Y, S_{min}, T^{-1}\}$ and remain so after another set of LMIs,

$$A_i Q + QA_i^T + B_i Y + Y^T B_i^T < rQ, \quad i = 1, 2, \ldots, l, \qquad (23)$$

is augmented with a pre-assigned $r < 0$ to set the minimum trajectory decay rate when all the actuators enter their linear range.

Since the only coupled variables in the above non-convex BMIs are Y and S_{min}, a popularly adopted approach is to solve them alternately. Hence, we propose the following two convex problems (CPs) for the given plant and saturation information $\{A(t), B(t)\}$, $\underline{S}_{min} = \mathrm{diag}(1/\bar{\eta}_1, 1/\bar{\eta}_2, \ldots, 1/\bar{\eta}_m)$, and $\bar{u}_j, j = 1, \ldots, m$:

CP1 $\min\limits_{0<\xi I_n \le Q, Y, 0<T^{-1}} -\xi$ subject to (21), (22), (23),

$S_{min} = S_{min}^*$ from CP2,

CP2 $\min\limits_{0<\xi I_n \le Q, \underline{S}_{min} < S_{min} \le I_m, 0<T^{-1}} -\xi$ subject to (21), (22), (23),

$Y = Y^*$ from CP1,

which are to be solved altenately. Note that minimizing the objective function "$-\xi$" is equivalent to maximizing the minimum eigenvalue of Q, which is proportional to the minimum semi-axis length of the ellipsoidal set $\mathcal{P} = \{x \in \mathcal{R}^n \mid x^T Px < 1\}$. This lets us obtain a large stability region. To begin, CP1 is solved first with S_{min} set to I_m or whatever are feasible values are deemed appropriate. To end the procedure, a stopping rule may be that the increment of ξ^* becomes insignificant. Because each iteration is the solution of a CP, theoretically, the resultant sequence of optimal ξ^*'s will be monotonically non-decreasing. However, as a whole, the sequence may converge to a local optimum.

Example 2. Consider the linearized equations of motion for a satellite studied in [7] and [12], which are in the form of a polytopic uncertain system with four states and two inputs. In [7], the authors found a linear state feedback controller that stabilized the system for any initial condition $x(0) \in \{x^T x \leq 1\}$ under constrained control. Subsequently, in [12] it was shown that the controller from [7] could actually stabilize the system for more initial conditions, and an ellipsoidal set $\mathcal{P}$ which had a volume 11.30 times larger than that of the unit sphere $\{x^T x \leq 1\}$ was shown to be a positively invariant set of the closed-loop system. Here, the problem is studied again to test the method proposed in this section. To let the test conditions be as close to those used in [7] and [12] as possible, $\underline{S}_{min}$ is set to 0 because there are no limits on how saturated the actuators are allowed to be, and r is set to -2 in (23) to match the decay rate of the closed-loop system of [7] and [12] when actuators are not saturated. We can solve CP1 and CP2 alternately from the simplest initial guess $S_{min} = I_2$ until ξ converges to a local optimum and get

$$K^* = \begin{bmatrix} -10.402 & -4.305 & 3.057 & -0.211 \\ -5.664 & -0.465 & -11.067 & -7.293 \end{bmatrix},$$

$$Q^* = \begin{bmatrix} 9.267 & -15.576 & -0.518 & -2.462 \\ -15.576 & 52.338 & 3.676 & -2.644 \\ -0.518 & 3.676 & 5.443 & -6.469 \\ -2.462 & -2.644 & -6.469 & 35.301 \end{bmatrix},$$

which give us a new state feedback gain matrix and a new positively invariant set $\mathcal{P}$ for the initial conditions. The volume of the new set is 179.956 times larger than that of the unit sphere. It is also interesting to note that $\eta_1^* = 1.576$, not equal to $\eta_2^* = 2.563$. Different initial guesses of S_{min} do result in different local optimal solutions or positively invariant sets $\mathcal{P}$ due to the non-convexity of BMIs. How to get the global optimal solution by adopting, for example, the branch and bound search method is still being studied. In Fig. 4, all the state variables starting from the initial condition $[0.25 \quad 4.0 \quad 1.0 \quad -4.0]^T$ are found to converge to zero. The system is assumed to be subject to a time-varying uncertain parameter $p(t)$ given in [12], which lies within the interval [0.5, 1.5] and is set to $1 + 0.5\sin(\frac{2\pi}{10})$ for the purpose of simulation. The two actuator outputs that drive all the system states to the origin of the state space are also plotted in Fig. 5. These outputs are limited to a saturation level of ±15.

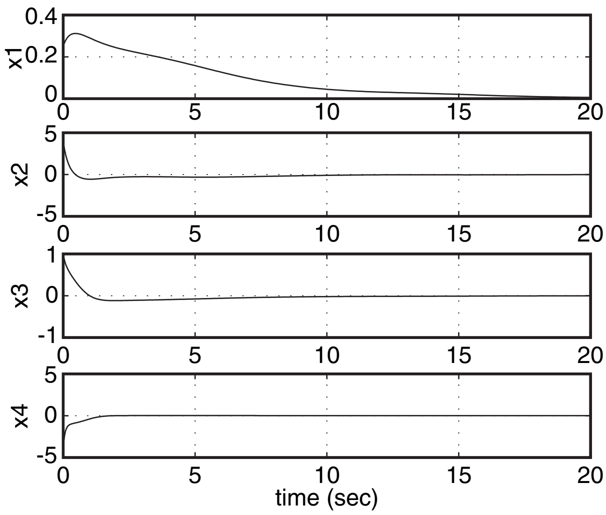

Fig. 4. State responses.

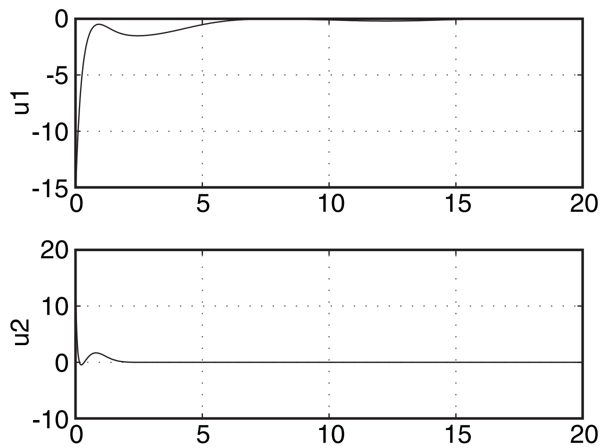

Fig. 5. Actuator output signals.

5. CONCLUSIONS

For polytopic uncertain systems with nonlinear actuators which satisfy generalized sector conditions, we have derived LMI conditions to guarantee the existence of robust state feedback controllers and to achieve the ultimate boundedness control. For the case in which actuators have the standard saturation characteristic, the proposed conditions can be adapted in order to find state feedback controllers that stabilize state trajectories from a large set of initial conditions. Examples have been provided to illustrate how these new methods are used.

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