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Robust FIR Filter Design with Envelope Constraints and Channel Uncertainty

Ching-Min Lee and I-Kong Fong

Abstract—In this note, a finite impulse response (FIR) filter design problem is considered. The signals to be filtered are assumed to be corrupted by the channel noise. In addition, the channel characteristics are assumed to contain uncertainties. The linear matrix inequalities approach is adopted to provide two optimization procedures for designing H_∞ optimal filters and robust H_∞ filters subject to filter output envelope constraints. A numerical example is presented to illustrate the proposed filter design methods.

Index Terms—Bounded stability, integral quadratic constraints, linear matrix inequality, robust FIR filter, time-domain envelope constraint.

I. INTRODUCTION

In the field of signal processing, many filter design problems can be cast as constrained optimization problems. The constraints are usually defined by the specifications of the desired filters, and these specifications arise either from the standards set by certain regulatory bodies or from practical considerations. The time-domain envelope-constrained filter design is one example of these problems, which often involve requirements on the transient responses, such as the pulse-shape requirements in digital data transmission systems. In particular, these kinds of filters may be seen in applications like the pulse compression for many communication and radar systems, the TV waveform equalization with respect to the K-mask, and the data channel equalization or deconvolution, [1]–[3].

As to the optimization part, the H_∞ optimization theory has been widely used in robust control and signal processing problems [4]–[8].
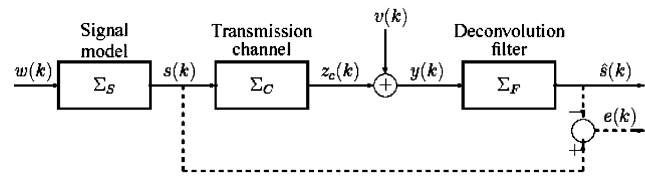
Fig. 1. Deconvolution filtering system.

Take the deconvolution filter design, for example. The objective of the problem may be set to the minimization of the H_∞ norm of the filtering error transfer function. In this approach, the system formulation allows the inclusion of the transmission channel and/or signal models. In addition, by utilizing methods developed in [7] and [9], the time-domain envelope constraints may be accommodated simultaneously. However, there are still other important factors to take care of, such as the system uncertainties. In the literature [10], [11], there are some discussions about input uncertainty of the filter, but consideration of input uncertainty bounds only [11] does not fully use information about system uncertainties that may be available. A more direct and complete approach for handling the transmission channel model uncertainties is desirable.

In this correspondence, the H_∞ optimal finite impulse response (FIR) filtering problem with envelope constraints and channel uncertainties is studied. The linear matrix inequality (LMI) framework is adopted, and the uncertainties in the channel are formulated as satisfying the integral quadratic constraints (IQCs) [12], [13]. The design method is also enhanced to ensure that the output of the filter is not too close to the constraining envelopes [8], [10], [14]. Compared with some existing method [14], which deals with the output envelope constraint problem by optimization procedures depending on the quasi-Newton method and golden section method, the LMI-based method handling a convex optimization problem is numerically more attractive. To illustrate the effectiveness of the proposed design method, a numerical example is presented.

II. PROBLEM FORMULATION

Consider the deconvolution filtering system shown in Fig. 1. In the system, the source signal $s(k) \in \mathcal{R}$ is assumed to be generated by the signal model

$$\Sigma_S : \begin{cases} x_s(k+1) = A_s x_s(k) + B_s w(k) \\ s(k) = C_s x_s(k) + D_s w(k) \end{cases} \quad (1)$$

where $x_s(k) \in \mathcal{R}^{n_s}$ is the model state vector, $w(k) \in l_2[0,\infty)$ is the driving signal of the model, and A_s, B_s, C_s, and D_s are known constant matrices of appropriate dimensions. The source signal is transmitted through a channel with an uncertain characteristic modeled by

$$\Sigma_C : \begin{cases} x_c(k+1) = A_c x_c(k) + B_c s(k) + \sum_{i=1}^p H_{c1i}\xi_{ci}(k) \\ z_c(k) = C_c x_c(k) + D_c s(k) + \sum_{i=1}^p H_{c2i}\xi_{ci}(k) \end{cases} \quad (2)$$

where $x_c(k) \in \mathcal{R}^{n_c}$ is the channel state vector, $\xi_{ci}(k) \in \mathcal{R}^{n_i}$, $i = 1, 2, \ldots, p$ is the ith uncertain vector satisfying the IQC [12]

$$\sum_{k=0}^\mu \|\xi_{ci}(k)\|^2 \le \sum_{k=0}^\mu \|E_{1i}x_c(k) + E_{2i}s(k) + E_{3i}\xi_c(k)\|^2 \quad (3)$$

as $\mu \to \infty$, $\xi_c^T(k) = [\xi_{c1}^T(k) \quad \cdots \quad \xi_{cp}^T(k)]$, and A_c, B_c, C_c, D_c, H_{c1i}, H_{c2i}, E_{1i}, E_{2i}, and E_{3i} are known constant matrices with appropriate dimensions. For subsequent usage, we define $H_{c1} = [H_{c11} \quad \cdots \quad H_{c1p}]$, $H_{c2} = [H_{c21} \quad \cdots \quad H_{c2p}]$, $E_1^T = [E_{11}^T \quad \cdots \quad E_{1p}^T]$, $E_2^T = [E_{21}^T \quad \cdots \quad E_{2p}^T]$, and $E_3^T = [E_{31}^T \quad \cdots \quad E_{3p}^T]$.

At the receiving end, the measured signal $y(k)$ is equal to $z_c(k) + v(k)$, where $v(k)$ is the energy-bounded channel noise. To optimally recover the source signal $s(k)$, the signal $y(k)$ is deconvoluted by a filter

$$\Sigma_F : \begin{cases} x_f(k+1) = A_f x_f(k) + B_f y(k) \\ \hat{s}(k) = C_f x_f(k) + D_f y(k) \end{cases} \qquad (4)$$

where $x_f(k) \in \mathcal{R}^{n_f}$ is the filter state vector, and A_f, B_f, C_f, and D_f are constant matrices to be designed. Let the filtering error be defined as $e(k) = s(k) - \hat{s}(k)$. Then, it satisfies

$$\Sigma_e : \begin{cases} x_e(k+1) = A_e x_e(k) + B_e w_e(k) + H_{1e}\xi_c(k) \\ e(k) = C_e x_e(k) + D_e w_e(k) + H_{2e}\xi_c(k) \end{cases} \qquad (5)$$

where $x_e^T(k) = [x_s^T(k)\ x_c^T(k)\ x_f^T(k)]$, $w_e^T(k) = [w^T(k)\ v^T(k)]$, and

$$A_e = \begin{bmatrix} A_s & 0 & 0 \\ B_c C_s & A_c & 0 \\ B_f D_c C_s & B_f C_c & A_f \end{bmatrix}$$

$$B_e = \begin{bmatrix} B_s & 0 \\ B_c D_s & 0 \\ B_f D_c D_s & B_f \end{bmatrix}, \quad H_{1e} = \begin{bmatrix} 0 \\ H_{c1} \\ B_f H_{c2} \end{bmatrix}$$

$$C_e = [C_s - D_f D_c C_s \quad -D_f C_c \quad -C_f],$$

$$D_e = [D_s - D_f D_c D_s \quad -D_f], \quad H_{2e} = -D_f H_{c2}. \qquad (6)$$

The purpose of this paper is to design an FIR filter Σ_F with three desired properties, of which the first one is given by the following definition.

Definition 1: [12] The filtering error dynamics Σ_e is called bounded stable if there exists a constant $\beta \geq 0$ such that $\|x_e(k)\| \leq \beta$ for all $k \geq 0$, no matter what the initial condition $x_e(0)$ and input $w_e(k) \in l_2[0, \infty)$ are.

The second desired property of the filter is that for a given filter input signal $y(k)$ of finite duration, the output signal $\hat{s}(k)$ has to be bounded above and below by

$$l_s(k) \leq \hat{s}(k) \leq u_s(k) \qquad (7)$$

for $k = 0, 1, \ldots, n$, where $l_s(k)$ and $u_s(k)$ are the lower and upper bounds of the time-domain mask, respectively, and $n+1$ is the duration of $\hat{s}(k)$. Finally, it is desired that when $x_e(0) = 0$, the filter has the H_∞ performance

$$\gamma^{-1} \sum_{k=0}^{\mu} \|e(k)\|^2 \leq \gamma \sum_{k=0}^{\mu} \|w_e(k)\|^2 \qquad (8)$$

for some scalar $\gamma > 0$ and all $w_e \neq 0$, as $\mu \to \infty$.

III. FIR FILTER DESIGN

To start, the $\mathcal{S}$-procedure [15] is applied to convert two of the above design objectives into a quadratic condition, and a theorem is derived.

Theorem 1: Under all admissible channel uncertainties satisfying the IQC (3), the error dynamics Σ_e is bounded stable and satisfies (8) when $x_e(0) = 0$ if there exists a symmetric positive definite matrix $X \in \mathcal{R}^{n_s + n_c + n_f}$ and positive scaling scalars $\tau_1, \tau_2, \ldots, \tau_p$, γ satisfying the following inequality:

$$(A_e x_e + B_e w_e + H_{1e}\xi_c)^T X (A_e x_e + B_e w_e + H_{1e}\xi_c)$$

$$- x_e^T X x_e + \sum_{i=1}^{p} \tau_i \left(\|E_{1i}x_c + E_{2i}s + E_{3i}\xi_c\|^2 - \|\xi_{c_i}\|^2 \right)$$

$$+ \gamma^{-1}\|C_e x_e + D_e w_e + H_{2e}\xi_c\|^2 - \gamma\|w_e\|^2 < 0 \qquad (9)$$

for all $[x_e^T, w_e^T, \xi_c^T]^T \neq 0$.

Proof: Summing up the left-hand side of the inequality in (9) along any trajectory of the error dynamics Σ_e, one gets

$$x_e^T(\mu+1)X x_e(\mu+1) - x_e^T(0)X x_e(0)$$

$$+ \sum_{i=1}^{p} \tau_i \left\{ \sum_{k=0}^{\mu} \|E_{1i}x_c(k) + E_{2i}s(k) + E_{3i}\xi_c(k)\|^2 \right.$$

$$\left. - \sum_{k=0}^{\mu} \|\xi_{ci}(k)\|^2 \right\}$$

$$+ \left\{ \gamma^{-1} \sum_{k=0}^{\mu} \|e(k)\|^2 - \gamma \sum_{k=0}^{\mu} \|w_e(k)\|^2 \right\} < 0 \qquad (10)$$

for all $w_e \neq 0$. From (3), it follows that

$$x_e^T(\mu+1)X x_e(\mu+1) < x_e^T(0)X x_e(0) + \gamma \sum_{k=0}^{\mu} \|w_e(k)\|^2$$

as $\mu \to \infty$. Thus, the bounded stability of the filtering error dynamics is implied. As to the H_∞ performance, when $x_e(0) = 0$, it follows from (3) and (10) that (8) holds. $\qquad \square$

A. H_∞ Optimal FIR Filter Design

To develop a set of easy-to-use conditions for designing the FIR filters, the parameter matrices of the filter Σ_F are assumed to be of the form [5]

$$A_f = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n_f \times n_f}, \quad B_f = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n_f \times 1}$$

$$C_f = [f(n_f) \quad f(n_{f-1}) \quad \cdots \quad f(1)], \quad D_f = f(0)$$

where $f(0), f(1), \cdots, f(n_f)$ are the parameters to be determined. Accordingly, the transfer function of the filter is

$$C_f(zI - A_f)^{-1}B_f + D_f$$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots + f(n_f)z^{-n_f}. \qquad (11)$$

At this point, it is appropriate to consider the second desired property of the filter. Suppose that a signal $\{y(0), y(1), y(2), \ldots, y(m), 0, 0, \ldots\}$ is given, and it is required that (7) be satisfied. Let [7]

$$y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(m) \end{bmatrix}, \quad f = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(n_f) \end{bmatrix}, \quad l_b = \begin{bmatrix} l_s(0) \\ l_s(1) \\ \vdots \\ l_s(n) \end{bmatrix}, \quad u_b = \begin{bmatrix} u_s(0) \\ u_s(1) \\ \vdots \\ u_s(n) \end{bmatrix} \qquad (12)$$

and

$$Y = \begin{bmatrix} y(0) & 0 & \cdots & 0 \\ y(1) & y(0) & \cdots & 0 \\ \vdots & y(1) & \cdots & \vdots \\ y(m) & \vdots & & y(0) \\ 0 & y(m) & \vdots & y(1) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & y(m) \end{bmatrix} \qquad (13)$$

where Y is an $n \times (n_f + 1)$ matrix, and $n = m + n_f + 1$. Therefore, the time-domain constraints in (7) are equivalent to

$$\text{diag}(l_b) \leq \text{diag}(Y f) \leq \text{diag}(u_b) \qquad (14)$$

where "diag($\cdot$)" denotes a diagonal matrix formed by putting the elements of the argument vector on the diagonal positions of the matrix. Based on the inequality (14) and Theorem 1, the following theorem can be established to provide a convex optimization problem with the LMI formulation, which may be easily solved for finding the filter parameters.

Theorem 2: An H_∞ optimal FIR filter of the form (11) satisfying the three desired properties formulated in Section II exists if the following optimization problem has solutions:

$$\min_{X,f,J} \gamma \tag{15}$$

subject to (16), shown at bottom of the page, and

$$\mathrm{diag}(l_b) \leq \mathrm{diag}(Yf) \tag{17}$$

$$\mathrm{diag}(Yf) \leq \mathrm{diag}(u_b) \tag{18}$$

$$X > 0, \quad J > 0, \quad \gamma > 0 \tag{19}$$

where $J = \mathrm{diag}(\tau_1 I_{n_1}, \cdots, \tau_p I_{n_p})$, $C_e = \begin{bmatrix} C_s & 0 & 0 \end{bmatrix} - D_f \begin{bmatrix} D_c C_s & C_c & 0 \end{bmatrix} - C_f \begin{bmatrix} 0 & 0 & I \end{bmatrix}$, $D_e = \begin{bmatrix} D_s & 0 \end{bmatrix} - D_f \begin{bmatrix} D_c D_s & I \end{bmatrix}$, $E_e = \begin{bmatrix} E_2 C_s & E_1 & 0 \end{bmatrix}$, and $E_w = \begin{bmatrix} E_2 D_s & 0 \end{bmatrix}$.

Proof: Note that (9) can be rewritten as

$$(A_e x_e + B_e w_e + H_{1e}\xi_c)^T X(A_e x_e + B_e w_e + H_{1e}\xi_c) - x_e^T X x_e$$
$$+ (E_e x_e + E_w w_e + E_3\xi_c)^T J(E_e x_e + E_w w_e + E_3\xi_c)$$
$$- \xi_c^T J\xi_c + \gamma^{-1}(C_e x_e + D_e w_e + H_{2e}\xi_c)^T$$
$$\times (C_e x_e + D_e w_e + H_{2e}\xi_c) - \gamma w_e^T w_e$$
$$= \begin{bmatrix} x_e^T & w_e^T & \xi_c^T \end{bmatrix} M \begin{bmatrix} x_e^T & w_e^T & \xi_c^T \end{bmatrix}^T < 0. \qquad \square$$

B. Robust H_∞ FIR Filter Design

When an H_∞ optimal FIR filter is designed using Theorem 2 in the above subsection, the time-domain constraints in (7) will be satisfied. However, in some cases, the output response of the filter may be very close to or even touch the constraint envelopes. In certain applications, this is regarded as undesired. To avoid this kind of situation, the envelope constraints and objective function considered in Theorem 2 can be modified as

$$m_s(k) - \eta \cdot e_s(k) \leq \hat{s}(k) \leq m_s(k) + \eta \cdot e_s(k) \tag{20}$$

for $k = 0, 1, \ldots, n$, and

$$\min_{X,f,J}(\rho_1\gamma + \rho_2\eta) \tag{21}$$

respectively, where

$$m_s(k) = \frac{1}{2}(u_s(k) + l_s(k)) \tag{22}$$

$$e_s(k) = \frac{1}{2}(u_s(k) - l_s(k)) \tag{23}$$

ρ_1 and ρ_2 are tunable weighting parameters, and $0 < \eta < 1$ is a new scaling parameter to be determined. Note that (20) is equivalent to the LMIs

$$\mathrm{diag}(m_b) - \eta \cdot \mathrm{diag}(e_b) \leq \mathrm{diag}(Yf) \tag{24}$$

and

$$\mathrm{diag}(Yf) \leq \mathrm{diag}(m_b) + \eta \cdot \mathrm{diag}(e_b) \tag{25}$$

where

$$m_b = \begin{bmatrix} m_s(0) \\ m_s(1) \\ \vdots \\ m_s(n) \end{bmatrix}, \quad e_b = \begin{bmatrix} e_s(0) \\ e_s(1) \\ \vdots \\ e_s(n) \end{bmatrix}$$

and f and Y are defined in (12) and (13).

IV. NUMERICAL EXAMPLE

In this section, an example is worked out to illustrate the proposed design algorithms. Suppose that the system shown in Fig. 1 has the signal model Σ_S with the following system matrices:

$$A_s^T = \begin{bmatrix} -2.3060 & 1 & 0 & 0 & 0 & 0 \\ -2.9625 & 0 & 1 & 0 & 0 & 0 \\ -2.2590 & 0 & 0 & 1 & 0 & 0 \\ -1.0922 & 0 & 0 & 0 & 1 & 0 \\ -0.3009 & 0 & 0 & 0 & 0 & 1 \\ -0.0325 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_s = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.0062 & 0.2170 \end{bmatrix}, \quad D_s = 0$$

and the channel model Σ_C with the following system matrices:

$$A_c = \begin{bmatrix} -1.90 & -1.18 & -0.24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -0.360 & -0.153 & -0.027 \end{bmatrix}, \quad D_c = 0.45.$$

To simplify the discussion, assume that there is only one uncertain variable in the channel. The related matrix parameters are $H_{c1}^T = H_{c11}^T = \begin{bmatrix} 0.02 & 0.03 & 0.01 \end{bmatrix}$, $H_{c2} = H_{c21} = 0.024$, $E_1 = E_{11} = \begin{bmatrix} 0.62 & 0.51 & 0.55 \end{bmatrix}$, $E_2 = E_{21} = 0.73$, and $E_3 = E_{31} = -0.082$. In addition, assume, to start, that there is no channel noise (i.e., $v \equiv 0$). In Fig. 2, the filter output constraint envelopes corresponding to a particular channel output $y(k)$ are shown. This $y(k)$ is generated by the unit-impulse response of the signal model when the uncertain variable of the channel model equals zero. Then, the H_∞ optimal filter is designed using the LMI Toolbox of Matlab [16] with n_f selected to be 15. The resultant optimal $\gamma = 3.5330$, and the filter output with respect to the above $y(k)$ is also displayed in Fig. 2. Clearly, the output is quite close to the lower envelope. Thus, the robust H_∞ filter is designed with $\rho_1 = 1$ and $\rho_2 = 1.5$. The resultant optimal γ and η are 3.8891 and 0.6243, respectively. In Fig. 3, the new filter output is displayed, which is very close to the envelope center line, as desired.

To see the robustness of the two filters, the filter outputs are computed again with respect to the same signal $s(k)$, but this time, a normally distributed white channel noise $v(k)$ with zero mean and variance equal to 1×10^{-4} is added. In addition, the channel uncertain variable $\xi_{c1}(k)$ is set to be $-3.356k/(k^2 + k + 1)$ for $k \geq 0$. The results are shown in Figs. 4 and 5. It is seen that most of the time, the filter

$$M = \begin{bmatrix} A_e^T X A_e - X + E_e^T J E_e & A_e^T X B_e + E_e^T J E_w & A_e^T X H_{1e} + E_e^T J E_3 & C_e^T \\ B_e^T X A_e + E_w^T J E_e & B_e^T X B_e + E_w^T X E_w - \gamma I & B_e^T X H_{1e} + E_w^T J E_3 & D_e^T \\ H_{1e}^T X A_e + E_3^T J E_e & H_{1e}^T X B_e + E_3^T J E_w & H_{1e}^T X H_{1e} + E_3^T J E_3 - J & H_{2e}^T \\ C_e & D_e & H_{2e} & -\gamma I \end{bmatrix} < 0 \tag{16}$$
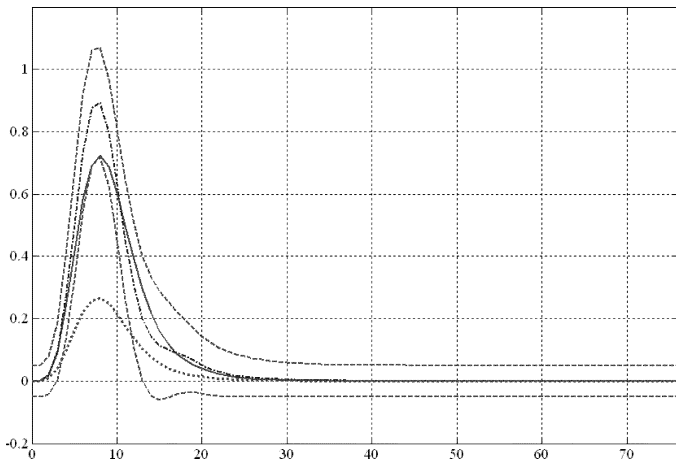
Fig. 2. Filter output $y(k)$ (dotted line), the corresponding filter output constraint envelopes (dashed lines), and output of the H_∞ optimal filter (solid line).
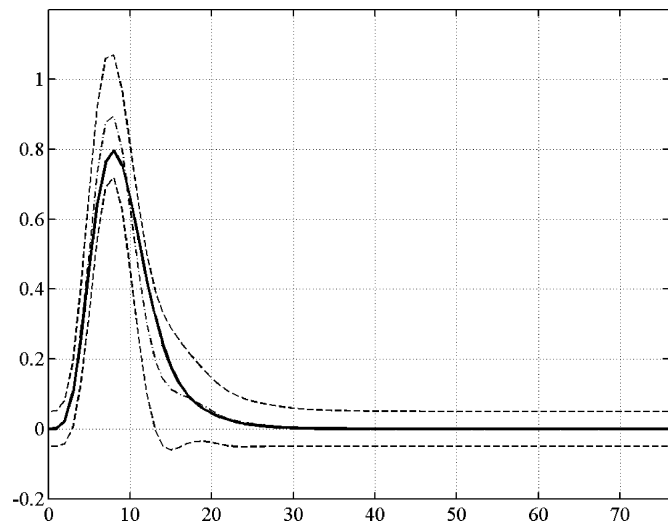

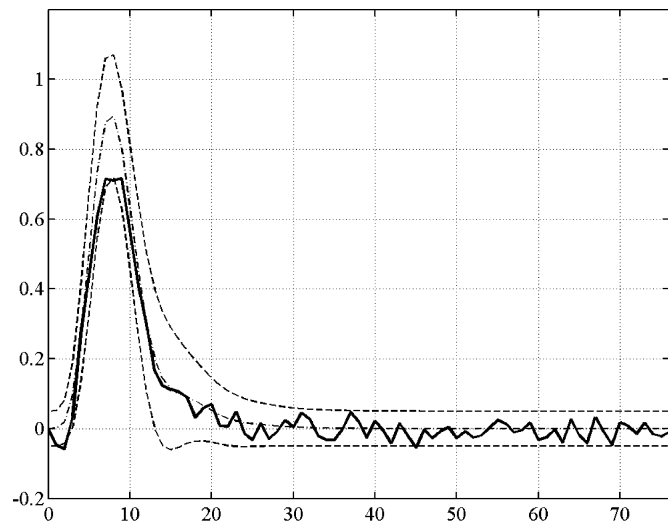Fig. 3. Center line of the mask (dash-dotted line) and output of the robust H_∞ filter (solid line).

Fig. 4. Filter output of the H_∞ optimal filter (solid line) with channel noise and uncertainty.
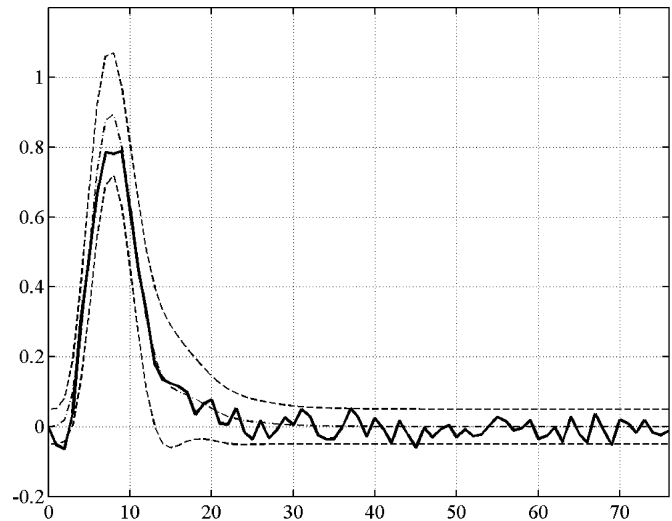

Fig. 5. Filter output of the robust H_∞ filter (solid line) with channel noise and uncertainty.

output of the robust H_∞ filter is kept within the envelopes, whereas the situation for the optimal H_∞ filter is relatively worse.

V. DISCUSSION AND CONCLUSION

In this correspondence, an LMI-based optimization approach is proposed to design FIR filters that satisfy prespecified time-domain envelope constraints at the output. In addition to the widely treated channel noise, the less-considered channel uncertainties are accommodated using the versatile IQC formulation. Both the H_∞ optimal filter and the robust H_∞ filter are offered so that the tradeoff between the H_∞ performance and constraint robustness may be carried out.

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Eigenstructure Approach for Complete Characterization of Linear-Phase FIR Perfect Reconstruction Analysis Length $2M$ Filterbanks

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Abstract—The eigenstructure based characterization of M-channel finite impulse response perfect reconstruction (FIR PR) filterbanks in a previous paper by the authors is extended here to the linear-phase case. Some results relating to linear-phase filterbanks is derived by finding appropriate restrictions on the eigenstructure of the analysis polyphase matrix. Consequently, a complete and minimal characterization for such filterbanks with all analysis length $2M$ and any synthesis length is developed. Parameterization and design examples are also presented.

Index Terms—Eigenstructure, FIR filterbank, linear phase.

I. INTRODUCTION

In [1], we used the *eigenstructure* representation of the *polyphase matrix* to propose complete characterizations of finite impulse response *perfect reconstruction* (FIR PR) M-channel filterbanks with first-order analysis polyphase matrix. *Linear-phase* FIR perfect reconstruction filterbanks (LPFBs) find application in many signal and image processing fields. In this correspondence, we extend the eigenstructure representation to obtain a complete characterization of linear-phase FIR perfect reconstruction M-channel filterbanks with all analysis filter lengths being $2M$ (hence first-order analysis polyphase matrix), which are referred to henceforth as FOLPFBs. The synthesis filter length in this characterization is not restricted to $2M$ as is conventionally done but may take a value up to M^2.

Characterization of a subclass of LPFBs, such as orthogonal, $M = 2$, or $M = 3$, has been reported on several occasions. The design of FOLPFBs with a multistage structure using the discrete cosine transform (DCT) such that a fast implementation exists is reported in [2], where it is called the *lapped biorthogonal transform*. Design of an LPFB of any order is reported in [3], when K filters are given such that the part polyphase matrix has rank K for all z^{-1} except $z^{-1} = 0$,

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and the remaining filters are designed. In [4], symbolic computation is used to characterize an LPFB of any order. However, none of the above characterizations are complete. In [5], a lattice structure is used to characterize the LPFB so that the analysis polyphase matrix of the FOLPFB becomes $(1/2\sqrt{2}) \begin{bmatrix} (1+z^{-1})\mathbf{U} & (1-z^{-1})\mathbf{U} \\ (1-z^{-1})\mathbf{V} & (1+z^{-1})\mathbf{V} \end{bmatrix}$ $\cdot \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0\mathbf{J}_{M/2} \\ \mathbf{V}_0\mathbf{J}_{M/2} & -\mathbf{V}_0 \end{bmatrix}$, where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{U}_0$, and $\mathbf{V}_0$ are nonsingular $M/2 \times M/2$ matrices, and $\mathbf{J}_k$ is a $k \times k$ counter identity matrix. It is shown to be complete for FOLPFBs with synthesis length $2M$ [6].

We briefly describe below the characterization of [1]. Replacing z^{-1} by λ, the lth-order analysis polyphase matrix $\mathbf{E}(z)$ is seen as a matrix polynomial $\mathbf{E}_l(\lambda)$. Any matrix polynomial may be characterized by the *Jordan pair* (or decomposable pair or spectral data) $(\mathbf{Y}, \mathbf{T}(\lambda))$ with $\mathbf{Y} = [\,\mathcal{X}_F \quad \mathcal{X}_R\,]$ and $\mathbf{T}(\lambda) = \mathrm{diag}(\mathbf{I}_\Gamma \lambda - \mathcal{J}_F, \mathcal{J}_R \lambda - \mathbf{I}_{Ml-\Gamma})$, where $\mathrm{diag}()$ represents a block diagonal matrix with the arguments as the blocks in sequence, $\mathbf{I}_k$ is the $k \times k$ identity matrix, and Γ is the degree of $|\mathbf{E}_l(\lambda)|$. $\mathcal{X}_F$ is the $M \times \Gamma$ canonical set of *Jordan chains*, and $\mathcal{J}_F$ is the $\Gamma \times \Gamma$ *Jordan form* of $\mathbf{E}_l(\lambda)$ (finite Jordan pair or finite spectrum). $\mathcal{J}_F$ is block diagonal with Jordan blocks of size $b_0, \ldots, b_n$ such that b_i are nonincreasing positive integers summing up to Γ. Each Jordan block's diagonal elements are eigenvalues of $\mathbf{E}_l(\lambda)$, upper off-diagonal elements are 1, and the remaining elements are 0.[1] $\mathcal{X}_R$ and $\mathcal{J}_R$ are the corresponding $M \times (Ml - \Gamma)$ and $(Ml - \Gamma) \times (Ml - \Gamma)$ matrices of the reversed matrix polynomial $\lambda^l \mathbf{E}_l(\lambda^{-1})$ for the zero eigenvalue (infinite Jordan pair or infinite spectrum). It follows that [7]

$$\mathbf{E}_l(\lambda) = \mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\mathbf{T}(\lambda)\mathbf{S}_{l-1}^{-1}\mathbf{Q}(\lambda) \tag{1}$$

where

$$\mathbf{S}_k = \begin{bmatrix} \mathcal{X}_F & \mathcal{X}_R \mathcal{J}_R^k \\ \mathcal{X}_F \mathcal{J}_F & \mathcal{X}_R \mathcal{J}_R^{k-1} \\ \vdots & \vdots \\ \mathcal{X}_F \mathcal{J}_F^k & \mathcal{X}_R \end{bmatrix}, \quad \mathbf{Q}(\lambda) = \begin{bmatrix} \mathbf{I}_M \\ \lambda \mathbf{I}_M \\ \vdots \\ \lambda^{l-1} \mathbf{I}_M \end{bmatrix}. \tag{2}$$

$\mathbf{A}$ is an $M \times Ml$ matrix such that $[\,\mathbf{S}_{l-2}^T \quad \mathbf{A}^T\,]^T$ is nonsingular, and $\mathbf{P} = \mathrm{diag}(\mathbf{I}_\Gamma, \mathcal{J}_R)\mathbf{S}_{l-1}^{-1}[\,\mathbf{I}_{Ml-M} \quad \mathbf{0}\,]^T\mathbf{S}_{l-2}$.

For the FIR inverse to exist, $\mathbf{E}_l(\lambda)$ has to be a matrix polynomial with a monomial determinant. This is equivalent to all eigenvalues of $\mathbf{E}_l(\lambda)$ being zero. Thus, $\mathcal{J}_F$ (as also $\mathcal{J}_R$) should have a zero diagonal. Further, for the first-order ($l = 1$) case, (1) simplifies to the *block diagonal* characterization [1]

$$\mathbf{E}_1(\lambda) = \mathbf{A}\mathbf{T}(\lambda)\mathbf{Y}^{-1} \tag{3}$$

where $\mathbf{A}$ and $\mathbf{Y}$ are any $M \times M$ nonsingular matrices. Note that $\mathcal{J}_F$, $\mathcal{J}_R$ are *nilpotent* matrices with indices of nilpotency n_F and n_R ($n_F = b_0$, size of the largest Jordan block of $\mathcal{J}_F$, etc.). Then, the synthesis polyphase becomes $\mathbf{E}_1^{-1}(\lambda) = \mathbf{Y}\mathrm{diag}(\mathbf{I}_\Gamma \lambda^{-1} + \sum_{i=2}^{n_F} \mathcal{J}_F^{i-1}\lambda^{-i}, -\mathbf{I}_{M-\Gamma} - \sum_{i=1}^{n_R-1} \lambda^i \mathcal{J}_R^i)\mathbf{A}^{-1}$. It follows that the maximum length of the synthesis filters is $(n_F + n_R)M$, and reconstruction delay is $(n_F + 1)M - 1$. The characterization allows unconstrained parameter optimization and provides control over the length of the synthesis filters and reconstruction delay.

[1]For example, for $\Gamma = 4$, $\{b_i\} = \{3, 1\}$ and zero eigenvalue $\mathcal{J}_F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$