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Robust stabilization of uncertain systems with persistent disturbance and a class of non-linear actuators

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Linear uncertain systems perturbed by persistent disturbances and driven by actuators subject to band-bounded non-linearities are considered. A set of linear matrix inequality based sufficient conditions are derived for designing state feedback controllers which assure ultimate boundedness of closed-loop state trajectories. The effect of the non-linear actuators is identified to that of the considered persistent disturbances. The control purpose is to minimize the ultimate boundedness region to which the state trajectories are eventually confined, while tolerating disturbances of given or the largest magnitudes. Finally, an experimental system, an inverted pendulum on a Stewart platform making translational movement, is arranged to demonstrate the feasibility and effectiveness of the derived results.

1. Introduction

In practical control systems, the actuators or driving devices like power amplifiers always have inherent non-linearities due to technological or physical constraints. For example, all actuators have a limited operation range. Though many actuators are manufactured to have pretty good linear characteristics over their operation ranges, non-linearities such as deadzone inevitably exist. If these non-linearities are not properly accounted for, they will deteriorate the overall performance or result in instability. Consequently, for many decades control problems with non-linear actuators have attracted considerable interest, and no less recently (Hanrion and Tarbouriech 1999, Henrion *et al*. 1999, Kapila *et al*. 1999 a,b, Fong and Hsu 2000).

Among all actuator non-linearities, the saturation is probably the most frequently encountered and studied (Bernstein and Michel 1995, Mahmoud 1997, Kapila and Pan 1999, Stoorvogel and Saberi 1999). As to still other non-linearities like deadzone, hysteresis, backlash, and quantization, commonly seen in hydraulic or electromagnetic devices, general discussions are rare. Traditionally, describing functions are used to approximate the frequency domain effects of many kinds of non-linearity, yet the main purpose is to investigate the possible existence of periodic solutions such as limit cycles (Khalil 1992). To ensure system stability, more precise analysis is needed. Individual results usually cover specific types of nonlinearity or use approximations (Choi and Kim 1996, Pare and How 1998).

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Besides actuator non-linearities, there are usually unwanted disturbance inputs produced by the environments of control systems that must be taken care of. For example, effects of persistent wind on a flying vehicle need to be compensated by the flight control system. There are plenty of research works about the factor, but most assume that the disturbance has finite signal energy or a certain probabilistic distribution, and adopt the $\boldsymbol{H}_\infty/\boldsymbol{H}_2$ theories to solve the related control problems. However, some disturbances can not be modelled by the above methods, such as persistent environmental temperature changes for a process control system. For these cases signal peak value is more appropriate for modelling the signal size, and the optimal $\ell_1/\boldsymbol{L}_1$ methods (Vidyasager 1986, Dahleh and Pearson 1987) naturally arise, which tries to minimize the worst-case peak to peak gain of the closed-loop system from exogenous inputs to regulated signals. In the last decade, many authors (Dahleh and Shamma 1992, Diaz-Bobillo and Dahleh 1992, Shamma 1993, 1996, Dahleh and Diaz-Bobillo 1995) have developed various optimal $\ell_1/\boldsymbol{L}_1$ control methods, though the proposed procedures for designing the optimal or sub-optimal controllers often need intensive computations, and the resultant controllers are either linear ones with high orders or nonlinear ones. To get simpler results, the optimal $\ell_1/\boldsymbol{L}_1$ problems are sometimes modified a little. In works like Abedor and Nagpal (1996) and Nguyen and Jabbari (1999) the considered signal norm is the supremum of the Euclidean norms of the signal vector at all time instants. This may facilitate the application of the linear matrix inequality (LMI) approach, and gives controller design methods with convenient computation tools.

In this paper, we consider both the effects from actuator non-linearities and external disturbances. We study the band-bounded non-linearities

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(Hsu and Fong 2001 a), which, unlike the sector-bounded non-linearities, are able to include the dead-zone and hysteresis as special cases. Furthermore, we allow disturbance in different channels to have different bounds. As to the regulated state variables, we adopt the ultimate boundedness control problem formulation, and try to design state feedback controllers which ensure that state trajectories reach and stay within a compact ultimate boundedness region around the origin. For given disturbances, the ultimate boundedness region is minimized. If the problem does not specify the bounds on disturbances beforehand, then the size of the ultimate boundedness region and the bounds of the tolerable disturbance can be obtained simultaneously in some optimal sense. Roughly speaking, this is a variant of the minimization problem of the closed-loop system peak to peak gain. We utilize the LMI approach here, so the proposed state feedback controller design procedure is very straightforward. Finally, we build an experimetal system to illustrate the effectiveness of the proposed approach.

Some notations are defined first. Let $\mathcal{R}$ be the field of real numbers, $\mathcal{R}^n$ the real vector space of n-tuple vectors, and $\mathcal{R}^{m\times n}$ the real vector space of $m\times n$ matrices. For any $X, Y \in \mathcal{R}^{n\times n}$, $X \geq Y$ means that X, Y are symmetric and $X - Y$ is positive semi-definite. Similar notations apply to symmetric positive/negative definite matrices. If $X > 0$, then $\lambda_{\max}(X)$ denotes its largest eigenvalue. The transpose of a real matrix X is denoted by X^T. I_n is the $n\times n$ identity matrix, and e_n^i is the ith column of I_n. In a symmetric block matrix, for simplicity the symbol $*$ will be used to replace the submatrices that lie above the diagonal. Finally, the notation $\mathbf{Co}\{\cdot\}$ represents the convex hull of the set in the argument.

2. Problem formulation

Consider the uncertain feedback system described by the mathematical model

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u(t)u(t) + B_d(t)d(t), \quad x(0) = x_o \\ \tilde{u}(t) &= Kx(t) \\ u(t) &= [\beta_1(t,\tilde{u}_1(t)) \quad \cdots \quad \beta_m(t,\tilde{u}_m(t))]^{\mathrm{T}}. \end{aligned}\right\} \quad (1)$$

In (1) the first linear state equation describes the plant to be controlled, where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the control input vector, and $d(t) \in \mathcal{R}^p$ is the exogenous disturbance input vector. The second equation represents the state feedback controller, where $K \in \mathcal{R}^{m\times n}$ is the state feedback gain matrix, and $\tilde{u}(t) \in \mathcal{R}^m$ is the controller output vector. Finally, the third equation stands for the non-linear characteristics of the actuators, where $\tilde{u}_i(t)$ is the ith component of $\tilde{u}(t)$, and $\beta_i(\cdot,\cdot) : [0,\infty) \times \mathcal{R} \to \mathcal{R}$ denotes the ith non-linear characteristic, illustrated in figure 1. Note that the dash–dotted curve (or curves, as the relationship may be time-varying) of $\beta_i(\cdot,\cdot)$ is only known to be confined to the band between the two parallel dashed lines which are characterized with unit slope and bias $\pm\delta_i$. Hence we call it the band-bounded non-linearity (Hsu and Fong 2001 a). This kind of non-linearity can accommodate characteristics such as backlash, hysteresis, deadzone, and quantization, which are not included by the sector-bounded non-linearity (Khalil 1992). It may be interpreted as that at the ith channel, $i = 1, 2, \ldots, m$, there is a non-linear actuator accepting the ith controller

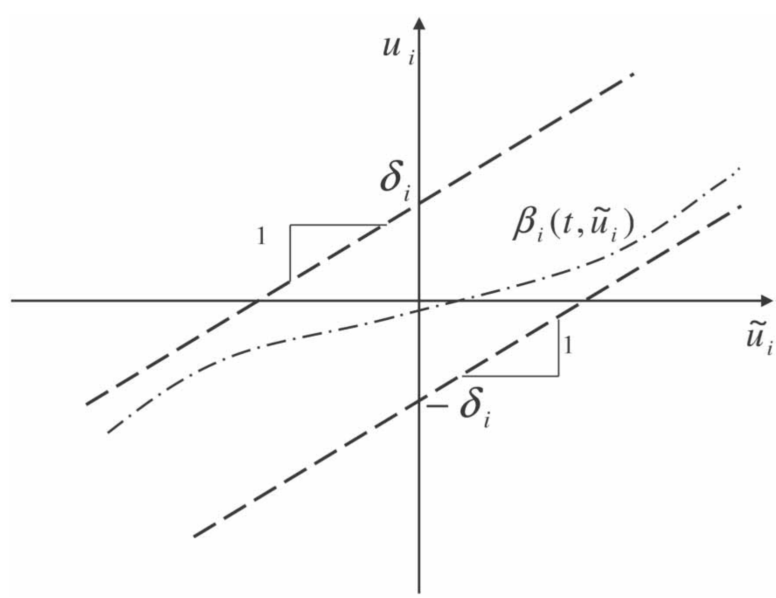

Figure 1. Actuator input–output characteristic with unity band gain.

output $(e_m^i)^T Kx(t)$ and producing the control input $(e_m^i)^{\mathrm{T}} Kx(t) + \Delta u_i(t)$ where $-\delta_i \leq \Delta u_i(t) \leq \delta_i$. In other words, the ith actuator has the input-output characteristic $\beta_i(\cdot,\cdot)$ which is only known to be bounded by $\tilde{u}_i - \delta_i \leq \beta_i(\cdot,\cdot) \leq \tilde{u}_i + \delta_i$ for all $\tilde{u}_i \in \mathcal{R}$.

Regarding this system, assume that the plant $\{A(t), B_u(t)\}$ is controllable, and there are two vectors $\underline{d}$, $\bar{d} \in \mathcal{R}^p$ such that $\underline{d}_i \leq d_i(t) \leq \bar{d}_i$ for all t and $i = 1, 2, \ldots, p$, where $\underline{d}_i$, $d_i(t)$, and $\bar{d}_i$ are the ith component of $\underline{d}$, $d(t)$, and $\bar{d}$ respectively. Note that there is no matching condition (Lunze 1998) assumption on the disturbance influence coefficient matrix $B_d(t)$, so the disturbance $d(t)$ may enter the system in a quite unrestrictive way. The constant vectors $\underline{d}$, $\bar{d}$ may be given or not, and if they are not given, then we wish to determine how large each of their components may be. This assumption allows one to handle disturbances with only the worst case information (Boyd and Barratt 1991). This is important in many real-world applications, since exogenous disturbance are often persistent, i.e. they continue acting on the system as long as the system is in operation.

As mentioned above, we see that $u(t)$ in (2) may be written as $u(t) = \tilde{u}(t) + \Delta u(t)$, and the ith component $\Delta u_i(t)$ of $\Delta u(t)$ satisfies $-\delta_i \leq \Delta u_i(t) \leq \delta_i$ for all $t \geq 0$. This allows us to re-write (1) as

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u(t)\tilde{u}(t) \\ &\quad + [B_u(t)\ B_d(t)]\begin{bmatrix} \Delta u(t) \\ d(t) \end{bmatrix}, \quad x(0) = x_0 \\ \tilde{u}(t) &= Kx(t). \end{aligned}\right\} \tag{2}$$

Now we simplify (2) by re-defining some notations: $\tilde{u}(t)$ is replaced by $u(t)$, $[B_u\ B_d]$ is replaced by B_w, $[\Delta u^{\mathrm{T}}(t)\ d^{\mathrm{T}}(t)]^{\mathrm{T}}$ is replaced by $w(t)$, and the dimension $m+p$ of the vector $[\Delta u^{\mathrm{T}}(t)\ d^{\mathrm{T}}(t)]^{\mathrm{T}}$ is replaced by p, where $p > m$. After these modifications, we can consider the feedback control system modelled by

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u(t)u(t) + B_w(t)w(t), \quad x(0) = x_0 \\ u(t) &= Kx(t). \end{aligned}\right\} \tag{3}$$

where $w(t) = [w_1(t) \cdots w_p(t)]$ and $\underline{w}_i \leq w_i(t) \leq \bar{w}_i$. Note that here $\bar{w}_i = -\underline{w}_i = \delta_i$ for $i = 1, 2, \ldots, m$, and $\bar{w}_{m+j}$, $\underline{w}_{m+j}$, respectively, equal to $\bar{d}_j$, $\underline{d}_j$ for $j \geq 1$. Clearly, by doing the above re-formulation, we show that the effect of band-bounded non-linear actuators is the same as that of exogenous disturbances, and may be considered together. Consequently, the assumption about $w_i(t)$ in (3) may be expressed in the convex form

$$w_i(t) = [1 - \pi_i(t)]\bar{w}_i + \pi_i(t)\underline{w}_i \tag{4}$$

where $0 \leq \pi_i(t) \leq 1$ for all t and $i = 1, 2, \ldots, p$. With $\Pi(t) = \mathrm{diag}(\pi_1(t), \ldots, \pi_p(t))$, we have a more compact expression

$$w(t) = [I_p - \Pi(t)]\bar{w} + \Pi(t)\underline{w}. \tag{5}$$

Let $\{\Pi_1, \Pi_2, \ldots, \Pi_{2^p}\}$ be the set of 2^p distinct diagonal $p \times p$ matrices whose diagonal elements are either 0 or 1, and let the convex hull (Horn and Johnson 1985) of the set be denoted by $\mathbf{Co}\{\Pi_1, \Pi_2, \ldots, \Pi_{2^p}\}$. With (5), the open-loop system of (3) can be represented by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u(t)u(t) \\ &\quad + B_w(t)\{[I_p - \Pi(t)]\bar{w} + \Pi(t)\underline{w}\} \end{aligned} \tag{6}$$

where $\Pi(t) \in \mathbf{Co}\{\Pi_1, \Pi_2, \ldots, \Pi_{2^p}\}$ for all t.

The parameters of the system (11) are assumed to subject to the *polytopic uncertainty*

$$\begin{aligned} &[A(t) \quad B_u(t) \quad B_w(t)] \\ &\quad \in \mathbf{Co}\{[A_1 \quad B_{u_1} \quad B_{w_1}], \ldots, [A_q \quad B_{u_q} \quad B_{w_q}]\}. \end{aligned} \tag{7}$$

Facing the parameter uncertainty, actuator non-linearities, and persistent disturbances, we intend to find state feedback controllers to achieve ultimate boundedness control for the closed-loop system (Corless and Leitmann 1981, Garofalo *et al.* 1989), which basically means making every state trajectories of (1) enter a neighbourhood (the ultimate boundedness region) of the origin of the state-space eventually. Here, we consider the ellipsoidal neighbourhood $\mathcal{E}_c = \{x^{\mathrm{T}}Px \leq c\}$ for some $P > 0$ and $c > 0$. Of course, it is the best if the 'size' (with any acceptable definition) of $\mathcal{E}_c$ can be guaranteed to the smallest while bounds $|\bar{d}_i|$ and $|\underline{d}_i|$, $i = 1, 2, \ldots, p$, of $d(t)$ in (1) are as large as possible if they are not given beforehand.

We end this section by presenting one lemma which will be useful subsequently.

Lemma 1 (Xie 1996): *For any real matrix function $F(t)$ of t such that $F^{\mathrm{T}}(t)F(t) \leq I$ for all t, and real constant matrices Σ_1, Σ_2, and $\Sigma > 0$ with matching dimensions, we have*

$$\Sigma_1^{\mathrm{T}}F(t)\Sigma_2 + \Sigma_2^{\mathrm{T}}F^{\mathrm{T}}(t)\Sigma_1 \leq \Sigma_1^{\mathrm{T}}\Sigma\Sigma_1 + \Sigma_2^{\mathrm{T}}\Sigma^{-1}\Sigma_2, \quad \text{for all } t.$$

3. The main results

The following theorem is our main result which pose LMIs for the polytopic uncertain system. Feasible solutions of the LMIs in the theorem give state feedback gains for the ultimate boundedness control purpose.

Theorem 1: *Consider the uncertain system* (3) *perturbed by bounded disturbances* (4) *and polytopic uncertainty* (7). *For a given $\gamma_o > 0$, if there exist matrices $Q \in \mathcal{R}^{n\times n}$,*

$Y \in \mathcal{R}^{m\times n}$, vectors $\bar{w} \in \mathcal{R}^p$, $\underline{w} \in \mathcal{R}^p$, and real scalars τ_1^{-1}, $\hat{\epsilon}$ satisfying the LMI conditions

$$Q \geq \hat{\epsilon} I_n > 0, \quad 1 > \tau_1^{-1} > 0, \quad \tau_1^{-1} + \lambda_{\max}(Q) < 2\gamma_o \tag{8}$$

$$e_p^i(\bar{w} - \underline{w}) > 0, \quad \forall\, i = 1, 2, \ldots, p \tag{9}$$

$$\begin{bmatrix} \Gamma_j & * & * \\ \kappa_{jl}^{\mathrm{T}} & -\hat{\epsilon} & * \\ Q & 0 & -\tau_1^{-1} I_n \end{bmatrix} < 0, \quad \forall\, j = 1, 2, \ldots, q \text{ and } l = 1, 2, \ldots, 2^p \tag{10}$$

where $\Gamma_j = A_j Q + Q A_j^{\mathrm{T}} + B_{u_j} Y + Y^{\mathrm{T}} B_{u_j}^{\mathrm{T}}$ and $\kappa_{jl} = B_{w_j}\{[I_p - \Pi_l]\bar{w} + \Pi_l \underline{w}\}$, then $u(t) = Kx(t) = YQ^{-1}x(t)$ makes every state trajectories of system (3) converge to and stay within the ultimate boundedness region $\mathcal{E}_{\tau_1^{-1}} = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} P x = x^{\mathrm{T}} Q^{-1} x \leq \tau_1^{-1} < 1\}$ eventually, and the length of the major semi-axis of $\mathcal{E}_{\tau_1^{-1}}$ is less than γ_0.

Proof: For $\mathcal{E}_c = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} P x \leq c\}$ with $P > 0$ to be an ultimate boundedness region of the closed-loop system (3), it suffices to ensure that the time derivative of the Lyapunov function candidate $v(x) = x^{\mathrm{T}} P x$ is negative along all state trajectories of (3) outside $\mathcal{E}_c$ for all considered $w(t)$. Here we require $\mathrm{d}v[x(t)]/\mathrm{d}t$ to be negative outside a ball $\mathcal{B}_\epsilon = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} x \leq \epsilon\}$ which is contained by the set $\mathcal{E}_c$ for all considered $w(t)$. This can be implied by the S-procedure (Boyd *et al.* 1994), or by the existence of $\tau_1 > 0$ such that

$$\begin{aligned} &x^{\mathrm{T}}(t)[A^{\mathrm{T}}(t)P + PA(t) + PB_u(t)K + K^{\mathrm{T}} B_u^{\mathrm{T}}(t)P]x(t) \\ &\quad + \{\bar{w}^{\mathrm{T}}[I_p - \Pi(t)] + \underline{w}^{\mathrm{T}} \Pi(t)\} B_w^{\mathrm{T}}(t) P x(t) \\ &\quad + x^{\mathrm{T}}(t) P B_w(t)\{[I_p - \Pi(t)]\bar{w} + \Pi(t)\underline{w}\} \\ &\quad - \tau_1[\epsilon - x^{\mathrm{T}}(t)x(t)] < 0 \end{aligned} \tag{11}$$

for all $[A(t)\ B_u(t)\ B_w(t)] \in \mathbf{Co}\{[A_1\ B_{u_1}\ B_{w_1}], \ldots, [A_q\ B_{u_q}\ B_{w_q}]\}$, $\Pi(t) \in \mathbf{Co}\{\Pi_1, \Pi_2, \ldots, \Pi_{2^p}\}$, and $t \geq 0$. The condition (11) is equivalent to

$$\begin{aligned} &x^{\mathrm{T}}(t)[A_j^{\mathrm{T}} P + PA_j + PB_{u_j}K + K^{\mathrm{T}} B_{u_j}^{\mathrm{T}} P]x(t) + \{\bar{w}^{\mathrm{T}}[I_p - \Pi_l] \\ &\quad + \underline{w}^{\mathrm{T}} \Pi_l\} B_{w_j}^{\mathrm{T}} P x(t) \\ &\quad + x^{\mathrm{T}}(t) P B_{w_j}\{[I_p - \Pi_l]\bar{w} + \Pi_l \underline{w}\} - \tau_1[\epsilon - x^{\mathrm{T}}(t)x(t)] < 0 \end{aligned} \tag{12}$$

for all $t \geq 0$, $j = 1, 2, \ldots, q$, and $l = 1, 2, \ldots, 2^p$. In the matrix form with the new variable $\hat{\epsilon} = \tau_1 \epsilon > 0$, the condition (12) may be expressed as

$$\begin{bmatrix} A_j^{\mathrm{T}} P + PA_j + PB_{u_j}K + K^{\mathrm{T}} B_{u_j}^{\mathrm{T}} P + \tau_1 I_n & * \\ \{\bar{w}^{\mathrm{T}}[I_p - \Pi_l] + \underline{w}^{\mathrm{T}} \Pi_l\} B_{w_j}^{\mathrm{T}} P & -\hat{\epsilon} \end{bmatrix} < 0. \tag{13}$$

Multiplying (13) from the left- and right-hand sides by $\mathrm{diag}(P^{-1}, 1)$, and letting $Q = P^{-1} > 0$, $Y = KQ$, further transform (13) into

$$\begin{bmatrix} QA_j^{\mathrm{T}} + A_j Q + B_{u_j} Y + Y^{\mathrm{T}} B_{u_j}^{\mathrm{T}} + \tau_1 Q^2 & * \\ \{\bar{w}^{\mathrm{T}}[I_p - \Pi_l] + \underline{w}^{\mathrm{T}} \Pi_l\} B_{w_j}^{\mathrm{T}} & -\hat{\epsilon} \end{bmatrix} < 0. \tag{14}$$

It follows from the Schur complement (Boyd *et al.* 1994) that (14) is equivalent to (10). The condition $\mathcal{B}_\epsilon = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} x \leq \epsilon\} \subseteq \mathcal{E}_c = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} P x \leq c\}$ is satisfied by setting $c = \tau_1^{-1}$ and requiring accordingly $Q \geq \hat{\epsilon} I_n > 0$. In addition, the major semi-axis length of $\mathcal{E}_c$ equals $\sqrt{\tau_1^{-1} \cdot \lambda_{\max}(Q)}$, which will be no greater than γ_o provided (8) holds, since $\sqrt{\tau_1^{-1} \cdot \lambda_{\max}(Q)} \leq \frac{1}{2}[\tau_1^{-1} + \lambda_{\max}(Q)]$. As to (9), it is easy to see that magnitudes for different disturbances should satisfy (9) under the assumptions. This concludes the proof. □

Closer inspection of the main LMIs (8), (9), and (10) reveals that the Theorem 1 is also applicable in the situation where the only information of $\{A(t), B_u(t), B_w(t)\}$ is known, because even if $\bar{w}$, and $\underline{w}$ are viewed as variables, (8), (9) and (10) are still LMIs. Naturally, it would the best if we could set an objective function for LMIs (8), (9) and (10) to form an optimization problem, enabling us to find the state feedback controller which results in the smallest ultimate boundedness region while tolerating the largest disturbance. Thus we form the following convex optimization problem

$$\left.\begin{aligned} &\min \quad \tau_1^{-1} + \|Y\| - \sum_{j=1}^{p} (\bar{w}_i - \underline{w}_i) \\ &\text{subject to} \quad (8),\ (9),\ \text{and } (10) \end{aligned}\right\} \tag{15}$$

where the two-norm of Y is included in the objective function to indirectly minimize the controller gain $K = YQ^{-1}$, while τ_1^{-1} is for reducing the size of $\mathcal{E}_{\tau_1^{-1}}$. Finally, the last two terms reflect our desire to withstand disturbances of the largest magnitudes. If in the problem to be studied the disturbance magnitudes $\bar{w}$ and $\underline{w}$ are given beforehand, then in Theorem 1, the main LMI (10) is with respect to less number of variables, and the term $-\sum_{i=1}^{p}(\bar{w}_i - \underline{w}_i)$ may be taken out from the objective function (15). Of course, proper weightings can also be multiplied to individual terms when deemed necessary.

4. Stabilization of an inverted pendulum on a Stewart platform

This section presents a practical application of the theory developed in § 3 to the stabilization of an inverted pendulum on a Stewart platform (Bhaskar and Mruthyunjaya 2000). A photograph of the experimental equipment is shown in figure 2. As clearly

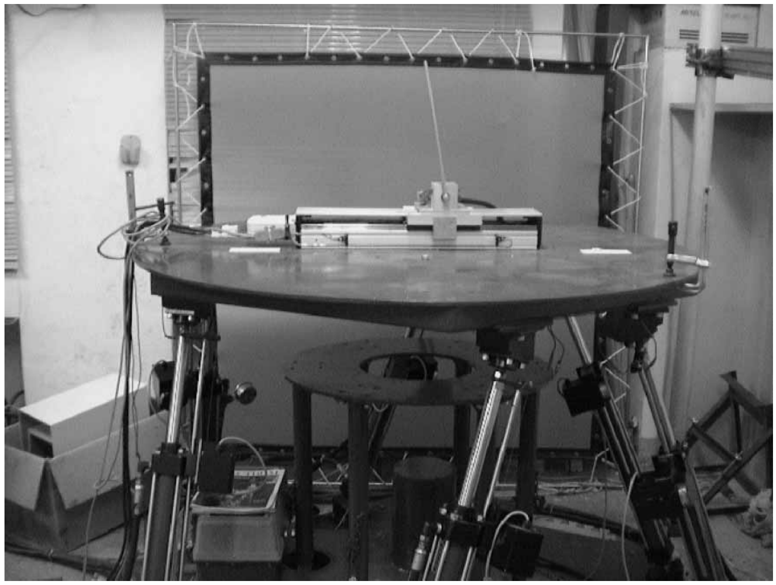

Figure 2. Photograph of the inverted pendulum system on a Stewart platform.

displayed in the photograph, the inverted pendulum itself has only single degree-of-freedom, but is mounted on a Stewart platform, which is the most celebrated parallel manipulator capable of performing six degree-of-freedom movement (Hsu and Fong 2001 b). The inverted pendulum system includes a ballscrew table cart and a stick freely pivoted at the cart. To balance the stick, the cart is moved back and forth by using an electric motor to drive the ballscrew table. Besides, optical encoder sensors are mounted at appropriate places to measure the angle between the stick and the vertical line, as well as rotation angle of the ballscrew, which is then converted to the position of the cart. Velocity of the cart and the angular velocity of the stick are obtained indirectly by computing the finite derivatives of the sensor outputs. The goal here is to design and test controllers that will stabilize the inverted pendulum at its vertical position, despite the presence of actuator non-linearities, or disturbance and uncertainty caused by the translational movement of the Stewart platform.

4.1. *Mathematical model of the experimental system*

A schematic diagram of the inverted pendulum system on the Stewart platform is shown in figure 3. Here, we attach frames $\{P\}$ and $\{B\}$ with origins O_P and O_B to the centroids of the upper platform and the lower base of the Stewart platform, respectively. The unit vectors of the frames $\{P\}$ and $\{B\}$ are $\{\mathbf{i}_P, \mathbf{j}_P, \mathbf{k}_P\}$ and $\{\mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B\}$, respectively. The relative position of the origin of the frame $\{P\}$ with respect to that of the frame $\{B\}$ can be described by $\overrightarrow{O_BO_P} = x_P\mathbf{i}_B + y_P\mathbf{j}_B + z_P\mathbf{k}_B$. Because in our experiment the upper platform will only move translationally, the three Roll–Pitch–Yaw Euler angles often used to characterize the orientation of the upper platform will not be needed in this study. Therefore, we have $\mathbf{i}_P = \mathbf{i}_B$, $\mathbf{j}_P = \mathbf{j}_B$, and $\mathbf{k}_P = \mathbf{k}_B$. The ballscrew table is placed to align with the $\mathbf{i}_P$ axis on the upper platform, and the centroids of both objects are assumed to coincide when the table is reset to the middle position.

In figure 3, the displacement of the table cart from the centroid O_P is denoted by x_c. Let $\mathbf{r}_{\text{cart}}(t)$ be the displacement vector of the cart from O_P, then

$$\mathbf{r}_{\text{cart}}(t) = x_c\mathbf{i}_P = x_c\mathbf{i}_B. \tag{16}$$

Let the position vector of the stick from O_P to its centre of mass be $\mathbf{r}_{\text{stick}}(t)$, then

$$\begin{aligned}\mathbf{r}_{\text{stick}}(t) &= [x_c + l_s \sin(\theta)]\,\mathbf{i}_P + l_s \cos(\theta)\,\mathbf{k}_P\\ &= [x_c + l_s \sin(\theta)]\,\mathbf{i}_B + l_s \cos(\theta)\,\mathbf{k}_B\end{aligned} \tag{17}$$

where l_s denotes the half length of the stick, and θ is the angle between the vertical line and the stick. In the mechanical system, the table cart is constrained to move only in the $\mathbf{i}_P$ direction on the track of the ballscrew, so any forces acting in the $\mathbf{j}_P$ and $\mathbf{k}_P$ directions have no effect on the motion of the cart. If the effect of friction is neglected, then the differential equation describing the motion of the cart can be obtained by summing all forces in the $\mathbf{i}_P$ direction. Thus we have

$$m_c\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{\text{cart}}(t) = (f - f_h + m_c\ddot{x}_P)\,\mathbf{i}_P = (f - f_h + m_c\ddot{x}_P)\,\mathbf{i}_B \tag{18}$$

where m_c is the mass of the table cart, f is the force applied to the cart in the $\mathbf{i}_P$ direction, and f_h is the reaction force from the stick in the $-\mathbf{i}_P$ direction.

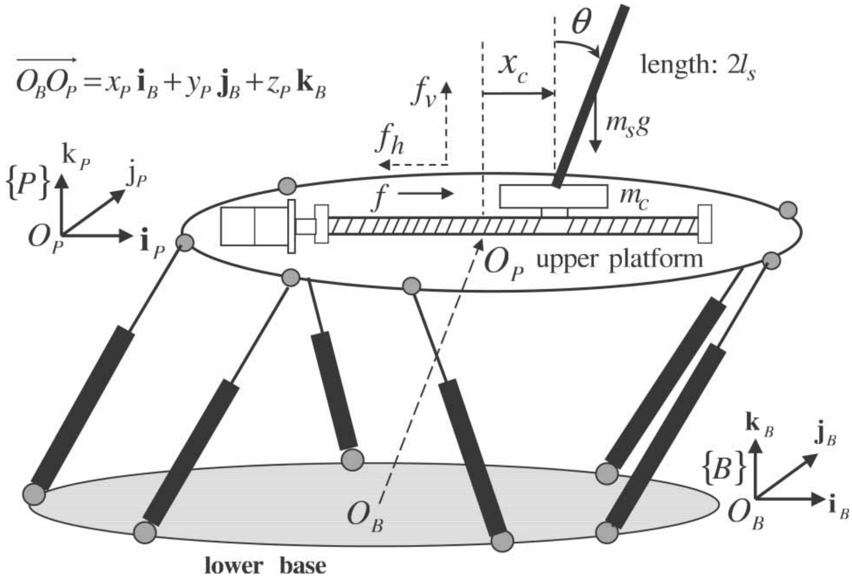

Figure 3. Schematic diagram of the inverted pendulum system on a Stewart platform.

Similarly, applying Newton's Second Law of Motion to the stick, we get

$$m_s \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{\text{stick}}(t) = (f_h + m_s\ddot{x}_P)\,\mathbf{i}_P + (f_v - m_s g + m_s\ddot{z}_P)\,\mathbf{k}_P$$
$$= (f_h + m_s\ddot{x}_P)\,\mathbf{i}_B + (f_v - m_s g + m_s\ddot{z}_P)\,\mathbf{k}_B \quad (19)$$

where m_s is the mass of the stick, g is the gravitational acceleration, and f_v is the reaction force from the cart in the $\mathbf{j}_P$ direction. From the rotational motion of the stick, one more equation of motion can be derived. Summing the moments about the stick's centre of mass yields

$$f_v l_s \sin(\theta) - f_h l_s \cos(\theta) = \frac{1}{3} m_s l_s^2 \ddot{\theta} \quad (20)$$

where $(1/3)m_s l_s^2$ is the stick's moment of inertia about its mass centre. By combining (18), (19) and (20), f_h and f_v can be eliminated to yield

$$(m_c + m_s)\ddot{x}_c + m_s l_s \cos(\theta)\ddot{\theta} = f + (m_c + m_s)\ddot{x}_P + m_s l_s \sin(\theta)\dot{\theta}^2 \quad (21)$$

$$\frac{3}{4} m_s \cos(\theta)\ddot{x}_c + m_s l_s\ddot{\theta} = \frac{3}{4} m_s g \sin(\theta) - \frac{3}{4} m_s \ddot{z}_P \sin(\theta) + \frac{3}{4} m_s \ddot{x}_P \cos(\theta). \quad (22)$$

Equations (21) and (22) form a set of non-linear differential equations that describe the motion of the inverted pendulum system on a Stewart platform making only translational movement. The set of equations may be linearized provided the stick is not allowed to fall too far from the vertical line, and to rotate too quickly. Under these conditions, sinusoidal terms in the equations can be approximated by $\sin(\theta) \cong \theta$ and $\cos(\theta) \cong 1$, and the square term can be approximated by $\dot{\theta}^2 \cong 0$. Thus (21) and (22) are approximated by

$$(m_c + m_s)\ddot{x}_c + m_s l_s\ddot{\theta} = f + (m_c + m_s)\ddot{x}_P, \quad (23)$$

$$\frac{3}{4} m_s\ddot{x}_c + m_s l_s\ddot{\theta} = \frac{3}{4} m_s g\theta - \frac{3}{4} m_s\ddot{z}_P\theta + \frac{3}{4} m_s\ddot{x}_P. \quad (24)$$

Let $x_1 = x_c$, $x_2 = \dot{x}_c$, $x_3 = \theta$, and $x_4 = \dot{\theta}$, then (23) and (24) can be cast into the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{3m_s g}{(m_s + 4m_c)} + \dfrac{3m_s\ddot{z}_P}{(m_s + 4m_c)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{3(m_s + m_c)g}{(m_s + 4m_c)l_s} - \dfrac{3(m_s + m_c)\ddot{z}_P}{(m_s + 4m_c)l_s} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{4}{(m_s + 4m_c)} \\ 0 \\ \dfrac{-3}{(m_s + 4m_c)l_s} \end{bmatrix} f + \begin{bmatrix} 0 \\ \ddot{x}_P \\ 0 \\ 0 \end{bmatrix}. \quad (25)$$

In (25), it is seen that the bounded accelerations $\ddot{z}_P$ in the direction of $\mathbf{k}_B$ and $\ddot{x}_P$ in the direction of $\mathbf{i}_B$ generated by the motion of the Stewart platform introduce parameter uncertainty and exogenous disturbance, respectively, to the state equation. As to the input term f, it is mentioned above that the cart is driven by an electric motor, actually an AC servo motor, connected to a ballscrew table, which transforms the motor torque into the driving force f. Hence, it is more convenient to use the torque τ generated by the AC servo motor instead of the force f as the control input variable. Let the screw drive patch be p_d (m/rev). According to the principle of conservation of mechanical energy, we have

$$f = \eta_b \frac{2\pi}{p_d} \tau \tag{26}$$

where η_b is the efficiency coefficient and is usually assumed to be between 0.8 and 0.9. Thus, equation (25) can be re-written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{3m_s g}{(m_s+4m_c)} + \frac{3m_s \ddot{z}_P}{(m_s+4m_c)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3(m_s+m_c)g}{(m_s+4m_c)l_s} - \frac{3(m_s+m_c)\ddot{z}_P}{(m_s+4m_c)l_s} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ \frac{4}{(m_s+4m_c)}(\frac{2\pi\eta_b}{p_d}) \\ 0 \\ \frac{-3}{(m_s+4m_c)l_s}(\frac{2\pi\eta_b}{p_d}) \end{bmatrix} \tau + \begin{bmatrix} 0 \\ \ddot{x}_P \\ 0 \\ 0 \end{bmatrix}. \tag{27}$$

State equation (27) is the linear model representing our plant, which will be driven by the actuator consisting of an AC servo motor and a power amplifier. The motor driver is operated in the torque mode, which means that the torque generated by the motor is supposed to be proportional to the input voltage to the driver. However, deadzone non-linearity exists between driver input and motor torque output due to mechanical friction and electrical factors. The width of the deadzone can be measured by applying small voltage to the motor driver incrementally until the ballscrew starts to move. The measured threshold voltage is 0.06 V, which corresponds to a torque control input of magnitude 0.0228 kg-m^2/s^2.

For this physical system, the cart is only allowed to move within a range of length 0.30 m due to finite ballscrew track length. This will restrict the magnitude of the state variable x_1. Therefore, stabilizing state feedback gains designed by conventional methods, such as the pole-placement method, may be affected by the non-linearities or may violate the constraint. Here, we will look for state feedback laws to stabilize the system despite the existence of actuator non-linearities, exogenous disturbance, and parameter uncertainty. Hence for this experimental system, it is expected to see that the stick is kept upright with some small swings, and the cart is restrained to move back and forth inside a narrow neighbourhood of its reset position. To facilitate future reference, we list some parameters values of the inverted pendulum system in table 1.

Parameter	Notation	Value	Unit
half stick length	l_s	0.16	m
stick mass	m_s	0.48	kg
cart mass	m_c	5.73	kg
ballscrew drive pitch	p_d	0.02	m/rev
ballscrew track length	—	0.30	m
ballscrew drive efficiency	η_b	0.9	—

Table 1. Various parameter values of the inverted pendulum system.

4.2. *Stabilization of the inverted pendulum*

Comparing the system models (3) and (27), we see that $x \in \mathcal{R}^4$, $u = \tau$, the motor torque output, and

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -\frac{3m_s g}{(m_s+4m_c)} - \frac{3m_s \ddot{z}_P(t)}{(m_s+4m_c)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3(m_s+4m_c)g}{(m_s+4m_c)l_s} + \frac{3(m_s+4m_c)\ddot{z}_P(t)}{(m_s+4m_c)l_s} & 0 \end{bmatrix},$$
$$B_u(t) = \begin{bmatrix} 0 \\ \frac{4}{(m_s+4m_c)}\left(\frac{2\pi\eta_b}{p_d}\right) \\ 0 \\ \frac{-3}{(m_s+4m_c)l_s}\left(\frac{2\pi\eta_b}{p_d}\right) \end{bmatrix}$$
$$B_w(t) = \begin{bmatrix} 0 & 0 \\ \frac{4}{(m_s+4m_c)}\left(\frac{2\pi\eta_b}{p_d}\right) & 1 \\ 0 & 0 \\ \frac{-3}{(m_s+4m_c)l_s}\left(\frac{2\pi\eta_b}{p_d}\right) & 0 \end{bmatrix}. \tag{28}$$

Note that the above $B_w(t)$ has two columns, though in (27) the only exogenous disturbance is $\ddot{x}_P(t)$. This is due to how we treat actuator non-linearities. The deadzone characteristic of the motor and driver is regarded as a kind of the band-bounded non-linearities. The effect of the deadzone on the control input, denoted as $\Delta u(t)$, is replaced by an equivalent disturbance input $w_1(t)$, which together with the "true" exogenous disturbance $w_2(t) = \ddot{x}_P(t)$ form the disturbance vector $w(t) \in \mathcal{R}^2$. For $w_1(t)$, we have $|w_1(t)| \le \bar{w}_1 = \delta = 0.0228$ kg-m^2/s^2, while for $w_2(t)$ we have $|w_2(t)| \le \bar{w}_2 = 0.0395$ m/s^2, because horizontally, the upper platform of Stewart platform is commanded to make the simple harmonic motion $x_P(t) = a_m \sin(2\pi t/T_m)$ and $y_P(t) = 0$, where $a_m = 0.1$ m and $T_m = 10$ s. If we let $\bar{w} = [\bar{w}_1 \;\; \bar{w}_2]^{\mathrm{T}}$, then it is possible to write

$$w(t) = [I_2 - 2\,\Pi(t)]\bar{w} \tag{29}$$

where $\Pi(t) \in \mathbf{Co}\{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$, and $\{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$ is the set of four distinct diagonal 2×2 matrices with either zero or unity diagonal elements. In addition to the horizontal movement, the upper platform of the Stewart platform is commanded to make a vertical simple harmonic motion $z_P(t) = 0.8 + a_m \sin(2\pi t/T_m)$ (m). Hence from the parameter values listed in table 1 and (28), we have

$$[A(t) \quad B_u(t) \quad B_w(t)] \in \mathbf{Co}\left\{ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.606 & 0 & 48.33 & 48.33 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 48.96 & 0 & -226.56 & -226.56 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.6 & 0 & 48.33 & 48.33 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 48.57 & 0 & -226.56 & -226.56 & 0 \end{bmatrix} \right\}. \tag{30}$$

Note that the units for the linear and angular displacements are (m) and (rad), respectively.

Up to now a mathematical model in consistent with the one formulated in a § 2 is derived. However, practically all state and control input variables in physical systems have upper bounds on their magnitudes. In this particular physical system, through some test runs it is discovered that the short ballscrew track length of 0.30 m is the most restrictive, as it limits the traveling range of the cart. Hence, the following state constraints

$$x(t) \in \mathcal{S}(h, \phi) = \{x \in \mathcal{R}^4 \mid h_i^{\mathrm{T}} x \le \phi_i,\ i = 1, 2\} \tag{31}$$

with $h_1 = [1 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$, $h_2 = [-1 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$, and $\phi_1 = \phi_2 = 0.15$ must be accommodated. To be able to do so, the set invariance concept (Blanchini 1999) is introduced here because in the literature (Boyd *et al.* 1994) it is successfully utilized in the quadratic stabilization problem subject to the same type of state constraints. More specifically, an ellipsoid positively invariant set $\mathcal{P} = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} P x < 1\}$ is considered here, and it is sufficient to require that $\mathcal{P}$ be contained by $\mathcal{S}(h, \phi)$, or equivalently to augment the following LMI

$$h_i^{\mathrm{T}} Q h_i \le \phi_i^2, \qquad i = 1, 2 \tag{32}$$

where $Q = P^{-1} > 0$, to the LMIs in Theorem 1.

Corollary 1: *Consider the polytopic uncertain system* (3) *with* (30) *that is subject to the state constraints* (31) *as well as the bounded disturbances* (29). *For a given $\gamma_o > 0$, if there exist matrices and scalars $Q \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{m\times n}$, $\tau_1^{-1} \in \mathcal{R}$, and $\hat{\epsilon} \in \mathcal{R}$ satisfying the LMIs* (8), (10) *and* (32), *then $u(t) = Kx(t) = YQ^{-1}x(t)$ makes every state trajectories of closed-loop system originating from the set $\{x_o \in \mathcal{R}^n \mid x_o^{\mathrm{T}} P x_o = x_o^{\mathrm{T}} Q^{-1} x_o \le 1\}$ converge to and stay within the ultimate boundedness region $\mathcal{E}_{\tau_1^{-1}} = \{x \in \mathcal{R}^n \mid x^{\mathrm{T}} P x = x^{\mathrm{T}} Q^{-1} x \le \tau_1^{-1} < 1\}$ eventually. Also, the length of the major semi-axis of $\mathcal{E}_{\tau_1^{-1}}$ is less than γ_0.*

Based on the feasibility problem stated in the above corollary, a convex optimization problem

$$\left.\begin{array}{ll} \min & \varpi_1 \tau_1^{-1} + \varpi_2 \| Y \| \\ \text{subject to} & (8),\ (10),\ \text{and}\ (32) \end{array}\right\} \tag{33}$$

is formed. In a representative application case we set $\gamma_o = 20$, $\varpi_1 = 50$, and $\varpi_2 = 1$. The resultant optimal solution of (33) is

$$K^* = \begin{bmatrix} 4.78 & 11.38 & 21.39 & 4.44 \end{bmatrix},$$

$$P^* = \begin{bmatrix} 64.97 & 16.21 & 41.73 & 3.6 \\ 16.21 & 20.06 & 35.19 & 4.72 \\ 41.73 & 35.19 & 87.73 & 8.26 \\ 3.6 & 4.72 & 8.26 & 1.18 \end{bmatrix}$$

$$\hat{\epsilon}^* = 0.0075, \quad \tau_1^{-1*} = 0.2.$$

Figure 4 shows the closed-loop system responses corresponding to the initial conditions $x_o = [-0.117 \;\; 0 \;\; -0.014 \;\; 0.128]^{\mathrm{T}}$ in the presence of the translational movement $x_P(t) = 0.1 \sin(2\pi t/10)$ m and $z_P(t) = 0.8 + 0.1 \sin(2\pi t/10)$ m of the upper platform of the Stewart platform. It is seen that the ultimate boundedness control is achieved.

5. Conclusions and discussions

For a class of linear uncertain systems subject to persistent disturbances and driven by the band-bounded non-linear actuators, an LMI based robust state feedback controller design method is proposed to guarantee ultimate boundedness of state trajectories. The problem formulation is flexible in that the effect of band-bounded nonlinear actuators may be accommodated like that of exogenous disturbances, and that every disturbance may have theirs own bounds.

Moreover, an inverted pendulum on a Stewart platform making translational movement is introduced, and its corresponding constrained control problem is discussed and solved by the proposed method. The experimental system can also be used as a good test bench for other controller design methods.

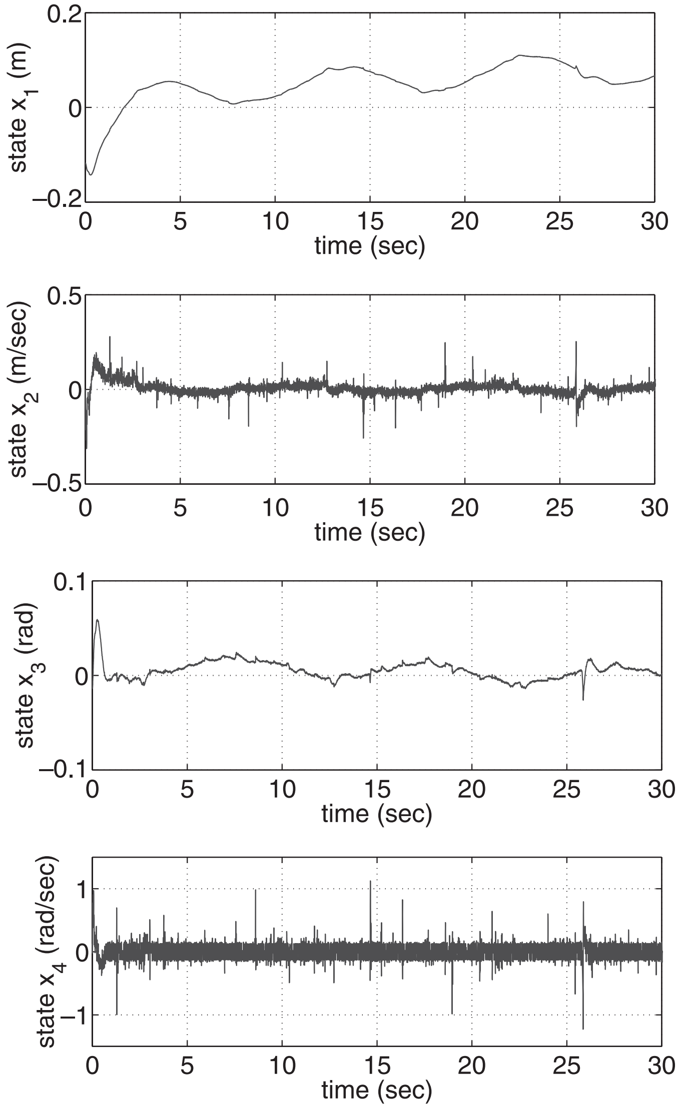

Figure 4. Experimental responses of the inverted pendulum on a moving Stewart platform.

It would be a challenge to stabilize the inverted pendulum on the Stewart platform making 6-DOF movements, and it is conjectured that more information, such as the rates of three Euler angles of the Stewart platform, which can be measured indirectly by three gyros, will be needed. Because this problem is similar to the stabilization of missiles vertically launched on a ship, it worths further investigations.

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