Robust Filtering for 2-D State-Delayed Systems With NFT Uncertainties

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***Abstract*—This paper is concerned with the robust filtering problem for two-dimensional (2-D) state-delayed systems with uncertainties represented by nonlinear fraction transformation. The authors first establish the stability $\mathcal{H}_\infty$ performance and generalized $\mathcal{H}_2$ performance criteria for the system. Based on the results, the authors propose efficient methods to solve the robust $\mathcal{H}_\infty$ filtering, generalized $\mathcal{H}_2$ filtering, and mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problems by using a parameter-dependent Lyapunov function approach. The methods involve solving linear matrix inequalities. Two examples are given to show the effectiveness of the proposed approach.**

***Index Terms*—Linear matrix inequality (LMI), nonlinear fraction transformation, robust filter, time-delay systems, two-dimensional (2-D) systems.**

I. Introduction

THE filtering problem of two-dimensional (2-D) systems has attracted increasing attentions due to its application as well as theoretical importance in the fields such as multidimensional digital filtering, linear image processing, and so on [6], [12]. In these applications, it is usually desirable to estimate the values of state variables from the system measurement data. Various schemes, such as the Kalman filter, the $\mathcal{H}_\infty$ filter, and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter have been addressed in the literature (see, e.g., [5], [6], [19], and [21], and references cited therein).

For the Kalman filtering scheme [19], it requires *a priori* information about the statistical properties of external noise. Without such *a priori* information, the Kalman filtering scheme is not applicable. To handle problems with unknown noise properties, an $\mathcal{H}_\infty$ filtering scheme is proposed in [5] and [6]. Recently, the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for 2-D systems with polytopic uncertainties is also reported in [21] by using a much less conservative parameter-dependent Lyapunov function approach [4]. In practical applications, however, the uncertain parameters may affect the system in a nonlinear fashion. To handle this class of uncertainties, a general uncertainty model, the nonlinear fraction transformation (NFT), is first proposed by Tuan *et al.* [22]. The NFT model can be transformed into a linear fraction transformation (LFT) model. However, by comparing the NFT model with other types of uncertainty model, such as the LFT and norm-bounded models, an advantage of the NFT model is that it can result in less conservative designs than other models.

As is well known, time delays of signal transmissions are frequently encountered in engineering and biological systems. Examples of 2-D systems with time delays include the material rolling process [20] and models described by the delayed lattice differential equation [11] and partial difference equations [23], [24]. In addition, certain 2-D systems containing digital processors that need finite numerical computation time [2], [18] display the delay phenomenon. Delays are often a source of instability and poor performance. Therefore, for the one-dimensional (1-D) state-delayed systems, there have been much literature on the robust filtering that offer various schemes (see, e.g., [8], [9], [13], and [16], and the references cited therein). In contrast, most results for the 2-D filtering problem focus on systems without delays, though for specific stability and control problems of uncertain 2-D discrete state-delayed systems research results [17], [18] start to appear.

In this paper, we propose a complete methodology of robust filter synthesis for 2-D state-delayed systems with uncertainties described by the NFT model. In the systems, it is assumed that time delays appear in both the horizontal and vertical directions. The achievements are summarized as follows. First, we present a computationally tractable sufficient linear matrix inequality (LMI) [1] condition for the stability of 2-D state-delayed systems. This LMI condition plays a crucial role throughout the paper. Second, we develop a less conservative LMI formulation for the $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ performance of the uncertain 2-D state-delayed systems. Finally, we provide an efficient way to solve the robust $\mathcal{H}_\infty$, generalized $\mathcal{H}_2$, and mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problems by using a parameter-dependent Lyapunov function approach, which enables us to obtain less conservative design results.

The notation used throughout the paper is quite standard. $\mathcal{Z}$ is the set of nonnegative integers, $\mathcal{R}^n$ is the n-dimensional Euclidean space, and $\mathcal{R}^{n\times m}$ is the set of $n\times m$ real matrices. Π^{T} stands for the transpose of a matrix Π, and $P>0(<0)$ means that the symmetric matrix P is positive definite (negative definite). The boldface characters represent matrix variables, and $\otimes$ is the Kronecker product. In symmetric block matrices, we use $*$ as an ellipsis for the terms that are implied by symmetry, and $\mathrm{diag}(\cdots)$ for block-diagonal matrices. The l_2 norm of a 2-D signal $w(i,j)$ is defined and denoted by $\|W\|_2=[\sum_{n=0}^{\infty}\|W_n\|_2^2]^{1/2}$, where

$$W_n=[w(n,0)\;w(n-1,1)\;\cdots\;w(0,n)]$$

$$\|W_n\|_2=\left[\sum_{j=0}^{n}\|w(n-j,j)\|^2\right]^{\frac{1}{2}}$$

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and $\|\cdot\|$ is the Euclidean vector norm. We say a 2-D signal $w(i,j) \in l_2$ if it has a bounded l_2 norm. Finally, we shall need the following definitions:

$$\begin{aligned}\|w(i,j)\|_{\mathbf{M}}^2 &\equiv w(i,j)^{\mathrm{T}}\mathbf{M}w(i,j)\\ \|W_n\|_{\mathbf{M}}^2 &\equiv \sum_{j=0}^{n}\|w(n-j,j)\|_{\mathbf{M}}^2.\end{aligned}$$

II. Preliminaries

Consider the uncertain 2-D state-delayed system described by the Fornasini–Marchesini second model [12]

$$\begin{aligned}x(i+1,j+1) &= A(\alpha)\begin{bmatrix}x(i+1,j)\\x(i,j+1)\end{bmatrix}\\ &\quad + A_d(\alpha)\begin{bmatrix}x(i+1,j-d_1)\\x(i-d_2,j+1)\end{bmatrix}\\ &\quad + B_\Delta(\alpha)\begin{bmatrix}w_\Delta(i+1,j)\\w_\Delta(i,j+1)\end{bmatrix}\\ &\quad + B(\alpha)\begin{bmatrix}w(i+1,j)\\w(i,j+1)\end{bmatrix}\\ z_\Delta(i,j) &= C_\Delta(\alpha)x(i,j) + D_{\Delta z}(\alpha)w_\Delta(i,j)\\ &\quad + D_z(\alpha)w(i.j)\\ y(i,j) &= C(\alpha)x(i,j) + D_\Delta(\alpha)w_\Delta(i,j)\\ &\quad + D(\alpha)w(i.j)\\ z(i,j) &= L(\alpha)x(i,j)\\ w_\Delta(i,j) &= \Delta(\alpha)z_\Delta(i,j)\end{aligned} \tag{1}$$

where $x \in \mathcal{R}^{n_x}$ is the state vector, $w \in \mathcal{R}^{n_w}$ is the disturbance input vector, $y \in \mathcal{R}^{n_y}$ is the measured output vector, $z \in \mathcal{R}^{n_z}$ is the signal vector to be estimated, $w_\Delta \in \mathcal{R}^{n_\Delta}$ and $z_\Delta \in \mathcal{R}^{n_\Delta}$ are introduced to handle the nonlinear parameter dependence of the system, d_1 and d_2 are positive integers denoting time delays along vertical and horizontal directions, respectively, and

$$\begin{aligned}A(\alpha) &= [A_1(\alpha) \quad A_2(\alpha)]\\ A_d(\alpha) &= [A_{d1}(\alpha) \quad A_{d2}(\alpha)]\\ B(\alpha) &= [B_1(\alpha) \quad B_2(\alpha)]\\ B_\Delta(\alpha) &= [B_{\Delta 1}(\alpha) \quad B_{\Delta 2}(\alpha)].\end{aligned} \tag{2}$$

It is assumed that

$$\begin{bmatrix}A_1(\alpha) & B_{\Delta 1}(\alpha) & B_1(\alpha) & A_{d1}(\alpha)\\ A_2(\alpha) & B_{\Delta 2}(\alpha) & B_2(\alpha) & A_{d2}(\alpha)\\ C(\alpha) & D_\Delta(\alpha) & D(\alpha) & 0\\ C_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) & 0\\ L(\alpha) & 0 & 0 & 0\\ 0 & \Delta(\alpha) & 0 & 0\end{bmatrix} = \sum_{j=1}^{m}\alpha_j\begin{bmatrix}A_1^{(j)} & B_{\Delta 1}^{(j)} & B_1^{(j)} & A_{d1}^{(j)}\\ A_2^{(j)} & B_{\Delta 2}^{(j)} & B_2^{(j)} & A_{d2}^{(j)}\\ C^{(j)} & D_\Delta^{(j)} & D^{(j)} & 0\\ C_\Delta^{(j)} & D_{\Delta z}^{(j)} & D_z^{(j)} & 0\\ L^{(j)} & 0 & 0 & 0\\ 0 & \Delta^{(j)} & 0 & 0\end{bmatrix} \tag{3}$$

where $\alpha = [\alpha_1, \cdots, \alpha_m]^{\mathrm{T}}$ is unknown in the unit simplex

$$\left\{[\alpha_1, \cdots, \alpha_m]^{\mathrm{T}} : \sum_{j=1}^{m}\alpha_j = 1, \alpha_j \geq 0\right\}. \tag{4}$$

Note that the equations of system (1) may be expressed by the NFT model

$$\begin{aligned}x(i+1,j+1) &= \mathcal{A}_1(\alpha)x(i+1,j) + \mathcal{A}_2(\alpha)x(i,j+1)\\ &\quad + \mathcal{A}_{d1}(\alpha)x(i+1,j-d_1)\\ &\quad + \mathcal{A}_{d2}(\alpha)x(i-d_2,j+1)\\ &\quad + \mathcal{B}_1(\alpha)w(i+1,j)\\ &\quad + \mathcal{B}_2(\alpha)w(i,j+1)\\ y(i,j) &= \mathcal{C}(\alpha)x(i,j) + \mathcal{D}(\alpha)w(i.j)\\ z(i,j) &= \mathcal{L}(\alpha)x(i,j)\end{aligned} \tag{5}$$

where

$$\begin{aligned}&\begin{bmatrix}\mathcal{A}_1(\alpha) & \mathcal{B}_1(\alpha) & \mathcal{A}_{d1}(\alpha)\\ \mathcal{A}_2(\alpha) & \mathcal{B}_2(\alpha) & \mathcal{A}_{d2}(\alpha)\\ \mathcal{C}(\alpha) & \mathcal{D}(\alpha) & 0\\ \mathcal{L}(\alpha) & 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}A_1(\alpha) & B_1(\alpha) & A_{d1}(\alpha)\\ A_2(\alpha) & B_2(\alpha) & A_{d2}(\alpha)\\ C(\alpha) & D(\alpha) & 0\\ L(\alpha) & 0 & 0\end{bmatrix} + \begin{bmatrix}B_{\Delta 1}(\alpha)\\ B_{\Delta 2}(\alpha)\\ D_\Delta(\alpha)\\ 0\end{bmatrix}\\ &\quad \times \Delta(\alpha)\left[I - D_{\Delta z}(\alpha)\Delta(\alpha)\right]^{-1}\left[C_\Delta(\alpha)\ D_z(\alpha)\ 0\right].\end{aligned} \tag{6}$$

Remark 1: When $\Delta^{(1)} = \cdots = \Delta^{(m)} = 0$, the NFT model (5) reduces to a polytopic uncertain system with only linear uncertain parameters. On the other hand, when only $\Delta(\alpha)$ depends on α in (5), it reduces to an LFT model.

In this paper, the basic objective is to find a filter of the form

$$\begin{aligned}x_f(i+1,j+1) &= \mathbf{A_{f1}}x_f(i+1,j) + \mathbf{A_{f2}}x_f(i,j+1)\\ &\quad + \mathbf{B_{f1}}y(i+1,j) + \mathbf{B_{f2}}y(i,j+1)\\ z_f(i,j) &= \mathbf{C_f}x_f(i,j)\end{aligned} \tag{7}$$

for the system (1). Define the augmented state vector $\hat{x}(i,j) = [x^{\mathrm{T}}(i,j)x_f^{\mathrm{T}}(i,j)]^{\mathrm{T}}$ and the filtering error output signal $\hat{z}(i,j) = z(i,j) - z_f(i,j)$. Then we have the error equations

$$\begin{aligned}\hat{x}(i+1,j+1) &= \hat{A}(\alpha)\begin{bmatrix}\hat{x}(i+1,j)\\\hat{x}(i,j+1)\end{bmatrix}\\ &\quad + \hat{A}_d(\alpha)\begin{bmatrix}\hat{x}(i+1,j-d_1)\\\hat{x}(i-d_2,j+1)\end{bmatrix}\\ &\quad + \hat{B}_\Delta(\alpha)\begin{bmatrix}w_\Delta(i+1,j)\\w_\Delta(i,j+1)\end{bmatrix}\\ &\quad + \hat{B}(\alpha)\begin{bmatrix}w(i+1,j)\\w(i,j+1)\end{bmatrix}\\ z_\Delta(i,j) &= \hat{C}_\Delta(\alpha)\hat{x}(i,j) + D_{\Delta z}(\alpha)w_\Delta(i,j)\\ &\quad + D_z(\alpha)w(i.j)\\ \hat{z}(i,j) &= \hat{C}(\alpha)\hat{x}(i,j)\\ w_\Delta(i,j) &= \Delta(\alpha)z_\Delta(i,j)\end{aligned} \tag{8}$$

where

$$\hat{A}(\alpha) = \begin{bmatrix} A_1(\alpha) & 0 & A_2(\alpha) & 0 \\ \mathbf{B_{f1}}C(\alpha) & \mathbf{A_{f1}} & \mathbf{B_{f2}}C(\alpha) & \mathbf{A_{f2}} \end{bmatrix}$$
$$\hat{A}_d(\alpha) = \begin{bmatrix} A_{d1}(\alpha) & 0 & A_{d2}(\alpha) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\hat{B}(\alpha) = \begin{bmatrix} B_1(\alpha) & B_2(\alpha) \\ \mathbf{B_{f1}}D(\alpha) & \mathbf{B_{f2}}D(\alpha) \end{bmatrix}$$
$$\hat{B}_\Delta(\alpha) = \begin{bmatrix} B_{\Delta 1}(\alpha) & B_{\Delta 2}(\alpha) \\ \mathbf{B_{f1}}D_\Delta(\alpha) & \mathbf{B_{f2}}D_\Delta(\alpha) \end{bmatrix}$$
$$\hat{C}_\Delta(\alpha) = [C_\Delta(\alpha) \quad 0],$$
$$\hat{C}(\alpha) = [L(\alpha) \quad -\mathbf{C_f}]. \tag{9}$$

III. Robust Stability and Performance Criteria

The main purpose of this section is to develop some robust stability and performance criteria for 2-D state-delayed systems. These criteria play important roles in solving the robust filtering problems to be discussed in the next section.

A. Stability Analysis

Consider the autonomous nominal 2-D state-delayed system described by

$$x(i+1,j+1) = A\begin{bmatrix} x(i+1,j) \\ x(i,j+1) \end{bmatrix} + A_d \begin{bmatrix} x(i+1,j-d_1) \\ x(i-d_2,j+1) \end{bmatrix} \tag{10}$$

where $A = [A_1\ A_2]$ and $A_d = [A_{d1}\ A_{d2}]$.

Definition 1: The 2-D state-delayed system (10) is asymptotically stable if $\lim_{r\to\infty} X_r = 0$ for every initial condition $X_0 < \infty$, where

$$X_r = \sup\{\|x(i,j)\| : i+j=r, \quad i,j \in \mathcal{Z}\}. \tag{11}$$

It is known [12] that the system (10) is asymptotically stable if and only if

$$\det\left(I - z_2A_1 - z_1A_2 - z_2^{d_1+1}A_{d1} - z_1^{d_2+1}A_{d2}\right) \neq 0 \tag{12}$$

for all $(z_1, z_2) \in \mathcal{U}$, where

$$\mathcal{U} = \{(z_1,z_2) \| z_1| \le 1, |z_2| \le 1\}. \tag{13}$$

The above condition is necessary and sufficient for the asymptotic stability of system (10). Unfortunately, the condition is frequency-dependent and must be checked on infinitely many points in $\mathcal{U}$. In the following theorem, a computationally tractable sufficient condition will be given to guarantee the asymptotic stability of the system (10).

Theorem 1: The 2-D state-delayed system (10) is asymptotically stable if there exist positive definite matrices $\mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1$, and $\mathbf{Q}_2$ such that

$$\mathbf{W} = \begin{bmatrix} \mathbf{P}_1 & 0 & 0 & 0 \\ 0 & \mathbf{P}_2 & 0 & 0 \\ 0 & 0 & \mathbf{Q}_1 & 0 \\ 0 & 0 & 0 & \mathbf{Q}_2 \end{bmatrix} - \tilde{A}^{\mathrm{T}}(\mathbf{P}_1+\mathbf{P}_2+\mathbf{Q}_1+\mathbf{Q}_2)\tilde{A} > 0 \tag{14}$$

where $\tilde{A} = [A_1\ A_2\ A_{d1}\ A_{d2}]$.

Proof: Suppose the condition (14) is satisfied but (10) is unstable. Then

$$\det\left(I - z_2A_1 - z_1A_2 - z_2^{d_1+1}A_{d1} - z_1^{d_2+1}A_{d2}\right) = 0 \tag{15}$$

for some $(z_1, z_2) \in \mathcal{U}$, and there exists a nonzero vector v such that

$$v = \tilde{A}\begin{bmatrix} z_2 I \\ z_1 I \\ z_2^{d_1+1} I \\ z_1^{d_2+1} I \end{bmatrix} v. \tag{16}$$

Thus, from (14) and (16)

$$\begin{aligned} v^*(\mathbf{P}_1+\mathbf{P}_2+\mathbf{Q}_1+\mathbf{Q}_2)v &= \bar{v}^*\tilde{A}^{\mathrm{T}}(\mathbf{P}_1+\mathbf{P}_2+\mathbf{Q}_1+\mathbf{Q}_2)\tilde{A}\bar{v} \\ &= \bar{v}^*\left\{\begin{bmatrix} \mathbf{P}_1 & 0 & 0 & 0 \\ 0 & \mathbf{P}_2 & 0 & 0 \\ 0 & 0 & \mathbf{Q}_1 & 0 \\ 0 & 0 & 0 & \mathbf{Q}_2 \end{bmatrix} - \mathbf{W}\right\}\bar{v} \end{aligned} \tag{17}$$

where v^* denotes the complex conjugate transpose of v, and

$$\bar{v}^* = v^*\left[z_2^* I \quad z_1^* I \quad \left(z_2^{d_1+1}\right)^* I \quad \left(z_1^{d_2+1}\right)^* I\right].$$

It follows from (17) that

$$v^*\left[\mathbf{P}_1\left(1-|z_2|^2\right) + \mathbf{P}_2\left(1-|z_1|^2\right) + \mathbf{Q}_1\left(1-\left|z_2^{d_1+1}\right|^2\right) + \mathbf{Q}_2\left(1-\left|z_1^{d_2+1}\right|^2\right)\right]v = -\bar{v}^*\mathbf{W}\bar{v}. \tag{18}$$

However $\mathbf{W} > 0$, $\mathbf{P}_1 > 0$, $\mathbf{P}_2 > 0$, $\mathbf{Q}_1 > 0$, $\mathbf{Q}_2 > 0$, and $(z_1, z_2) \in \mathcal{U}$ imply that the right-hand side and the left-hand side of (18) are negative and nonnegative, respectively. This leads to a contradiction and concludes the proof. ■

Remark 2: With the notational change $\mathbf{P} = \mathbf{P}_1+\mathbf{P}_2+\mathbf{Q}_1+\mathbf{Q}_2$ and $\mathbf{Q} = \mathbf{P}_2$, Theorem 1 coincides with Theorem 3 of [18]. Here, the proof is different, and the formulation is arranged for the easy integration with the subsequent $\mathcal{H}_\infty/\mathcal{H}_2$ performance criteria. In addition, when $A_d = 0$ in (10), the stability condition in Theorem 1 reduces to the well-established stability conditions [14], [15] for 2-D systems without delays.

Remark 3: Clearly, Theorem 1 is a delay-independent stability condition, which in general is more conservative than delay-dependent results. For stability judgment and state feedback stabilization problems of 2-D state-delayed systems, some delay-dependent results have been derived [3], and the corresponding filter synthesis problems are currently under investigations.

B. Robust $\mathcal{H}_\infty$ Performance

Definition 2: The $\mathcal{H}_\infty$-norm of the 2-D state-delayed system (8) is defined as

$$\sup_{n\ge 1,\ w\in l_2,\ \hat{X}_o=0} \frac{\left(\sum_{j=0}^{n}\|\hat{Z}_j\|^2\right)^{\frac{1}{2}}}{\left(\sum_{j=0}^{n}\|W_j\|^2\right)^{\frac{1}{2}}}. \tag{19}$$

By the above definition, the $\mathcal{H}_\infty$-norm of the 2-D delay system (8) is less than or equal to γ if and only if

$$\sum_{j=0}^{n}\|\hat{Z}_j\|^2 \le \gamma^2 \sum_{j=0}^{n}\|W_j\|^2 \tag{20}$$

for all $n \geq 1$, $w \in l_2$, α satisfying (4), and $\hat{x}(0,0)=0$. In the following theorem, a sufficient $\mathcal{H}_\infty$ performance condition for the filtering error dynamics (8) is derived.

Theorem 2: Given a scalar $\gamma > 0$, the $\mathcal{H}_\infty$ performance constraint (20) holds for (8) if there exist matrices $\mathbf{P}(\alpha) = \mathrm{diag}[\mathbf{P}_1(\alpha), \mathbf{P}_2(\alpha)]$, $\mathbf{Q}(\alpha) = \mathrm{diag}[\mathbf{Q}_1(\alpha), \mathbf{Q}_2(\alpha)]$, $\mathbf{E}_1(\alpha) = \mathrm{diag}[\mathbf{E}_{11}(\alpha), \mathbf{E}_{12}(\alpha)]$, and $\mathbf{F}_1(\alpha) = \mathrm{diag}[\mathbf{F}_{11}(\alpha), \mathbf{F}_{12}(\alpha)]$ such that

$$\begin{bmatrix} \mathbf{E}_1(\alpha) & * \\ I_2 \otimes \Delta(\alpha) & -\mathbf{F}_1^{-1}(\alpha) \end{bmatrix} > 0 \tag{21}$$

and (22), shown at the bottom of the page, for all α in the unit simplex (4), where $\mathbf{M}(\alpha) = \mathbf{P}_1(\alpha)+\mathbf{P}_2(\alpha)+\mathbf{Q}_1(\alpha)+\mathbf{Q}_2(\alpha)$.

Proof: See Appendix I. ■

Remark 4: Theorem 2 provides a new robust $\mathcal{H}_\infty$ performance criterion for 2-D state-delayed systems with NFT uncertainties. In the simpler case where there are no uncertainties in the system, Theorem 2 reduces to Theorem 5 of [17]. In another simpler case [21] where the 2-D system has only polytopic uncertainties and no state delays, i.e., $\Delta(\alpha) = 0$ and $A_{d1}(\alpha) = A_{d2}(\alpha) = 0$, Theorem 2 with $\mathbf{P}_1 = \mathbf{P}_{11}$ and $\mathbf{P}_2 = \mathbf{P}_{22}$ reduces to Theorem 1 of [21] with $\mathbf{P}_{12} = 0$.

C. Robust Generalized $\mathcal{H}_2$ Performance

Definition 3: The generalized $\mathcal{H}_2$-norm of the 2-D state-delayed system (8) is defined as

$$\sup_{n\geq 1,\, w\in l_2,\, \hat{X}_o=0} \frac{\|\hat{Z}_n\|}{\left(\sum_{j=0}^{n} \|W_j\|^2\right)^{\frac{1}{2}}}. \tag{23}$$

By the above definition, the generalized $\mathcal{H}_2$-norm of the 2-D delay system (8) is less than or equal to $\sqrt{\mu}$ if and only if

$$\|\hat{Z}_n\|^2 \leq \mu \sum_{j=0}^{n} \|W_j\|^2 \tag{24}$$

for all $n \geq 1$, $w \in l_2$, α satisfying (4), and $\hat{x}(0,0)=0$. In the following theorem, a sufficient $\mathcal{H}_2$ performance condition for the filtering error dynamics (8) is derived.

Theorem 3: Given a scalar $\mu > 0$, the generalized $\mathcal{H}_2$ performance constraint (24) holds for (8) if there exist a scalar ε and matrices $\mathbf{X}(\alpha) = \mathrm{diag}[\mathbf{X}_1(\alpha), \mathbf{X}_2(\alpha)]$, $\mathbf{Y}(\alpha) = \mathrm{diag}[\mathbf{Y}_1(\alpha), \mathbf{Y}_2(\alpha)]$, $\mathbf{E}_2(\alpha)$, $\mathbf{F}_2(\alpha)$, $\mathbf{E}_3(\alpha) = \mathrm{diag}[\mathbf{E}_{31}(\alpha), \mathbf{E}_{32}(\alpha)]$, and $\mathbf{F}_3(\alpha) = \mathrm{diag}[\mathbf{F}_{31}(\alpha), \mathbf{F}_{32}(\alpha)]$ such that

$$\begin{bmatrix} \mathbf{E}_2(\alpha) & * \\ \Delta(\alpha) & -\mathbf{F}_2^{-1}(\alpha) \end{bmatrix} > 0, \tag{25}$$

$$\begin{bmatrix} \mathbf{E}_3(\alpha) & * \\ I_2 \otimes \Delta(\alpha) & -\mathbf{F}_3^{-1}(\alpha) \end{bmatrix} > 0, \tag{26}$$

$$\begin{bmatrix} -\mathbf{N}(\alpha) & * & * & * & * \\ 0 & \mathbf{F}_2(\alpha) & * & * & * \\ 0 & 0 & -\varepsilon I & * & * \\ \hat{C}_\Delta(\alpha) & D_{\Delta z}(\alpha) & D_z(\alpha) & -\mathbf{E}_2^{-1}(\alpha) & * \\ \hat{C}(\alpha) & 0 & 0 & 0 & -\mu I \end{bmatrix} < 0 \tag{27}$$

and (28), shown at the bottom of the page, for all α in the unit simplex (4), where $\mathbf{N}(\alpha) = \mathbf{X}_1(\alpha)+\mathbf{X}_2(\alpha)+\mathbf{Y}_1(\alpha)+\mathbf{Y}_2(\alpha)$.

Proof: See Appendix II. ■

Remark 5: Theorem 3 provides a new robust generalized $\mathcal{H}_2$ performance criteria for 2-D state-delayed systems with NFT uncertainties. In the simpler case in which $\Delta(\alpha) = 0$ and $A_{d1}(\alpha) = A_{d1}(\alpha) = 0$, Theorem 3 with $\mathbf{P}_1 = \mathbf{P}_{11}$ and $\mathbf{P}_2 = \mathbf{P}_{22}$ reduces to Theorem 2 of [21] with $\mathbf{P}_{12} = 0$.

$$\begin{bmatrix} -\mathbf{P}(\alpha) & * & * & * & * & * & * \\ 0 & -\mathbf{Q}(\alpha) & * & * & * & * & * \\ 0 & 0 & \mathbf{F}_1(\alpha) & * & * & * & * \\ 0 & 0 & 0 & -\gamma I & * & * & * \\ \hat{A}(\alpha) & \hat{A}_d(\alpha) & \hat{B}_\Delta(\alpha) & \hat{B}(\alpha) & -\mathbf{M}^{-1}(\alpha) & * & * \\ I_2 \otimes \hat{C}(\alpha) & 0 & 0 & 0 & 0 & -\gamma I & * \\ I_2 \otimes \hat{C}_\Delta(\alpha) & 0 & I_2 \otimes D_{\Delta z}(\alpha) & I_2 \otimes D_z(\alpha) & 0 & 0 & -\mathbf{E}_1^{-1}(\alpha) \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} -\mathbf{X}(\alpha) & * & * & * & * & * \\ 0 & -\mathbf{Y}(\alpha) & * & * & * & * \\ 0 & 0 & \mathbf{F}_3(\alpha) & * & * & * \\ 0 & 0 & 0 & -\frac{(1-\varepsilon)}{2} I & * & * \\ \hat{A}(\alpha) & \hat{A}_d(\alpha) & \hat{B}_\Delta(\alpha) & \hat{B}(\alpha) & -\mathbf{N}^{-1}(\alpha) & * \\ I_2 \otimes \hat{C}_\Delta(\alpha) & 0 & I_2 \otimes D_{\Delta z}(\alpha) & I_2 \otimes D_z(\alpha) & 0 & -\mathbf{E}_3^{-1}(\alpha) \end{bmatrix} < 0 \tag{28}$$

IV. Synthesis of Robust Filters

In this section, the LMI approach is adopted to develop convex optimization methods for synthesizing robust filters. First, some shorthand notations are brought in as follows:

$$\Pi = \begin{bmatrix} I \\ 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 0 \\ I \end{bmatrix}, \ \tilde{C}^{(j)} = \begin{bmatrix} C^{(j)} & 0 \\ 0 & I \end{bmatrix}$$
$$\tilde{D}^{(j)} = \begin{bmatrix} D^{(j)} \\ 0 \end{bmatrix}, \ \Phi = [I \quad I], \ \tilde{D}_\Delta^{(j)} = \begin{bmatrix} D_\Delta^{(j)} \\ 0 \end{bmatrix}. \quad (29)$$

With these shorthand notations, system matrices of the filtering error dynamics (8) can be rewritten as

$$\hat{A}(\alpha) = \sum_{j=1}^{m} \alpha_j \hat{A}^{(j)}, \quad \hat{A}_d(\alpha) = \sum_{j=1}^{m} \alpha_j \hat{A}_d^{(j)}$$
$$\hat{B}(\alpha) = \sum_{j=1}^{m} \alpha_j \hat{B}^{(j)}, \quad \hat{C}(\alpha) = \sum_{j=1}^{m} \alpha_j \hat{C}^{(j)}$$
$$\hat{B}_\Delta(\alpha) = \sum_{j=1}^{m} \alpha_j \hat{B}_\Delta^{(j)}, \quad \hat{C}_\Delta(\alpha) = \sum_{j=1}^{m} \alpha_j \hat{C}_\Delta^{(j)}$$
$$D_{\Delta z}(\alpha) = \sum_{j=1}^{m} \alpha_j D_{\Delta z}^{(j)}, \quad D_z(\alpha) = \sum_{j=1}^{m} \alpha_j D_z^{(j)}$$
$$\Delta(\alpha) = \sum_{j=1}^{m} \alpha_j \Delta^{(j)} \quad (30)$$

where

$$\hat{A}^{(j)} = \Pi A^{(j)} (I_2 \otimes \Pi^{\mathrm{T}}) + \Lambda \mathbf{G} \left(I_2 \otimes \tilde{C}^{(j)} \right)$$
$$\hat{A}_d^{(j)} = \Pi A_d^{(j)} (I_2 \otimes \Pi^{\mathrm{T}})$$
$$\hat{B}^{(j)} = \Pi B^{(j)} + \Lambda \mathbf{G} \left(I_2 \otimes \tilde{D}^{(j)} \right)$$
$$\hat{B}_\Delta^{(j)} = \Pi B_\Delta^{(j)} + \Lambda \mathbf{G} \left(I_2 \otimes \tilde{D}_\Delta^{(j)} \right)$$
$$\hat{C}^{(j)} = L^{(j)} \Pi^{\mathrm{T}} - \mathbf{C_f} \Lambda^{\mathrm{T}}$$
$$\hat{C}_\Delta^{(j)} = C_\Delta^{(j)} \Pi^{\mathrm{T}}$$
$$A^{(j)} = \begin{bmatrix} A_1^{(j)} & A_2^{(j)} \end{bmatrix}, \quad A_d^{(j)} = \begin{bmatrix} A_{d1}^{(j)} & A_{d2}^{(j)} \end{bmatrix}$$
$$B^{(j)} = \begin{bmatrix} B_1^{(j)} & B_2^{(j)} \end{bmatrix}, \quad B_\Delta^{(j)} = \begin{bmatrix} B_{\Delta 1}^{(j)} & B_{\Delta 2}^{(j)} \end{bmatrix}$$
$$\mathbf{G} = [\mathbf{B_{f1}} \quad \mathbf{A_{f1}} \quad \mathbf{B_{f2}} \quad \mathbf{A_{f2}}]. \quad (31)$$

It is noted that $\hat{A}^{(j)}$, $\hat{B}_\Delta^{(j)}$, $\hat{B}^{(j)}$, and $\hat{C}^{(j)}$ are affine functions of the filter system matrix variables $\mathbf{G}$ and $\mathbf{C}_f$, respectively. This fact is useful in our subsequent development.

A. Robust $\mathcal{H}_\infty$ Filter Synthesis

The robust $\mathcal{H}_\infty$ filtering problem addressed in this paper is as follows. Given a scalar $\gamma > 0$, find a filter (7) such that the filtering error dynamics (8) is asymptotically stable and the $\mathcal{H}_\infty$ performance constraint (20) is satisfied for all admissible uncertainties. In order to reduce the conservatism of the resultant filter synthesis method, a parameter-dependent Lyapunov function, which is quadratic in the uncertain parameters [10], will be utilized implicitly. At this point, it is noted that (21) and (22) are not LMIs with respect to the variable matrices but can be converted into ones in the following Lemma.

Lemma 1: If there exist matrices $\mathbf{P}^{(j)} = \mathrm{diag}(\mathbf{P}_1^{(j)}, \mathbf{P}_2^{(j)})$, $\mathbf{Q}^{(j)} = \mathrm{diag}(\mathbf{Q}_1^{(j)}, \mathbf{Q}_2^{(j)})$, $\mathbf{E}_1^{(j)} = \mathrm{diag}(\mathbf{E}_{11}^{(j)}, \mathbf{E}_{12}^{(j)})$, $\mathbf{F}_1^{(j)} = \mathrm{diag}(\mathbf{F}_{11}^{(j)}, \mathbf{F}_{12}^{(j)})$, $\mathbf{V}_1 = \mathrm{diag}(\mathbf{V}_{11}, \mathbf{V}_{12})$, $\mathbf{K}_1 = \mathrm{diag}(\mathbf{K}_{11}, \mathbf{K}_{12})$, and $\mathbf{H}$ such that

$$\begin{bmatrix} \mathbf{E}_1^{(j)} & * \\ \mathbf{V}_1 \left(I_2 \otimes \Delta^{(j)} \right) & \mathbf{F}_1^{(j)} + \left(\mathbf{V}_1 + \mathbf{V}_1^{\mathrm{T}} \right) \end{bmatrix} > 0 \quad (32)$$

and (33), shown at the bottom of the page, for $j = 1, \ldots, m$, where $\mathbf{M}^{(j)} = \mathbf{P}_1^{(j)} + \mathbf{P}_2^{(j)} + \mathbf{Q}_1^{(j)} + \mathbf{Q}_2^{(j)}$, then $\mathbf{P}(\alpha) = \sum_{j=1}^{m} \alpha_j \mathbf{P}^{(j)}$, $\mathbf{Q}(\alpha) = \sum_{j=1}^{m} \alpha_j \mathbf{Q}^{(j)}$, $\mathbf{E}_1(\alpha) = \sum_{j=1}^{m} \alpha_j \mathbf{E}_1^{(j)}$, and $\mathbf{F}_1(\alpha) = \sum_{j=1}^{m} \alpha_j \mathbf{F}_1^{(j)}$ satisfy (21) and (22) of Theorem 2 for all α in the unit simplex (4).

Proof: Note first that $\mathbf{F}_1^{(j)} + (\mathbf{V}_1 + \mathbf{V}_1^{\mathrm{T}}) > 0$ in (32) and $\mathbf{F}_1^{(j)} < 0$ in (33) together ensure that $\mathbf{V}_1$ is nonsingular. With $\mathbf{E}_1(\alpha) = \sum_{j=1}^{m} \alpha_j \mathbf{E}_1^{(j)}$ and $\mathbf{F}_1(\alpha) = \sum_{j=1}^{m} \alpha_j \mathbf{F}_1^{(j)}$, (32) implies

$$\begin{bmatrix} \mathbf{E}_1(\alpha) & * \\ \mathbf{V}_1 [I_2 \otimes \Delta(\alpha)] & \mathbf{F}_1(\alpha) + \left(\mathbf{V}_1 + \mathbf{V}_1^{\mathrm{T}} \right) \end{bmatrix} > 0 \quad (34)$$

for all α in the unit simplex (4). Performing the congruence transformation $\mathrm{diag}(I, \mathbf{V}_1^{-\mathrm{T}})$ to (34) yields

$$\begin{bmatrix} \mathbf{E}_1(\alpha) & * \\ I_2 \otimes \Delta(\alpha) & \mathbf{V}_1^{-1} \mathbf{F}_1(\alpha) \mathbf{V}_1^{-\mathrm{T}} + \mathbf{V}_1^{-1} + \mathbf{V}_1^{-\mathrm{T}} \end{bmatrix} > 0. \quad (35)$$

Since $[\mathbf{F}_1^{-1}(\alpha) + \mathbf{V}_1^{-1}] \mathbf{F}_1(\alpha) [\mathbf{F}_1^{-1}(\alpha) + \mathbf{V}_1^{-1}]^{\mathrm{T}} \le 0$, one has

$$-\mathbf{F}_1^{-1}(\alpha) \ge \mathbf{V}_1^{-1} \mathbf{F}_1(\alpha) \mathbf{V}_1^{-\mathrm{T}} + \mathbf{V}_1^{-1} + \mathbf{V}_1^{-\mathrm{T}}. \quad (36)$$

Thus, (21) holds for all α in the unit simplex (4), provided the conditions of this Lemma are satisfied. The proof of (22) is similar, with the relevant congruence transformation $\mathrm{diag}(I, I, I, I, \mathbf{H}^{-\mathrm{T}}, I, \mathbf{K}_1^{-\mathrm{T}})$ and inequalities $[\mathbf{M}^{-1}(\alpha) - \mathbf{H}^{-1}] \mathbf{M}(\alpha) [\mathbf{M}^{-1}(\alpha) - \mathbf{H}^{-1}]^{\mathrm{T}} \ge 0$ and $[\mathbf{E}_1^{-1}(\alpha) - \mathbf{K}_1^{-1}] \mathbf{E}_1(\alpha) [\mathbf{E}_1^{-1}(\alpha) - \mathbf{K}_1^{-1}]^{\mathrm{T}} \ge 0$. The detail steps are omitted for the sake of brevity. ■

$$\begin{bmatrix} -\mathbf{P}^{(j)} & * & * & * & * & * & * \\ 0 & -\mathbf{Q}^{(j)} & * & * & * & * & * \\ 0 & 0 & \mathbf{F}_1^{(j)} & * & * & * & * \\ 0 & 0 & 0 & -\gamma I & * & * & * \\ \mathbf{H}\hat{A}^{(j)} & \mathbf{H}\hat{A}_d^{(j)} & \mathbf{H}\hat{B}_\Delta^{(j)} & \mathbf{H}\hat{B}^{(j)} & \mathbf{M}^{(j)} - (\mathbf{H} + \mathbf{H}^{\mathrm{T}}) & * & * \\ I_2 \otimes \hat{C}^{(j)} & 0 & 0 & 0 & 0 & -\gamma I & * \\ \mathbf{K}_1 \left(I_2 \otimes \hat{C}_\Delta^{(j)} \right) & 0 & \mathbf{K}_1 \left(I_2 \otimes D_{\Delta z}^{(j)} \right) & \mathbf{K}_1 \left(I_2 \otimes D_z^{(j)} \right) & 0 & 0 & \mathbf{E}_1^{(j)} - \left(\mathbf{K}_1 + \mathbf{K}_1^{\mathrm{T}} \right) \end{bmatrix} < 0 \quad (33)$$

Although (32) and (33) in the above lemma are LMIs with respect to the variable matrices, further transformations are necessary to get LMIs from which filter system matrices can be conveniently obtained. This is accomplished in the following Theorem.

Theorem 4: For the system (1), if there exist matrices

$$\begin{aligned}\hat{\mathbf{P}}^{(j)} &= \mathrm{diag}(\hat{\mathbf{P}}_1^{(j)}, \hat{\mathbf{P}}_2^{(j)}), \hat{\mathbf{Q}}^{(j)} = \mathrm{diag}(\hat{\mathbf{Q}}_1^{(j)}, \hat{\mathbf{Q}}_2^{(j)}),\\ \mathbf{E}_1^{(j)} &= \mathrm{diag}(\mathbf{E}_{11}^{(j)}, \mathbf{E}_{12}^{(j)}), \mathbf{F}_1^{(j)} = \mathrm{diag}(\mathbf{F}_{11}^{(j)}, \mathbf{F}_{12}^{(j)}),\\ \mathbf{V}_1 &= \mathrm{diag}(\mathbf{V}_{11}, \mathbf{V}_{12}), \mathbf{K}_1 = \mathrm{diag}(\mathbf{K}_{11}, \mathbf{K}_{12}), \hat{\mathbf{G}}, \hat{\mathbf{C}}_\mathbf{f}\end{aligned}$$

and $\hat{\mathbf{H}} = \begin{bmatrix} \hat{\mathbf{H}}_1 & \hat{\mathbf{H}}_3 \\ \hat{\mathbf{H}}_2 & \hat{\mathbf{H}}_3 \end{bmatrix}$, such that (32) and the LMI (37), shown at bottom of the page, hold for $j = 1, \ldots, m$, where

$$\begin{aligned}\Psi_{51} &= \hat{\mathbf{H}}\Pi A^{(j)}(I_2 \otimes \Pi^\mathrm{T}) + \Phi^\mathrm{T}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{C}^{(j)}\right)\\ \Psi_{52} &= \hat{\mathbf{H}}\Pi A_d^{(j)}(I_2 \otimes \Pi^\mathrm{T})\\ \Psi_{53} &= \hat{\mathbf{H}}\Pi B_\Delta^{(j)} + \Phi^\mathrm{T}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{D}_\Delta^{(j)}\right)\\ \Psi_{54} &= \hat{\mathbf{H}}\Pi B^{(j)} + \Phi^\mathrm{T}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{D}^{(j)}\right)\\ \Psi_{55} &= \hat{\mathbf{P}}_1^{(j)} + \hat{\mathbf{P}}_2^{(j)} + \hat{\mathbf{Q}}_1^{(j)} + \hat{\mathbf{Q}}_2^{(j)} - (\hat{\mathbf{H}} + \hat{\mathbf{H}}^\mathrm{T})\\ \Psi_{61} &= I_2 \otimes \left(L^{(j)}\Pi^\mathrm{T}\right) - I_2 \otimes (\hat{\mathbf{C}}_\mathbf{f}\Lambda^\mathrm{T})\\ \Psi_{71} &= \mathbf{K}_1\left[I_2 \otimes \left(C_\Delta^{(j)}\Pi^\mathrm{T}\right)\right]\\ \Psi_{73} &= \mathbf{K}_1\left(I_2 \otimes D_{\Delta z}^{(j)}\right)\\ \Psi_{74} &= \mathbf{K}_1\left(I_2 \otimes D_z^{(j)}\right)\end{aligned} \tag{38}$$

then the robust $\mathcal{H}_\infty$ filtering problem stated at the beginning of this subsection is solvable, and the filter system matrices in (7) can be obtained from any feasible $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}} = [\hat{\mathbf{B}}_{\mathbf{f}1}\ \hat{\mathbf{A}}_{\mathbf{f}1}\ \hat{\mathbf{B}}_{\mathbf{f}2}\ \hat{\mathbf{A}}_{\mathbf{f}2}]$ as $\mathbf{G} = [\hat{\mathbf{B}}_{\mathbf{f}1}\ \hat{\mathbf{A}}_{\mathbf{f}1}\hat{\mathbf{H}}_3^{-1}\ \hat{\mathbf{B}}_{\mathbf{f}2}\ \hat{\mathbf{A}}_{\mathbf{f}2}\hat{\mathbf{H}}_3^{-1}]$ and $\mathbf{C}_\mathbf{f} = \hat{\mathbf{C}}_\mathbf{f}\hat{\mathbf{H}}_3^{-1}$.

Proof: From the negative definiteness of $\Psi_{55}, -\hat{\mathbf{P}}^{(j)}$, and $-\hat{\mathbf{Q}}^{(j)}$ in (37), it is seen that $\hat{\mathbf{H}}_3$ is nonsingular. Applying the congruence transformation $\mathrm{diag}(\Omega_1^{-\mathrm{T}}, \Omega_1^{-\mathrm{T}}, I, I, \Omega^{-\mathrm{T}}, I, I)$ to (37), where $\Omega_1 = \mathrm{diag}(\Omega, \Omega)$ and $\Omega = \mathrm{diag}(I, \hat{\mathbf{H}}_3^\mathrm{T})$, one can obtain (33) with

$$\begin{aligned}\mathbf{P}^{(j)} &= \Omega_1^{-1}\hat{\mathbf{P}}^{(j)}\Omega_1^{-\mathrm{T}},\\ \mathbf{Q}^{(j)} &= \Omega_1^{-1}\hat{\mathbf{Q}}^{(j)}\Omega_1^{-\mathrm{T}},\\ \mathbf{H} &= \begin{bmatrix} \hat{\mathbf{H}}_1 & I \\ \hat{\mathbf{H}}_3^{-\mathrm{T}}\hat{\mathbf{H}}_2 & \hat{\mathbf{H}}_3^{-\mathrm{T}} \end{bmatrix}\end{aligned} \tag{39}$$

by using the identities

$$\begin{aligned}\Pi^\mathrm{T}\Omega^\mathrm{T} &= \Pi^\mathrm{T},\\ \Omega^\mathrm{T}\tilde{D}^{(j)} &= \tilde{D}^{(j)},\\ \Omega^\mathrm{T}\tilde{D}_\Delta^{(j)} &= \tilde{D}_\Delta^{(j)}\\ \Omega^\mathrm{T}\tilde{C}^{(j)} &= \tilde{C}^{(j)}\Omega^\mathrm{T}\\ \Omega\mathbf{H}\Lambda &= \Phi^\mathrm{T}\\ \Omega\mathbf{H}\hat{A}^{(j)}\Omega_1^\mathrm{T} &= \hat{\mathbf{H}}\Pi A^{(j)}(I_2 \otimes \Pi^\mathrm{T})\\ &\quad + \Phi^\mathrm{T}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{C}^{(j)}\right)\\ \Omega\mathbf{H}\hat{A}_d^{(j)}\Omega_1^\mathrm{T} &= \hat{\mathbf{H}}\Pi A_d^{(j)}(I_2 \otimes \Pi^\mathrm{T})\\ \Omega\mathbf{H}\hat{B}_\Delta^{(j)} &= \hat{\mathbf{H}}\Pi B_\Delta^{(j)} + \Phi^\mathrm{T}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{D}_\Delta^{(j)}\right)\\ \Omega\mathbf{H}\hat{B}^{(j)} &= \hat{\mathbf{H}}\Pi B^{(j)} + \Phi^\mathrm{T}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{D}^{(j)}\right)\\ \left(I_2 \otimes \hat{C}^{(j)}\right)\Omega_1^\mathrm{T} &= I_2 \otimes \left(L^{(j)}\Pi^\mathrm{T}\right) - I_2 \otimes (\hat{\mathbf{C}}_\mathbf{f}\Lambda^\mathrm{T})\\ \mathbf{K}_1\left(I_2 \otimes \hat{C}_\Delta^{(j)}\right)\Omega_1^\mathrm{T} &= \mathbf{K}_1\left[I_2 \otimes \left(C_\Delta^{(j)}\Pi^\mathrm{T}\right)\right].\end{aligned} \tag{40}$$

In addition, from (31) and (40), it is easy to check that

$$\mathbf{G} = \hat{\mathbf{G}}\Omega_1^{-\mathrm{T}} \quad \text{and} \quad \mathbf{C}_\mathbf{f} = \hat{\mathbf{C}}_\mathbf{f}\hat{\mathbf{H}}_3^{-1}. \tag{41}$$

Thus, the proof is complete. ■

Remark 6: In Theorem 4, γ is regarded as given. However, (37) is still an LMI when γ is also a variable. Thus, it is possible to formulate the following convex optimization problem to find a filter with the smallest $\mathcal{H}_\infty$ norm:

$$\min \gamma \quad \text{subject to (32) and (37) for } j = 1, \ldots, m \tag{42}$$

with respect to γ and the variables stated in Theorem 4.

B. Robust Generalized $\mathcal{H}_2$ Filter Synthesis

The robust generalized $\mathcal{H}_2$ filtering problem addressed in this paper is as follows. Given a scalar $\mu > 0$, find a filter (7) such that the filtering error dynamics (8) is asymptotically stable and the generalized $\mathcal{H}_2$ performance constraint (24) is satisfied for all admissible uncertainties. Because the ideas and procedures involved in proving the following Lemma 2 and Theorem 5 are similar to the above arguments for Lemma 1 and Theorem 4, respectively, Lemma 2 and Theorem 5 are stated without proof for the sake of brevity.

Lemma 2: If there exist a scalar ε and matrices $\mathbf{X}^{(j)} = \mathrm{diag}(\mathbf{X}_1^{(j)}, \mathbf{X}_2^{(j)})$, $\mathbf{Y}^{(j)} = \mathrm{diag}(\mathbf{Y}_1^{(j)}, \mathbf{Y}_2^{(j)})$, $\mathbf{E}_2^{(j)}$, $\mathbf{F}_2^{(j)}$, $\mathbf{V}_2$,

$$\begin{bmatrix} -\hat{\mathbf{P}}^{(j)} & * & * & * & * & * & * \\ 0 & -\hat{\mathbf{Q}}^{(j)} & * & * & * & * & * \\ 0 & 0 & \mathbf{F}_1^{(j)} & * & * & * & * \\ 0 & 0 & 0 & -\gamma I & * & * & * \\ \Psi_{51} & \Psi_{52} & \Psi_{53} & \Psi_{54} & \Psi_{55} & * & * \\ \Psi_{61} & 0 & 0 & 0 & 0 & -\gamma I & * \\ \Psi_{71} & 0 & \Psi_{73} & \Psi_{74} & 0 & 0 & \mathbf{E}_1^{(j)} - \left(\mathbf{K}_1 + \mathbf{K}_1^\mathrm{T}\right) \end{bmatrix} < 0 \tag{37}$$

$\mathbf{K}_2$, $\mathbf{E}_3^{(j)} = \mathrm{diag}(\mathbf{E}_{31}^{(j)}, \mathbf{E}_{32}^{(j)})$, $\mathbf{F}_3^{(j)} = \mathrm{diag}(\mathbf{F}_{31}^{(j)}, \mathbf{F}_{32}^{(j)})$, $\mathbf{V}_3 = \mathrm{diag}(\mathbf{V}_{31}, \mathbf{V}_{32})$, $\mathbf{K}_3 = \mathrm{diag}(\mathbf{K}_{31}, \mathbf{K}_{32})$, and $\mathbf{J}$ such that

$$\begin{bmatrix} \mathbf{E}_2^{(j)} & * \\ \mathbf{V}_2\Delta^{(j)} & \mathbf{F}_2^{(j)} + (\mathbf{V}_2 + \mathbf{V}_2^{\mathrm{T}}) \end{bmatrix} > 0 \quad (43)$$

$$\begin{bmatrix} \mathbf{E}_3^{(j)} & * \\ \mathbf{V}_3\left(I_2 \otimes \Delta^{(j)}\right) & \mathbf{F}_3^{(j)} + (\mathbf{V}_3 + \mathbf{V}_3^{\mathrm{T}}) \end{bmatrix} > 0 \quad (44)$$

and (45) and (46), shown at the bottom of the page, for $j = 1, \ldots, m$, where $\mathbf{N}^{(j)} = \mathbf{X}_1^{(j)} + \mathbf{X}_2^{(j)} + \mathbf{Y}_1^{(j)} + \mathbf{Y}_2^{(j)}$, then ε, $\mathbf{X}(\alpha) = \sum_{j=1}^m \alpha_j \mathbf{X}^{(j)}$, $\mathbf{Y}(\alpha) = \sum_{j=1}^m \alpha_j \mathbf{Y}^{(j)}$, $\mathbf{E}_2(\alpha) = \sum_{j=1}^m \alpha_j \mathbf{E}_2^{(j)}$, $\mathbf{F}_2(\alpha) = \sum_{j=1}^m \alpha_j \mathbf{F}_2^{(j)}$, $\mathbf{E}_3(\alpha) = \sum_{j=1}^m \alpha_j \mathbf{E}_3^{(j)}$, and $\mathbf{F}_3(\alpha) = \sum_{j=1}^m \alpha_j \mathbf{F}_3^{(j)}$ satisfy (25)–(28) of Theorem 3 for all α in the unit simplex (4).

Theorem 5: For the system (1), if there exist a scalar ε and matrices $\hat{\mathbf{X}}^{(j)} = \mathrm{diag}(\hat{\mathbf{X}}_1^{(j)}, \hat{\mathbf{X}}_2^{(j)})$, $\hat{\mathbf{Y}}^{(j)} = \mathrm{diag}(\hat{\mathbf{Y}}_1^{(j)}, \hat{\mathbf{Y}}_2^{(j)})$, $\mathbf{E}_2^{(j)}$, $\mathbf{F}_2^{(j)}$, $\mathbf{V}_2$, $\mathbf{K}_2$, $\mathbf{E}_3^{(j)} = \mathrm{diag}(\mathbf{E}_{31}^{(j)}, \mathbf{E}_{32}^{(j)})$, $\mathbf{F}_3^{(j)} = \mathrm{diag}(\mathbf{F}_{31}^{(j)}, \mathbf{F}_{32}^{(j)})$, $\mathbf{V}_3 = \mathrm{diag}(\mathbf{V}_{31}, \mathbf{V}_{32})$, $\mathbf{K}_3 = \mathrm{diag}(\mathbf{K}_{31}, \mathbf{K}_{32})$, $\hat{\mathbf{G}}$, $\hat{\mathbf{C}}_f$, and

$$\hat{\mathbf{J}} = \begin{bmatrix} \hat{\mathbf{J}}_1 & \hat{\mathbf{J}}_3 \\ \hat{\mathbf{J}}_2 & \hat{\mathbf{J}}_3 \end{bmatrix}$$

such that the LMIs (43) and (44) (see (47) and (48), shown at bottom of the page) hold for $j = 1, \ldots, m$, where

$$\begin{aligned}
\Theta_{51} &= \hat{\mathbf{J}}\Pi A^{(j)}(I_2 \otimes \Pi^{\mathrm{T}}) + \Phi^{\mathrm{T}}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{C}^{(j)}\right) \\
\Theta_{52} &= \hat{\mathbf{J}}\Pi A_d^{(j)}(I_2 \otimes \Pi^{\mathrm{T}}) \\
\Theta_{53} &= \hat{\mathbf{J}}\Pi B_\Delta^{(j)} + \Phi^{\mathrm{T}}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{D}_\Delta^{(j)}\right) \\
\Theta_{54} &= \hat{\mathbf{J}}\Pi B^{(j)} + \Phi^{\mathrm{T}}\hat{\mathbf{G}}\left(I_2 \otimes \tilde{D}^{(j)}\right) \\
\Theta_{55} &= \hat{\mathbf{X}}_1^{(j)} + \hat{\mathbf{X}}_2^{(j)} + \hat{\mathbf{Y}}_1^{(j)} + \hat{\mathbf{Y}}_2^{(j)} - (\hat{\mathbf{J}} + \hat{\mathbf{J}}^{\mathrm{T}}) \\
\Theta_{61} &= \mathbf{K}_3\left[I_2 \otimes \left(C_\Delta^{(j)}\Pi^{\mathrm{T}}\right)\right] \\
\Theta_{63} &= \mathbf{K}_3\left(I_2 \otimes D_{\Delta z}^{(j)}\right) \\
\Theta_{64} &= \mathbf{K}_3\left(I_2 \otimes D_z^{(j)}\right)
\end{aligned} \quad (49)$$

then the robust generalized $\mathcal{H}_2$ filtering problem stated at the beginning of this subsection is solvable, and the filter system matrices in (7) can be obtained from any feasible $\hat{\mathbf{J}}$ and $\hat{\mathbf{G}} = [\hat{\mathbf{B}}_{\mathbf{f}1}\ \hat{\mathbf{A}}_{\mathbf{f}1}\ \hat{\mathbf{B}}_{\mathbf{f}2}\ \hat{\mathbf{A}}_{\mathbf{f}2}]$ as $\mathbf{G} = [\hat{\mathbf{B}}_{\mathbf{f}1}\ \hat{\mathbf{A}}_{\mathbf{f}1}\hat{\mathbf{J}}_3^{-1}\ \hat{\mathbf{B}}_{\mathbf{f}2}\ \hat{\mathbf{A}}_{\mathbf{f}2}\hat{\mathbf{J}}_3^{-1}]$ and $\mathbf{C}_f = \hat{\mathbf{C}}_{\mathbf{f}}\hat{\mathbf{J}}_3^{-1}$.

Remark 7: Similar to that explained in Remark 6, it is possible to formulate the following convex optimization problem to find a filter with the smallest $\mathcal{H}_2$ norm, as follows:

$$\min \mu \quad \text{subject to (43), (44), (47), and (48) for } j = 1, \ldots, m \quad (50)$$

with respect to μ and the variables stated in Theorem 5.

$$\begin{bmatrix} -\mathbf{N}^{(j)} & * & * & * & * \\ 0 & \mathbf{F}_2^{(j)} & * & * & * \\ 0 & 0 & -\varepsilon I & * & * \\ \mathbf{K}_2\hat{C}_\Delta^{(j)} & \mathbf{K}_2 D_{\Delta z}^{(j)} & \mathbf{K}_2 D_z^{(j)} & \mathbf{E}_2^{(j)} - (\mathbf{K}_2 + \mathbf{K}_2^{\mathrm{T}}) & * \\ \hat{C}^{(j)} & 0 & 0 & 0 & -\mu I \end{bmatrix} < 0 \quad (45)$$

$$\begin{bmatrix} -\mathbf{X}^{(j)} & * & * & * & * & * \\ 0 & -\mathbf{Y}^{(j)} & * & * & * & * \\ 0 & 0 & \mathbf{F}_3^{(j)} & * & * & * \\ 0 & 0 & 0 & -\frac{(1-\varepsilon)}{2}I & * & * \\ \mathbf{J}\hat{A}^{(j)} & \mathbf{J}\hat{A}_d^{(j)} & \mathbf{J}\hat{B}_\Delta^{(j)} & \mathbf{J}\hat{B}^{(j)} & \mathbf{N}^{(j)} - (\mathbf{J} + \mathbf{J}^{\mathrm{T}}) & * \\ \mathbf{K}_3\left(I_2 \otimes \hat{C}_\Delta^{(j)}\right) & 0 & \mathbf{K}_3\left(I_2 \otimes D_{\Delta z}^{(j)}\right) & \mathbf{K}_3\left(I_2 \otimes D_z^{(j)}\right) & 0 & \mathbf{E}_3^{(j)} - (\mathbf{K}_3 + \mathbf{K}_3^{\mathrm{T}}) \end{bmatrix} < 0 \quad (46)$$

$$\begin{bmatrix} -\left(\hat{\mathbf{X}}_1^{(j)} + \hat{\mathbf{X}}_2^{(j)} + \hat{\mathbf{Y}}_1^{(j)} + \hat{\mathbf{Y}}_2^{(j)}\right) & * & * & * & * \\ 0 & \mathbf{F}_2^{(j)} & * & * & * \\ 0 & 0 & -\varepsilon I & * & * \\ \mathbf{K}_2 C_\Delta^{(j)}\Pi^{\mathrm{T}} & \mathbf{K}_2 D_{\Delta z}^{(j)} & \mathbf{K}_2 D_z^{(j)} & \mathbf{E}_2^{(j)} - (\mathbf{K}_2 + \mathbf{K}_2^{\mathrm{T}}) & * \\ L^{(j)}\Pi^{\mathrm{T}} - \hat{\mathbf{C}}_{\mathbf{f}}\Lambda^{\mathrm{T}} & 0 & 0 & 0 & -\mu I \end{bmatrix} < 0, \quad (47)$$

$$\begin{bmatrix} -\hat{\mathbf{X}}^{(j)} & * & * & * & * & * \\ 0 & -\hat{\mathbf{Y}}^{(j)} & * & * & * & * \\ 0 & 0 & \mathbf{F}_3^{(j)} & * & * & * \\ 0 & 0 & 0 & -\frac{(1-\varepsilon)}{2}I & * & * \\ \Theta_{51} & \Theta_{52} & \Theta_{53} & \Theta_{54} & \Theta_{55} & * \\ \Theta_{61} & 0 & \Theta_{63} & \Theta_{64} & 0 & \mathbf{E}_3^{(j)} - (\mathbf{K}_3 + \mathbf{K}_3^{\mathrm{T}}) \end{bmatrix} < 0 \quad (48)$$

C. Robust Mixed Generalized $\mathcal{H}_2/\mathcal{H}_\infty$ Filter Synthesis

By integrating the above results, a robust mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem can be addressed as follows. Find a filter (7) for (1) to

$$\min_{\gamma>0,\,\mu>0} \rho\gamma + (1-\rho)\mu$$
$$\text{subject to } \sum_{j=0}^{n} \|\hat{Z}_j\|^2 \le \gamma^2 \sum_{j=0}^{n} \|W_j\|^2$$

and

$$\|\hat{Z}_n\|^2 \le \mu \sum_{j=0}^{n} \|W_j\|^2 \tag{51}$$

for all $n \ge 1$, $w \in l_2$, and $\hat{x}(0,0) = 0$, where $\rho \in [0,1]$ is a preselected weighting constant for the tradeoff between the $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ performances. An upper bound for the optimal objective function value of this problem may be found by applying the following Theorem, which is a combination of Theorems 4 and 5.

Theorem 6: An upper bound for the objective function (51) in the robust mixed generalized $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem can be obtained by solving the following convex optimization problem:

$$\min \rho\gamma + (1-\rho)\mu$$
$$\text{subject to (32), (37), (43), (44), (47), and (48)} \tag{52}$$

with respect to ε,

$$\begin{aligned}
\hat{\mathbf{P}}^{(j)} &= \mathrm{diag}(\hat{\mathbf{P}}_1^{(j)}, \hat{\mathbf{P}}_2^{(j)}), \hat{\mathbf{Q}}^{(j)} = \mathrm{diag}(\hat{\mathbf{Q}}_1^{(j)}, \hat{\mathbf{Q}}_2^{(j)}),\\
\hat{\mathbf{X}}^{(j)} &= \mathrm{diag}(\hat{\mathbf{X}}_1^{(j)}, \hat{\mathbf{X}}_2^{(j)}), \hat{\mathbf{Y}}^{(j)} = \mathrm{diag}(\hat{\mathbf{Y}}_1^{(j)}, \hat{\mathbf{Y}}_2^{(j)}),\\
\mathbf{E}_1^{(j)} &= \mathrm{diag}(\mathbf{E}_{11}^{(j)}, \mathbf{E}_{12}^{(j)}), \mathbf{E}_2^{(j)}, \mathbf{E}_3^{(j)} = \mathrm{diag}(\mathbf{E}_{31}^{(j)}, \mathbf{E}_{32}^{(j)}),\\
\mathbf{F}_1^{(j)} &= \mathrm{diag}(\mathbf{F}_{11}^{(j)}, \mathbf{F}_{12}^{(j)}), \mathbf{F}_2^{(j)}, \mathbf{F}_3^{(j)} = \mathrm{diag}(\mathbf{F}_{31}^{(j)}, \mathbf{F}_{32}^{(j)}),\\
\mathbf{V}_1 &= \mathrm{diag}(\mathbf{V}_{11}, \mathbf{V}_{12}), \mathbf{V}_2, \mathbf{V}_3 = \mathrm{diag}(\mathbf{V}_{31}, \mathbf{V}_{32}),\\
\mathbf{K}_1 &= \mathrm{diag}(\mathbf{K}_{11}, \mathbf{K}_{12}), \mathbf{K}_2, \mathbf{K}_3 = \mathrm{diag}(\mathbf{K}_{31}, \mathbf{K}_{32}),
\end{aligned}$$

$\hat{\mathbf{G}}$, $\hat{\mathbf{C}}_f$, and $\hat{\mathbf{H}} = \hat{\mathbf{J}}$ for $j = 1, \ldots, m$. The filter system matrices in (7) can be obtained from the optimal $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}} = [\hat{\mathbf{B}}_{\mathbf{f}1}\ \hat{\mathbf{A}}_{\mathbf{f}1}\ \hat{\mathbf{B}}_{\mathbf{f}2}\ \hat{\mathbf{A}}_{\mathbf{f}2}]$ as $\mathbf{G} = [\hat{\mathbf{B}}_{\mathbf{f}1}\ \hat{\mathbf{A}}_{\mathbf{f}1}\hat{\mathbf{H}}_3^{-1}\ \hat{\mathbf{B}}_{\mathbf{f}2}\ \hat{\mathbf{A}}_{\mathbf{f}2}\hat{\mathbf{H}}_3^{-1}]$ and $\mathbf{C}_f = \hat{\mathbf{C}}_{\mathbf{f}}\hat{\mathbf{H}}_3^{-1}$.

V. Two Examples

Example 1: Consider the uncertain 2-D state-delayed system

$$\begin{aligned}
x(i+1,j+1) &= \mathcal{A}_1(\alpha)x(i+1,j) + \mathcal{A}_2(\alpha)x(i,j+1)\\
&\quad + A_{d1}x(i+1,j-d_1)\\
&\quad + A_{d2}x(i-d_2,j+1)\\
&\quad + B_1w(i+1,j) + B_2w(i,j+1)\\
y(i,j) &= Cx(i,j) + Dw(i.j)\\
z(i,j) &= Lx(i,j)
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
\mathcal{A}_1(\alpha) &= \Gamma_0 + \alpha_1\Gamma_1 + \alpha_2\Gamma_2 + \alpha_3\Gamma_3\\
&\quad + \alpha_1^2\Gamma_4 + \alpha_2^2\Gamma_5 + \alpha_3^2\Gamma_6
\end{aligned} \tag{54}$$

$$\begin{aligned}
\mathcal{A}_2(\alpha) &= \Upsilon_0 + \alpha_1\Upsilon_1 + \alpha_2\Upsilon_2 + \alpha_3\Upsilon_3\\
&\quad + \alpha_1^2\Upsilon_4 + \alpha_2^2\Upsilon_5 + \alpha_3^2\Upsilon_6
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
&\Gamma_0 = \begin{bmatrix} 0 & 0.08 \\ -0.1 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.02 & 0.12 \\ 0.1 & 0.16 \end{bmatrix}\\
&\Gamma_2 = \begin{bmatrix} -0.02 & 0.12 \\ 0.1 & 0.16 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & 0.12 \\ 0.1 & 0.16 \end{bmatrix}\\
&\Gamma_4 = \begin{bmatrix} 0.02 & 0.06 \\ 0.01 & 0 \end{bmatrix}, \quad \Gamma_5 = \begin{bmatrix} 0.05 & 0 \\ 0.01 & 0.08 \end{bmatrix}\\
&\Gamma_6 = \begin{bmatrix} 0.02 & 0.05 \\ 0.01 & 0.06 \end{bmatrix}, \quad \Upsilon_0 = \begin{bmatrix} 0.10 & 0 \\ 0.02 & 0.06 \end{bmatrix}\\
&\Upsilon_1 = \begin{bmatrix} 0.12 & 0.1 \\ 0.1 & 0.15 \end{bmatrix}, \quad \Upsilon_2 = \begin{bmatrix} -0.12 & 0.1 \\ 0.12 & 0.15 \end{bmatrix}\\
&\Upsilon_3 = \begin{bmatrix} 0 & 0.1 \\ 0.12 & 0.15 \end{bmatrix}, \quad \Upsilon_4 = \begin{bmatrix} 0.07 & 0 \\ 0.08 & 0.06 \end{bmatrix}\\
&\Upsilon_5 = \begin{bmatrix} 0.09 & 0.05 \\ 0.03 & 0 \end{bmatrix}, \quad \Upsilon_6 = \begin{bmatrix} 0 & 0.06 \\ 0.03 & 0.05 \end{bmatrix}\\
&A_{d1} = \begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 0.06 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.031 & 0.02 \\ 0.02 & 0.01 \end{bmatrix}\\
&B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\\
&C = [0.5 \quad 0], \quad D = 0, \quad L = [1 \quad 1].
\end{aligned} \tag{56}$$

It is noted that the system can be represented by the NFT and LFT models as follows:

- NFT model:

$$\begin{aligned}
\mathcal{A}_1(\alpha) &= \Gamma_0 + \alpha_1\Gamma_1 + \alpha_2\Gamma_2 + \alpha_3\Gamma_3\\
&\quad + [\alpha_1 I_2 \quad \alpha_2 I_2 \quad \alpha_3 I_2] \cdot \begin{bmatrix} \Gamma_4 & 0 & 0 \\ 0 & \Gamma_5 & 0 \\ 0 & 0 & \Gamma_6 \end{bmatrix}\\
&\quad \times \begin{bmatrix} \alpha_1 I_2 & 0 & 0 \\ 0 & \alpha_2 I_2 & 0 \\ 0 & 0 & \alpha_3 I_2 \end{bmatrix} \begin{bmatrix} I_2 \\ I_2 \\ I_2 \end{bmatrix}
\end{aligned} \tag{57}$$

$$\begin{aligned}
\mathcal{A}_2(\alpha) &= \Upsilon_0 + \alpha_1\Upsilon_1 + \alpha_2\Upsilon_2 + \alpha_3\Upsilon_3\\
&\quad + [\alpha_1 I_2 \quad \alpha_2 I_2 \quad \alpha_3 I_2] \begin{bmatrix} \Upsilon_4 & 0 & 0 \\ 0 & \Upsilon_5 & 0 \\ 0 & 0 & \Upsilon_6 \end{bmatrix}\\
&\quad \times \begin{bmatrix} \alpha_1 I_2 & 0 & 0 \\ 0 & \alpha_2 I_2 & 0 \\ 0 & 0 & \alpha_3 I_2 \end{bmatrix} \begin{bmatrix} I_2 \\ I_2 \\ I_2 \end{bmatrix}
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
A_1(\alpha) &= \alpha_1(\Gamma_0 + \Gamma_1) + \alpha_2(\Gamma_0 + \Gamma_2) + \alpha_3(\Gamma_0 + \Gamma_3)\\
A_2(\alpha) &= \alpha_1(\Upsilon_0 + \Upsilon_1) + \alpha_2(\Upsilon_0 + \Upsilon_2) + \alpha_3(\Upsilon_0 + \Upsilon_3)\\
B_{\Delta 1}(\alpha) &= [\alpha_1 I_2 \quad \alpha_2 I_2 \quad \alpha_3 I_2] \begin{bmatrix} \Gamma_4 & 0 & 0 \\ 0 & \Gamma_5 & 0 \\ 0 & 0 & \Gamma_6 \end{bmatrix}\\
B_{\Delta 2}(\alpha) &= [\alpha_1 I_2 \quad \alpha_2 I_2 \quad \alpha_3 I_2] \begin{bmatrix} \Upsilon_4 & 0 & 0 \\ 0 & \Upsilon_5 & 0 \\ 0 & 0 & \Upsilon_6 \end{bmatrix}\\
\Delta(\alpha) &= \begin{bmatrix} \alpha_1 I_2 & 0 & 0 \\ 0 & \alpha_2 I_2 & 0 \\ 0 & 0 & \alpha_3 I_2 \end{bmatrix}\\
C_\Delta(\alpha) &= \begin{bmatrix} I_2 \\ I_2 \\ I_2 \end{bmatrix}\\
D_{\Delta z}(\alpha) &= 0, \quad D_z(\alpha) = 0, \quad D_\Delta(\alpha) = 0, \quad n_\Delta = 6.
\end{aligned} \tag{59}$$

- LFT model:

$$\mathcal{A}_1(\alpha)=\Gamma_0+[\Gamma_4\,\Gamma_1-\Gamma_4\,\Gamma_5\,\Gamma_2-\Gamma_5\,\Gamma_6\,\Gamma_3-\Gamma_6]\Delta(\alpha) \cdot \left[I_{12}-\begin{bmatrix}0&I_2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&I_2&0&0\\0&0&0&0&0&0\\0&0&0&0&0&I_2\\0&0&0&0&0&0\end{bmatrix}\Delta(\alpha)\right]^{-1}\begin{bmatrix}I_2\\I_2\\I_2\\I_2\\I_2\\I_2\end{bmatrix} \tag{60}$$

$$\mathcal{A}_2(\alpha)=\Upsilon_0+[\Upsilon_4\,\Upsilon_1-\Upsilon_4\,\Upsilon_5\,\Upsilon_2-\Upsilon_5\,\Upsilon_6\,\Upsilon_3-\Upsilon_6]\Delta(\alpha) \cdot \left[I_{12}-\begin{bmatrix}0&I_2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&I_2&0&0\\0&0&0&0&0&0\\0&0&0&0&0&I_2\\0&0&0&0&0&0\end{bmatrix}\Delta(\alpha)\right]^{-1}\begin{bmatrix}I_2\\I_2\\I_2\\I_2\\I_2\\I_2\end{bmatrix} \tag{61}$$

and

$$\begin{aligned} A_1 &=\Gamma_0, \quad A_2=\Upsilon_0\\ B_{\Delta 1} &=[\Gamma_4\,\Gamma_1-\Gamma_4\,\Gamma_5\,\Gamma_2-\Gamma_5\,\Gamma_6\,\Gamma_3-\Gamma_6]\\ B_{\Delta 2} &=[\Upsilon_4\,\Upsilon_1-\Upsilon_4\,\Upsilon_5\,\Upsilon_2-\Upsilon_5\,\Upsilon_6\,\Upsilon_3-\Upsilon_6]\\ C_\Delta &=\begin{bmatrix}I_2\\I_2\\I_2\\I_2\\I_2\\I_2\end{bmatrix}, \quad D_{\Delta z}=\begin{bmatrix}0&I_2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&I_2&0&0\\0&0&0&0&0&0\\0&0&0&0&0&I_2\\0&0&0&0&0&0\end{bmatrix}\\ D_z &=0, \quad D_\Delta=0, \quad n_\Delta=12\\ \Delta(\alpha) &=\begin{bmatrix}\alpha_1 I_4&0&0\\0&\alpha_2 I_4&0\\0&0&\alpha_3 I_4\end{bmatrix}. \end{aligned} \tag{62}$$

By using the MATLAB LMI Control Toolbox [7], the optimization problems (42), (50), and (52) are solved for the system (53)–(56) in both NFT and LFT models. The results are shown in Table I, where it is seen that better performances are obtained for the NFT model than the LFT model. In fact for the optimization problem (52) with the LFT model, no feasible results exist. It is also worthy noting that for every problem, the computation time for the NFT model is much shorter than that for the LFT model. For instance, the problem (42) can be computed in less than 12 min for the NFT model using a Pentium IV PC, and in about one and half hours for the LFT model. In Table II, the tradeoff between the $\mathcal{H}_\infty$ and generalized $\mathcal{H}_2$ performances is displayed with three different weighting values of ρ. Clearly, larger ρ results in smaller optimal γ and larger optimal μ.

Note that when $\Delta(\alpha)=0$ and $A_{d1}(\alpha)=A_{d2}(\alpha)=0$, Theorems 2 and 3 with $\mathbf{P}_1=\mathbf{P}_{11}$ and $\mathbf{P}_2=\mathbf{P}_{22}$ reduce, respectively, to Theorems 1 and 2 of [21] with $\mathbf{P}_{12}=0$. For $A_{d1}(\alpha)=A_{d2}(\alpha)=0$ and $\Delta(\alpha)=0$ in the problem (42) of this example, it is found that the minimum γ's from the above method and the method given by [21] are 2.588 and 2.586, respectively. Similarly, the minimum μ's for the problem (50) are 5.090 and 5.088, respectively. It is seen that due to the absence of the variable $\mathbf{P}_{12}$, the proposed method gives a slightly larger γ and μ.

TABLE I
OPTIMAL OBJECTIVE FUNCTION VALUES FOR DIFFERENT SYSTEM MODELS

model\ optimization problem	(42)	(50)	(52) with $\rho=0.5$
NFT	3.9154	7.1333	10.6095
LFT	6.8388	11.0732	$+\infty$

TABLE II
OPTIMAL RESULTS OF (52) WITH DIFFERENT ρ FOR THE NFT MODEL

ρ	γ	μ	$\rho\gamma+(1-\rho)\mu$
0.1	16.7141	7.3085	8.2490
0.5	12.2100	9.0090	10.6095
0.9	10.2058	14.7696	10.6622

Example 2: Consider a heat diffusion system along a line described by the partial differential equation

$$\frac{\partial^2}{\partial\xi^2}u(\xi,t)=c(\alpha)\frac{\partial}{\partial t}u(\xi,t)+f(\xi,t)+w(\xi,t) \tag{63}$$

where $\xi\in[0,\bar{\xi}]$ is the spatial variable, $t\in[0,\infty)$ is the time variable, $u(\xi,t)$ is the temperature of the line at ξ and t, $c(\alpha)$ is the thermal diffusivity depending on an uncertain parameter vector $\alpha=[\alpha_1\;\alpha_2]^{\mathrm{T}}$, $f(\xi,t)$ is the control input, and $w(\xi,t)$ is the noise input. Suppose $c(\alpha)$ depends on α nonlinearly as

$$c(\alpha)=0.8-0.4\alpha_1+0.32\alpha_2+0.16\alpha_1^2-0.12\alpha_2^2 \tag{64}$$

and the system is controlled by a "mixed" state feedback law $f(\xi,t)=-200u(\xi,t)+102u(\xi-xi_0,t)$, where $\xi_0=0.1$. Using the central and back difference approximations

$$\frac{\partial^2}{\partial\xi^2}u(\xi,t)\simeq\frac{1}{(\Delta\xi)^2}\left[u(i\Delta\xi+\Delta\xi,j\Delta t)-2u(i\Delta\xi,j\Delta t)+u(i\Delta\xi-\Delta\xi,j\Delta t)\right] \tag{65}$$

$$\frac{\partial}{\partial t}u(\xi,t)\simeq\frac{1}{\Delta t}\left[u(i\Delta\xi,j\Delta t)-u(i\Delta\xi,j\Delta t-\Delta t)\right] \tag{66}$$

we obtain a discretized approximation of (63)

$$\begin{aligned} u(i+1,j)=&\left[\frac{(\Delta\xi)^2}{\Delta t}c(\alpha)-200(\Delta\xi)^2+2\right]u(i,j)\\ &-\frac{(\Delta\xi)^2}{\Delta t}c(\alpha)u(i,j-1)\\ &-\left[1-102(\Delta\xi)^2\right]u(i-1,j)+(\Delta\xi)^2w(i,j) \end{aligned} \tag{67}$$

where $u(i,j)=u(i\Delta\xi,j\Delta t)$ and $\Delta\xi$ is selected to be equal to ξ_0. For $\Delta t=0.1$, (67) can be converted into the Fornasini–Marchesini second model of the form (53) with

$$\begin{aligned} x(i,j)&=[u(i,j)\;0.5c(\alpha)u(i,j-1)+0.5u(i-1,j)]^{\mathrm{T}}\\ z(i,j)&=u(i,j)\\ y(i,j)&=c(\alpha)u(i,j-1)+u(i-1,j)\\ \mathcal{A}_1(\alpha)&=\begin{bmatrix}0&0\\0.5c(\alpha)&0\end{bmatrix}, \quad \mathcal{A}_2(\alpha)=\begin{bmatrix}0.1c(\alpha)&-0.2\\0.5&0\end{bmatrix}\\ A_{d1}&=0, \quad A_{d2}=\begin{bmatrix}0.12&0\\0&0\end{bmatrix}\\ B_1&=0, \quad B_2=\begin{bmatrix}0.01\\0\end{bmatrix}\\ C&=[0\quad 2], \quad D=0, \quad L=[1\quad 0] \quad \text{and} \quad d_2=1. \end{aligned}$$

The matrices $\mathcal{A}_1(\alpha)$ and $\mathcal{A}_2(\alpha)$ can be represented respectively by

$$\mathcal{A}_1(\alpha) = \Lambda_0 + \alpha_1\Lambda_1 + \alpha_2\Lambda_2 + \alpha_1^2\Lambda_3 + \alpha_2^2\Lambda_4 \quad (68)$$
$$\mathcal{A}_2(\alpha) = \Pi_0 + \alpha_1\Pi_1 + \alpha_2\Pi_2 + \alpha_1^2\Pi_3 + \alpha_2^2\Pi_4 \quad (69)$$

where

$$\Lambda_0 = \begin{bmatrix} 0 & 0 \\ 0.4 & 0 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0 & 0 \\ -0.2 & 0 \end{bmatrix}$$
$$\Lambda_2 = \begin{bmatrix} 0 & 0 \\ 0.16 & 0 \end{bmatrix}, \quad \Lambda_3 = \begin{bmatrix} 0 & 0 \\ 0.08 & 0 \end{bmatrix}$$
$$\Lambda_4 = \begin{bmatrix} 0 & 0 \\ -0.06 & 0 \end{bmatrix}, \quad \Pi_0 = \begin{bmatrix} 0.08 & -0.2 \\ 0.5 & 0 \end{bmatrix}$$
$$\Pi_1 = \begin{bmatrix} -0.04 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0.032 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Pi_3 = \begin{bmatrix} 0.016 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_4 = \begin{bmatrix} -0.012 & 0 \\ 0 & 0 \end{bmatrix}. \quad (70)$$

Note that (68) can be expressed by the NFT model

$$\begin{bmatrix} \mathcal{A}_1(\alpha) \\ \mathcal{A}_2(\alpha) \end{bmatrix} = \begin{bmatrix} A_1(\alpha) \\ A_2(\alpha) \end{bmatrix} + \begin{bmatrix} B_{\Delta 1}(\alpha) \\ B_{\Delta 2}(\alpha) \end{bmatrix} \Delta(\alpha) C_\Delta(\alpha) \quad (71)$$

where

$$A_1(\alpha) = \alpha_1(\Lambda_0 + \Lambda_1) + \alpha_2(\Lambda_0 + \Lambda_2)$$
$$A_2(\alpha) = \alpha_1(\Pi_0 + \Pi_1) + \alpha_2(\Pi_0 + \Pi_2)$$
$$B_{\Delta 1}(\alpha) = [\alpha_1 I_2 \quad \alpha_2 I_2] \begin{bmatrix} \Lambda_3 & 0 \\ 0 & \Lambda_4 \end{bmatrix}$$
$$B_{\Delta 2}(\alpha) = [\alpha_1 I_2 \quad \alpha_2 I_2] \begin{bmatrix} \Pi_3 & 0 \\ 0 & \Pi_4 \end{bmatrix}$$
$$\Delta(\alpha) = \begin{bmatrix} \alpha_1 I_2 & 0 \\ 0 & \alpha_2 I_2 \end{bmatrix}, \quad C_\Delta(\alpha) = \begin{bmatrix} I_2 \\ I_2 \end{bmatrix}$$
$$D_{\Delta z}(\alpha) = 0, \quad D_z(\alpha) = 0$$
$$D_\Delta(\alpha) = 0, \quad n_\Delta = 4. \quad (72)$$

For simplicity, we only consider the robust $\mathcal{H}_\infty$ filter design. By solving the optimization problem (42) in this paper, we can obtain the minimum noise attenuation level bound $\gamma^* = 0.0189$, and the corresponding filter matrices are

$$\mathbf{A_{f1}} = \begin{bmatrix} -0.0008 & 0.0379 \\ 0.0020 & -0.0744 \end{bmatrix}$$
$$\mathbf{A_{f2}} = \begin{bmatrix} 0.0265 & -0.5810 \\ 0.0103 & -0.2047 \end{bmatrix}$$
$$\mathbf{B_{f1}} = \begin{bmatrix} 0.0797 \\ -0.1642 \end{bmatrix}, \quad \mathbf{B_{f2}} = \begin{bmatrix} 12.0487 \\ -1.4029 \end{bmatrix}$$
$$\mathbf{C_f} = [-0.0091 \quad -0.0065].$$

Fig. 1 shows the magnitude plot of the filtering error dynamics over grid frequencies in the range of $[-\pi, \pi]$ for $\alpha_1 = 1$ and $\alpha_2 = 0$. It can be seen that the maximum magnitude is below the guaranteed noise attenuation level bound. This is also true for other checked uncertainties $\alpha = [0\ 1]^{\mathrm{T}}, [0.1\ 0.9]^{\mathrm{T}}, \ldots, [0.9\ 0.1]^{\mathrm{T}}$.

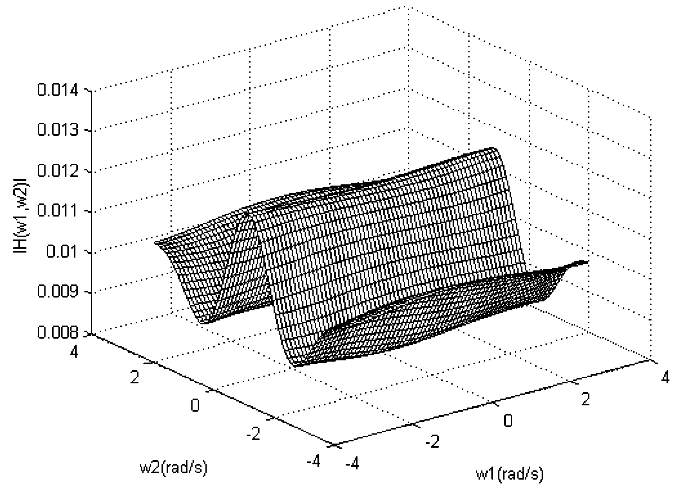

Fig. 1. Magnitudes of the filtering error transfer dynamics at different frequencies for $\alpha_1 = 1$ and $\alpha_2 = 0$.

VI. Conclusion

For 2-D systems with state delays and uncertainties described by the NFT model, this paper proposes convex optimization based filter synthesis methods. Sufficient conditions are developed in terms of LMI's for the stability, $\mathcal{H}_\infty$ performance, and generalized $\mathcal{H}_2$ performance of the considered 2-D systems. Then, it is shown how to convert the LMIs so that filter gain matrices can be obtained efficiently to satisfy the $\mathcal{H}_\infty$ and/or generalized $\mathcal{H}_2$ performance constraints. Two examples are given to illustrate the usage of the proposed methods, as well as the advantages of using the NFT model over the LFT model.

Appendix I
Proof of Theorem 2

First, the asymptotic stability of system (8) is established. For all α in the unit simplex (4), (22) implies that

$$\begin{bmatrix} -\mathbf{P}(\alpha) & * & * \\ 0 & -\mathbf{Q}(\alpha) & * \\ \hat{A}(\alpha) & \hat{A}_d(\alpha) & -\mathbf{M}^{-1}(\alpha) \end{bmatrix} < 0. \quad (73)$$

By the Schur's complement, (73) is equivalent to

$$\begin{bmatrix} -\mathbf{P}(\alpha) & 0 \\ 0 & -\mathbf{Q}(\alpha) \end{bmatrix} + \left[\hat{A}(\alpha) \quad \hat{A}_d(\alpha)\right]^{\mathrm{T}} \mathbf{M}(\alpha) \left[\hat{A}(\alpha) \quad \hat{A}_d(\alpha)\right] < 0. \quad (74)$$

It follows from Theorem 1 that the system (8) is asymptotically stable.

Next, the $\mathcal{H}_\infty$ performance is considered. By the Schur's complement, (21) is equivalent to $\mathbf{E}_1(\alpha) + [I_2 \otimes \Delta(\alpha)]^{\mathrm{T}} \mathbf{F}_1(\alpha) [I_2 \otimes \Delta(\alpha)] > 0$, which implies

$$\begin{bmatrix} z_\Delta(i+1,j) \\ z_\Delta(i,j+1) \end{bmatrix}^{\mathrm{T}} \left\{ \mathbf{E}_1(\alpha) + [I_2 \otimes \Delta(\alpha)]^{\mathrm{T}} \mathbf{F}_1(\alpha) [I_2 \otimes \Delta(\alpha)] \right\} \times \begin{bmatrix} z_\Delta(i+1,j) \\ z_\Delta(i,j+1) \end{bmatrix} \geq 0$$

for all z_Δ, or

$$\left\|\begin{bmatrix} w_\Delta(i+1,j) \\ w_\Delta(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{F}_1(\alpha)} + \left\|\begin{bmatrix} z_\Delta(i+1,j) \\ z_\Delta(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{E}_1(\alpha)} \geq 0 \quad (75)$$

for all w_Δ and z_Δ satisfying the last equation of (8). Moreover, (22) implies that

$$\begin{aligned}&\eta^{\mathrm{T}}(i,j)\{\mathrm{diag}[-\mathbf{P}(\alpha), -\mathbf{Q}(\alpha), \mathbf{F}_1(\alpha), -\gamma I] \\ &+ \left[\hat{A}(\alpha)\ \hat{A}_d(\alpha)\ \hat{B}_{\Delta 1}(\alpha)\ \hat{B}(\alpha)\right]^{\mathrm{T}}\mathbf{M}(\alpha) \\ &\times \left[\hat{A}(\alpha)\ \hat{A}_d(\alpha)\ \hat{B}_{\Delta 1}(\alpha)\ \hat{B}(\alpha)\right] \\ &+ \left[I_2\otimes \hat{C}(\alpha)\ 0\ 0\ 0\right]^{\mathrm{T}}\gamma^{-1}I\left[I_2\otimes \hat{C}(\alpha)\ 0\ 0\ 0\right] \\ &+ \left[I_2\otimes \hat{C}_\Delta(\alpha)\ 0\ I_2\otimes D_{\Delta z}(\alpha)\ I_2\otimes D_z(\alpha)\right]^{\mathrm{T}}\mathbf{E}_1(\alpha) \\ &\times \left[I_2\otimes \hat{C}_\Delta(\alpha)\ 0\ I_2\otimes D_{\Delta z}(\alpha)\ I_2\otimes D_z(\alpha)\right]\Big\}\eta(i,j) \leq 0\end{aligned} \quad (76)$$

where $\eta(i,j) = [\bar{x}(i,j)\ \bar{x}_d(i,j)\ \bar{w}_\Delta(i,j)\ \bar{w}(i,j)]^{\mathrm{T}}$, $\bar{x}(i,j) = [\hat{x}(i+1,j)^{\mathrm{T}}\ \hat{x}(i,j+1)^{\mathrm{T}}]$, $\bar{x}_d(i,j) = [\hat{x}(i+1,j-d_1)^{\mathrm{T}}\ \hat{x}(i-d_2,j+1)^{\mathrm{T}}]$, $\bar{w}_\Delta(i,j) = [w_\Delta(i+1,j)^{\mathrm{T}}\ w_\Delta(i,j+1)^{\mathrm{T}}]$, and $\bar{w}(i,j) = [w(i+1,j)^{\mathrm{T}}\ w(i,j+1)^{\mathrm{T}}]$. It follows from (8) that

$$\begin{aligned}&\|\hat{x}(i+1,j+1)\|^2_{\mathbf{M}(\alpha)} - \left\|\begin{bmatrix}\hat{x}(i+1,j)\\ \hat{x}(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{P}(\alpha)} \\ &- \left\|\begin{bmatrix}\hat{x}(i+1,j-d_1)\\ \hat{x}(i-d_2,j+1)\end{bmatrix}\right\|^2_{\mathbf{Q}(\alpha)} \\ &+ \left\|\begin{bmatrix} w_\Delta(i+1,j) \\ w_\Delta(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{F}_1(\alpha)} + \left\|\begin{bmatrix} z_\Delta(i+1,j) \\ z_\Delta(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{E}_1(\alpha)} \\ &+ \gamma^{-1}\left\|\begin{bmatrix}\hat{z}(i+1,j)\\ \hat{z}(i,j+1)\end{bmatrix}\right\|^2 - \gamma\left\|\begin{bmatrix}w(i+1,j)\\ w(i,j+1)\end{bmatrix}\right\|^2 \leq 0\end{aligned}$$

which, together with (75), implies

$$\begin{aligned}&\|\hat{x}(i+1,j+1)\|^2_{\mathbf{M}(\alpha)} - \left\|\begin{bmatrix}\hat{x}(i+1,j)\\ \hat{x}(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{P}(\alpha)} \\ &- \left\|\begin{bmatrix}\hat{x}(i+1,j-d_1)\\ \hat{x}(i-d_2,j+1)\end{bmatrix}\right\|^2_{\mathbf{Q}(\alpha)} + \gamma^{-1}\left\|\begin{bmatrix}\hat{z}(i+1,j)\\ \hat{z}(i,j+1)\end{bmatrix}\right\|^2 \\ &- \gamma\left\|\begin{bmatrix}w(i+1,j)\\ w(i,j+1)\end{bmatrix}\right\|^2 \leq 0.\end{aligned} \quad (77)$$

Now, for any integers $k \geq 0$, (77) leads to

$$\begin{aligned}&\|\hat{x}(k+1,0)\|^2_{\mathbf{M}(\alpha)} - \|\hat{x}(k,0)\|^2_{\mathbf{P}_2(\alpha)} - \|\hat{x}(k-d_2,0)\|^2_{\mathbf{Q}_2(\alpha)} \\ &\quad + \gamma^{-1}\left\|\begin{bmatrix}0\\ \hat{z}(k,0)\end{bmatrix}\right\|^2 - \gamma\left\|\begin{bmatrix}0\\ w(k,0)\end{bmatrix}\right\|^2 \leq 0 \\ &\|\hat{x}(k,1)\|^2_{\mathbf{M}(\alpha)} - \|\hat{x}(k,0)\|^2_{\mathbf{P}_1(\alpha)} - \|\hat{x}(k-1,1)\|^2_{\mathbf{P}_2(\alpha)} \\ &\quad - \|\hat{x}(k-1-d_2,1)\|^2_{\mathbf{Q}_2(\alpha)} \\ &\quad + \gamma^{-1}\left\|\begin{bmatrix}\hat{z}(k,0)\\ \hat{z}(k-1,1)\end{bmatrix}\right\|^2 - \gamma\left\|\begin{bmatrix}w(k,0)\\ w(k-1,1)\end{bmatrix}\right\|^2 \leq 0 \\ &\qquad\vdots \\ &\|\hat{x}(0,k+1)\|^2_{\mathbf{M}(\alpha)} - \|\hat{x}(0,k)\|^2_{\mathbf{P}_1(\alpha)} - \|\hat{x}(0,k-d_1)\|^2_{\mathbf{Q}_1(\alpha)} \\ &\quad + \gamma^{-1}\left\|\begin{bmatrix}\hat{z}(0,k)\\ 0\end{bmatrix}\right\|^2 - \gamma\left\|\begin{bmatrix}w(0,k)\\ 0\end{bmatrix}\right\|^2 \leq 0.\end{aligned} \quad (78)$$

Summing up the above inequalities gives

$$\begin{aligned}&\|\hat{X}_{k+1}\|^2_{\mathbf{M}(\alpha)} - \|\hat{X}_k\|^2_{\mathbf{P}_1(\alpha)} - \|\hat{X}_k\|^2_{\mathbf{P}_2(\alpha)} - \left\|\hat{X}_{k-d_1}\right\|^2_{\mathbf{Q}_1(\alpha)} \\ &\quad - \left\|\hat{X}_{k-d_2}\right\|^2_{\mathbf{Q}_2(\alpha)} + 2\gamma^{-1}\|\hat{Z}_k\|^2 - 2\gamma\|W_k\|^2 \leq 0\end{aligned} \quad (79)$$

for $k \geq 0$. Then, summing up (79) for $k \geq 0$ results in

$$\begin{aligned}2\gamma^{-1}\sum_{j=0}^{n}\|\hat{Z}_j\|^2 &\leq 2\gamma\sum_{j=0}^{n}\|W_j\|^2 - \|\hat{X}_{n+1}\|^2_{\mathbf{M}(\alpha)} + \|\hat{X}_0\|^2_{\mathbf{M}(\alpha)} \\ &\quad - \sum_{j=0}^{d_1}\|\hat{X}_{n+1-j}\|^2_{\mathbf{Q}_1(\alpha)} - \sum_{j=0}^{d_2}\|\hat{X}_{n+1-j}\|^2_{\mathbf{Q}_2(\alpha)}.\end{aligned} \quad (80)$$

Clearly, with $\hat{X}_0 = 0$

$$\gamma^{-1}\sum_{j=0}^{n}\|\hat{Z}_j\|^2 \leq \gamma\sum_{j=0}^{n}\|W_j\|^2 \quad (81)$$

for all $n \geq 1$. Thus, the $\mathcal{H}_\infty$-norm of the system (8) is no greater than γ. This completes the proof. ■

APPENDIX II
PROOF OF THEOREM 3

First, (28) implies

$$\begin{bmatrix} -\mathbf{X}(\alpha) & * & * \\ 0 & -\mathbf{Y}(\alpha) & * \\ \hat{A}(\alpha) & \hat{A}_d(\alpha) & -\mathbf{N}^{-1}(\alpha)\end{bmatrix} < 0 \quad (82)$$

which can be used to establish the asymptotic stability of system (8), just like how the stability is established from (73) in the proof of Theorem 2. In addition, imitating the argument from (22) to (77) in the proof of Theorem 2, one can show that (28) leads to

$$\begin{aligned}&\|\hat{x}(i+1,j+1)\|^2_{\mathbf{N}(\alpha)} - \left\|\begin{bmatrix}\hat{x}(i+1,j)\\ \hat{x}(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{X}(\alpha)} \\ &- \left\|\begin{bmatrix}\hat{x}(i+1,j-d_1)\\ \hat{x}(i-d_2,j+1)\end{bmatrix}\right\|^2_{\mathbf{Y}(\alpha)} + \left\|\begin{bmatrix} w_\Delta(i+1,j) \\ w_\Delta(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{F}_3(\alpha)} \\ &+ \left\|\begin{bmatrix} z_\Delta(i+1,j) \\ z_\Delta(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{E}_3(\alpha)} - \frac{1-\varepsilon}{2}\left\|\begin{bmatrix}w(i+1,j)\\ w(i,j+1)\end{bmatrix}\right\|^2 \leq 0.\end{aligned} \quad (83)$$

For all w_Δ and z_Δ satisfying (8), the same reasoning showing that (21) implies (75) also shows that (26) implies the nonnegativeness of the sum of the fourth and fifth terms in (83). Hence,

$$\begin{aligned}&\|\hat{x}(i+1,j+1)\|^2_{\mathbf{N}(\alpha)} - \left\|\begin{bmatrix}\hat{x}(i+1,j)\\ \hat{x}(i,j+1)\end{bmatrix}\right\|^2_{\mathbf{X}(\alpha)} \\ &- \left\|\begin{bmatrix}\hat{x}(i+1,j-d_1)\\ \hat{x}(i-d_2,j+1)\end{bmatrix}\right\|^2_{\mathbf{Y}(\alpha)} - \frac{1-\varepsilon}{2}\left\|\begin{bmatrix}w(i+1,j)\\ w(i,j+1)\end{bmatrix}\right\|^2 \leq 0\end{aligned} \quad (84)$$

which, with similar steps from (77) to (80), leads to

$$\begin{aligned}&\|\hat{X}_{n+1}\|^2_{\mathbf{N}(\alpha)} + \sum_{j=0}^{d_1}\|\hat{X}_{n+1-j}\|^2_{\mathbf{Y}_1(\alpha)} + \sum_{j=0}^{d_2}\|\hat{X}_{n+1-j}\|^2_{\mathbf{Y}_2(\alpha)} \\ &\quad + \varepsilon\sum_{j=0}^{n}\|W_j\|^2 \leq \sum_{j=0}^{n}\|W_j\|^2\end{aligned} \quad (85)$$

for $\hat{X}_0 = 0$. By the Shur's complement, (27) is equivalent to

$$\begin{aligned} &\operatorname{diag}\left[-\mathbf{N}(\alpha),\ \mathbf{F}_2(\alpha),\ -\varepsilon I\right] \\ &\quad + \left[\hat{C}(\alpha) \quad 0 \quad 0\right]^{\mathrm{T}} \mu^{-1} I \left[\hat{C}(\alpha) \quad 0 \quad 0\right] \\ &\quad + \left[\hat{C}_\Delta(\alpha) \quad D_{\Delta z}(\alpha) \quad D_z(\alpha)\right]^{\mathrm{T}} \mathbf{E}_2(\alpha) \\ &\quad \times \left[\hat{C}_\Delta(\alpha) \quad D_{\Delta z}(\alpha) \quad D_z(\alpha)\right] < 0. \end{aligned} \tag{86}$$

Again, with procedures similar to those adopted above, one gets

$$\begin{aligned} \mu^{-1} \left\|\hat{z}(i,j)\right\|^2 + \left\|w_\Delta(i,j)\right\|^2_{\mathbf{F}_2(\alpha)} + \left\|z_\Delta(i,j)\right\|^2_{\mathbf{E}_2(\alpha)} \\ \le \left\|\hat{x}(i,j)\right\|^2_{\mathbf{N}(\alpha)} + \varepsilon \left\|w(i,j)\right\|^2 \end{aligned} \tag{87}$$

from (86), and $\|w_\Delta(i,j)\|^2_{\mathbf{F}_2}(\alpha) + \|z_\Delta(i,j)\|^2_{\mathbf{E}_2}(\alpha) \ge 0$ from (25). Then

$$\mu^{-1} \left\|\hat{z}(i,j)\right\|^2 \le \left\|\hat{x}(i,j)\right\|^2_{\mathbf{N}(\alpha)} + \varepsilon \left\|w(i,j)\right\|^2 \tag{88}$$

and

$$\mu^{-1}\|\hat{Z}_n\|^2 \le \|\hat{X}_n\|^2_{\mathbf{N}(\alpha)} + \varepsilon\|W_n\|^2. \tag{89}$$

Together with (85), one finally obtains

$$\begin{aligned} \mu^{-1}\|\hat{Z}_n\|^2 &\le \|\hat{X}_n\|^2_{\mathbf{N}(\alpha)} + \varepsilon\|W_n\|^2 \\ &\le \|\hat{X}_{n+1}\|^2_{\mathbf{N}(\alpha)} + \sum_{j=0}^{d_1} \|\hat{X}_{n+1-j}\|^2_{\mathbf{Y}_1(\alpha)} \\ &\quad + \sum_{j=0}^{d_2} \|\hat{X}_{n+1-j}\|^2_{\mathbf{Y}_2(\alpha)} + \varepsilon \sum_{j=0}^{n} \|W_j\|^2 \\ &\le \sum_{j=0}^{n} \|W_j\|^2 \end{aligned} \tag{90}$$

and completes the proof. ■

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