H_∞ optimal singular and normal filter design for uncertain singular systems

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Abstract: The robust H_∞ optimal filtering problem for singular systems with norm-bounded uncertainties in all system matrices is considered. On the basis of the admissibility assumption of the uncertain singular systems, one singular and two normal filter design methods are proposed under the linear-matrix inequality framework. A numerical example is provided to illustrate the application of all three proposed methods.

1 Introduction

In the past decade, the H_∞ optimal filtering problem for singular systems has been an important research topic. This is due, not only to the theoretical interest, but also to the relevance of the topic in various engineering applications. Many works, such as Xu *et al.* [1] and Yue and Han [2], consider the filters with the same structure as the concerned singular system in the derivative term. Indeed, this is often a natural and convenient choice. However, it is more suitable for the cases in which the coefficient matrix for the derivative term has no uncertainty. In practical applications, more flexibility will be gained if all system matrices are allowed to contain uncertainties.

In this paper, the robust H_∞ optimal filtering problem is studied for singular systems, which contain norm-bounded uncertainties in all system matrices. First of all, a filter design method that generates singular filters is proposed as an extension of the method by Xu *et al.* [1]. Then, two kinds of normal filters (i.e. those with the system matrix for the derivative term being the identity matrix) [3] are considered. For singular as well as normal filters, the goal is to minimise the H_∞ performance level of the corresponding filtering error dynamics. The consideration of normal filters is beneficial when sometimes the physical realisations of singular filters are not easy [3, 4]. In order to realise a singular system, often it needs special algorithms [5] to convert a singular system model into a normal state-space form.

To handle systems with uncertainties in the system matrix for the derivative term, it is assumed that the uncertain systems are admissible and the concept of the restricted system equivalence (r.s.e.) [3] is applied. In addition, some preliminary results in Lin *et al.* [6], which considers the stabilisation problem for singular systems using the algebraic Riccati equation method, provide the further assumptions for the theoretical development of this paper. For easier applications, all proposed filter design methods are formulated within the linear-matrix inequality (LMI) framework.

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Some notations to be used subsequently are introduced here. The inequality $\boldsymbol{X} \geq \boldsymbol{0}$ means that the matrix $\boldsymbol{X}$ is symmetric and positive semi-definite, and $\boldsymbol{X} \geq \boldsymbol{Y}$ means $\boldsymbol{X} - \boldsymbol{Y} \geq \boldsymbol{0}$. Similar definitions apply to symmetric positive/negative definite matrices. For a matrix $\boldsymbol{M}$, $\|\boldsymbol{M}\|$ denotes the spectral norm of $\boldsymbol{M}$, and for a stable continuous-time transfer function matrix $\boldsymbol{G}(s)$, $\|\boldsymbol{G}\|_\infty = \sup_{\omega\in[0,\infty)}\|\boldsymbol{G}(j\omega)\|$ is its H_∞ norm. $\boldsymbol{I}_r$ is the identity matrix with dimension r, the superscript T represents the transpose of a matrix and diag($\boldsymbol{X}, \boldsymbol{Y}, \ldots, \boldsymbol{Z}$) is the block diagonal matrix with diagonal elements $\boldsymbol{X}, \boldsymbol{Y}, \ldots, \boldsymbol{Z}$. Finally, $*$ is used to simplify the presentation of symmetric matrices.

2 Preliminaries and problem formulation

First, consider the following nominal singular system

$$\Sigma_{\mathrm{n}}: \begin{cases} \boldsymbol{E}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{z}(t) = \boldsymbol{L}\boldsymbol{x}(t) \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathcal{R}^n$ and rank$\boldsymbol{E} = r < n$. The unforced singular system pair $(\boldsymbol{E}, \boldsymbol{A})$ of (1) with $\boldsymbol{u} = \boldsymbol{0}$ is regular if det$(s\boldsymbol{E} - \boldsymbol{A})$ is not identically zero. If deg(det$(s\boldsymbol{E} - \boldsymbol{A})$) = rank $\boldsymbol{E}$, then $(\boldsymbol{E}, \boldsymbol{A})$ is said to be impulse free. The pair $(\boldsymbol{E}, \boldsymbol{A})$ is stable if all the roots of det$(s\boldsymbol{E} - \boldsymbol{A}) = \boldsymbol{0}$ have negative real parts. Finally, $(\boldsymbol{E}, \boldsymbol{A})$ is admissible if it is regular, impulse free and stable. For Σ_{n}, its transfer function matrix from $\boldsymbol{u}(t)$ to $\boldsymbol{z}(t)$ is $\boldsymbol{G}(s) = \boldsymbol{L}(s\boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{B}$.

Definition 1 [3]: Suppose Σ_{n} in (1) is regular. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two $n \times n$ non-singular matrices and $\boldsymbol{E}_{\mathrm{r}} = \boldsymbol{PEQ}$, $\boldsymbol{A}_{\mathrm{r}} = \boldsymbol{PAQ}$, $\boldsymbol{B}_{\mathrm{r}} = \boldsymbol{PB}$, $\boldsymbol{L}_{\mathrm{r}} = \boldsymbol{LQ}$. The system

$$\Sigma_{\mathrm{r}}: \begin{cases} \boldsymbol{E}_{\mathrm{r}}\dot{\boldsymbol{x}}_{\mathrm{r}}(t) = \boldsymbol{A}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}}(t) + \boldsymbol{B}_{\mathrm{r}}\boldsymbol{u}(t) \\ \boldsymbol{z}(t) = \boldsymbol{L}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}}(t) \end{cases} \tag{2}$$

with $\boldsymbol{x}_{\mathrm{r}}(t) = \boldsymbol{Q}^{-1}\boldsymbol{x}(t)$ is r.s.e. to Σ_{n}.

For any given regular Σ_{n}, there exist [3] non-singular matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{E}_{\mathrm{r}} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{A}_{\mathrm{r}} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{r}} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}, \quad \boldsymbol{L}_{\mathrm{r}} = [\boldsymbol{L}_1 \quad \boldsymbol{L}_2] \tag{3}$$

Now consider the following uncertain singular system

$$\Sigma_u: \begin{cases} E\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)u(t) \\ z(t) = (L + \delta L)x(t) + (J + \delta J)u(t) \end{cases} \quad (4)$$

where δA, δB, δL and δJ are constant uncertainties.

Definition 2 [6, 7]: The unforced pair $(E, A + \delta A)$ in Σ_u is said to be quadratically admissible for all uncertainty δA if there exists a matrix X such that

$$E^T X = X^T E \geq 0 \quad (5)$$

$$(A + \delta A)^T X + X^T(A + \delta A) < 0 \quad (6)$$

The quadratic admissibility is also called the generalised quadratic stability by Xu and Yang [8] and Xu et al. [9], and it implies [7] the admissibility of a system.

Definition 3 [7]: Given a $\mu > 0$, the system Σ_u is said to be quadratically admissible with disturbance attenuation μ for all uncertainties δA, δB, δL and δJ if there exists a matrix X such that

$$E^T X = X^T E \geq 0 \quad (7)$$

$$\begin{bmatrix} (A+\delta A)^T X + X^T(A+\delta A) & * & * \\ (B+\delta B)^T X & -\mu^2 I & * \\ L + \delta L & J + \delta J & -I \end{bmatrix} < 0 \quad (8)$$

It is known [7] that quadratically admissible with disturbance attenuation μ means that the H_∞ norm of the transfer function matrix from $u(t)$ to $z(t)$ is less than μ.

2.1 Problem formulation

The uncertain singular system to be discussed is

$$\Sigma: \begin{cases} (E_0 + \delta E)\dot{x}(t) = (A_0 + \delta A)x(t) + (B_0 + \delta B)u(t) \\ y(t) = (C_0 + \delta C)x(t) + (D_0 + \delta D)u(t) \\ z(t) = (L_0 + \delta L)x(t) + (J_0 + \delta J)u(t) \end{cases} \quad (9)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $y(t) \in \mathcal{R}^p$ the measured output vector, $z(t) \in \mathcal{R}^q$ the vector to be estimated and $u(t) \in \mathcal{R}^m$ the disturbance input vector. The matrices E_0, A_0, B_0, C_0, D_0, L_0 and J_0 are known real constant matrices with appropriate dimensions. The constant uncertainty matrices satisfy

$$\begin{bmatrix} \delta A & \delta B \\ \delta C & \delta D \\ \delta L & \delta J \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \Delta [N_x \ N_u] \quad (10)$$

$\delta E = M\Delta N$ and $\Delta^T \Delta \leq I$. Suppose that the pair $(E_0 + \delta E, A_0 + \delta A)$ is admissible and $\text{rank}(E_0 + \delta E) = \text{rank}\, E_0 = r < n$ for all Δ under discussion. Let the non-singular P and Q be such that

$$PE_0Q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad PA_0Q = \begin{bmatrix} A_{r1} & A_{r2} \\ A_{r3} & A_{r4} \end{bmatrix}$$

$$PM = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad NQ = [N_1 \quad N_2] \quad (11)$$

By Lin *et al.* [6], it can be assumed without loss of generality that either $M_2 = 0$ or $N_2 = 0$. Here, it is further assumed that $\|N_1 M_1\| < 1$.

To estimate $z(t)$, the following filter is adopted

$$\Sigma_f: \begin{cases} E_f\dot{x}_f(t) = A_f x_f(t) + B_f y(t) \\ z_f(t) = C_f x_f(t) + D_f y(t) \end{cases} \quad (12)$$

where $x_f(t) \in \mathcal{R}^{n_f}$ and $z_f(t) \in \mathcal{R}^q$. The matrices E_f, A_f, B_f, C_f and D_f are to be determined. From Σ in (9) and Σ_f in (12), the filtering error dynamics may be written as

$$\Sigma_e: \begin{cases} E_e\dot{x}_e(t) = A_e x_e(t) + B_e u(t) \\ e(t) = C_e x_e(t) + D_e u(t) \end{cases} \quad (13)$$

where $e(t) = z(t) - z_f(t)$, $x_e^T(t) = [x^T(t) \ x_f^T(t)]$ and

$$E_e = \begin{bmatrix} E_0 + \delta E & 0 \\ 0 & E_f \end{bmatrix}, \quad A_e = \begin{bmatrix} A_0 + \delta A & 0 \\ B_f(C_0 + \delta C) & A_f \end{bmatrix}$$

$$B_e = \begin{bmatrix} B_0 + \delta B \\ B_f(D_0 + \delta D) \end{bmatrix} \quad (14)$$

$$C_e = [(L_0 + \delta L) - D_f(C_0 + \delta C) \quad -C_f]$$

$$D_e = (J_0 + \delta J) - D_f(D_0 + \delta D)$$

The purpose here is to design the filter Σ_f such that the pair (E_f, A_f) is admissible and

$$\sup_\Delta \|C_e(sE_e - A_e)^{-1}B_e + D_e\|_\infty < \mu_e \quad (15)$$

for a prescribed H_∞-norm bound $\mu_e > 0$.

2.2 Restricted system equivalence

Under the assumption that the pair $(E_0 + \delta E, A_0 + \delta A)$ is admissible, there exist [6] non-singular matrices P, Q, P_Δ and Q_Δ such that the system Σ in (9) is r.s.e. to the system

$$\tilde{\Sigma}_r: \begin{cases} \tilde{E}_r\dot{\tilde{x}}(t) = \tilde{A}_r\tilde{x}(t) + \tilde{B}_r u(t) \\ y(t) = \tilde{C}_r\tilde{x}(t) + \tilde{D}_r u(t) \\ z(t) = \tilde{L}_r\tilde{x}(t) + \tilde{J}_r u(t) \end{cases} \quad (16)$$

where

$$\tilde{E}_r = E_{0r} = PE_0Q, \quad \tilde{A}_r = P_\Delta(A_{0r} + M_{xr}\Delta N_{xr})Q_\Delta$$

$$\tilde{B}_r = P_\Delta(B_{0r} + M_{xr}\Delta N_{ur}), \quad \tilde{C}_r = (C_{0r} + M_{yr}\Delta N_{xr})Q_\Delta$$

$$\tilde{D}_r = (D_{0r} + M_{yr}\Delta N_{ur}), \quad \tilde{L}_r = (L_{0r} + M_{zr}\Delta N_{xr})Q_\Delta$$

$$\tilde{J}_r = (J_{0r} + M_{zr}\Delta N_{ur}) \quad (17)$$

and

$$A_{0r} = PA_0Q, \quad B_{0r} = PB_0, \quad C_{0r} = C_0Q$$

$$D_{0r} = D_0, \quad L_{0r} = L_0Q, \quad J_{0r} = J_0$$

$$M_{xr} = PM_x, \quad M_{yr} = M_y, \quad M_{zr} = M_z \quad (18)$$

$$N_{xr} = N_xQ, \quad N_{ur} = N_u$$

More specifically, for $N_2 = 0$ in (11), $P_\Delta = I_n - M_{0r}\tilde{\Delta}N_{0r}$ and $Q_\Delta = I_n$, and for $M_2 = 0$ in (11), $P_\Delta = I_n$ and $Q_\Delta = I_n - M_{0r}\tilde{\Delta}N_{0r}$, where $M_{0r} = PM$, $N_{0r} = NQ$ and $\tilde{\Delta} = \Delta(I + N_1M_1\Delta)^{-1}$.

Thus, the error dynamics $\tilde{\Sigma}_e$ from $\tilde{\Sigma}_r$ in (16) and Σ_f in (12) is

$$\tilde{\Sigma}_e: \begin{cases} \tilde{E}_e\dot{\tilde{x}}_e(t) = \tilde{A}_e\tilde{x}_e(t) + \tilde{B}_e u(t) \\ e(t) = \tilde{C}_e\tilde{x}_e(t) + \tilde{D}_e u(t) \end{cases} \quad (19)$$

where $\tilde{x}_e^T(t) = [\tilde{x}^T(t) \quad x_f^T(t)]$ and

$$\tilde{E}_e = \begin{bmatrix} E_{0r} & 0 \\ 0 & E_f \end{bmatrix}, \quad \tilde{A}_e = \begin{bmatrix} \tilde{A}_r & 0 \\ B_f\tilde{C}_r & A_f \end{bmatrix}, \quad \tilde{B}_e = \begin{bmatrix} \tilde{B}_r \\ B_f\tilde{D}_r \end{bmatrix},$$

$$\tilde{C}_e = [\tilde{L}_r - D_f\tilde{C}_r \quad -C_f], \quad \tilde{D}_e = \tilde{J}_r - D_f\tilde{D}_r \tag{20}$$

Lemma 1: Suppose there exist non-singular matrices P_s and Q_s such that the regular systems Σ_{s1} and Σ_{s2} are r.s.e., and there exist non-singular matrices P_f and Q_f such that the filters Σ_{f1} and Σ_{f2} are r.s.e., where for $i = 1, 2$

$$\Sigma_{si}: \begin{cases} E_{si}\dot{x}_{si}(t) = A_{si}x_{si}(t) + B_{si}u_s(t) \\ y_s(t) = C_{si}x_{si}(t) + D_s u_s(t) \\ z_s(t) = L_{si}x_{si}(t) + J_s u_s(t) \end{cases} \tag{21}$$

$$\Sigma_{fi}: \begin{cases} E_{fi}\dot{x}_{fi}(t) = A_{fi}x_{fi}(t) + B_{fi}y_s(t) \\ \hat{z}_f(t) = C_{fi}x_{fi}(t) + D_f y_s(t) \end{cases} \tag{22}$$

Then, the corresponding error dynamics Σ_{e1} and Σ_{e2} are also r.s.e., where for $i = 1, 2$

$$\Sigma_{ei}: \begin{cases} E_{ei}\dot{x}_{ei}(t) = A_{ei}x_{ei}(t) + B_{ei}u_s(t) \\ \hat{e}(t) = C_{ei}x_{ei}(t) + D_e u_s(t) \end{cases} \tag{23}$$

$\hat{e}(t) = z_s(t) - \hat{z}_f(t)$, $x_{ei}^T(t) = [x_{si}^T(t) \quad x_{fi}^T(t)]$ and

$$E_{ei} = \begin{bmatrix} E_{si} & 0 \\ 0 & E_{fi} \end{bmatrix}, \quad A_{ei} = \begin{bmatrix} A_{si} & 0 \\ B_{fi}C_{si} & A_{fi} \end{bmatrix}$$

$$B_{ei} = \begin{bmatrix} B_{si} \\ B_{fi}D_s \end{bmatrix}, \quad C_{ei} = [L_{si} - D_{fi}C_{si} \quad -C_{fi}] \tag{24}$$

$$D_e = J_s - D_f D_s$$

Moreover, the transfer function matrices $G_{e1}(s)$ and $G_{e2}(s)$ of Σ_{e1} and Σ_{e2}, respectively, are the same.

Proof: By the assumptions, there exist non-singular matrices P_s, Q_s, P_f and Q_f such that

$$E_{\xi 2} = P_\xi E_{\xi 1} Q_\xi, \quad A_{\xi 2} = P_\xi A_{\xi 1} Q_\xi, \quad B_{\xi 2} = P_\xi B_{\xi 1},$$

$$C_{\xi 2} = C_{\xi 1} Q_\xi, \quad L_{\xi 2} = L_{\xi 1} Q_\xi \tag{25}$$

for $\xi = s$ and f. It is easy to see that the matrices in (24) satisfy

$$E_{e2} = \hat{P}_e E_{e1} \hat{Q}_e, \quad A_{e2} = \hat{P}_e A_{e1} \hat{Q}_e$$

$$B_{e2} = \hat{P}_e B_{e1}, \quad C_{e2} = C_{e1} \hat{Q}_e \tag{26}$$

where $\hat{P}_e = \mathrm{diag}(P_s, P_f)$ and $\hat{Q}_e = \mathrm{diag}(Q_s, Q_f)$. Therefore Σ_{e1} and Σ_{e2} are also r.s.e. By (26)

$$\begin{aligned} G_{e2}(s) &= C_{e2}(sE_{e2} - A_{e2})^{-1}B_{e2} + D_e \\ &= C_{e1}\hat{Q}_e(s\hat{P}_e E_{e1}\hat{Q}_e - \hat{P}_e A_{e1}\hat{Q}_e)^{-1}\hat{P}_e B_{e1} + D_e \\ &= C_{e1}(sE_{e1} - A_{e1})^{-1}B_{e1} + D_e = G_{e1}(s) \end{aligned} \tag{27}$$

□

On Lemma 1, the filtering problem formulated in the last section can be reformulated as follows: design the filter Σ_f in (12) such that the pair (E_f, A_f) is admissible and

$$\sup_{\Delta} \|\tilde{C}_e(s\tilde{E}_e - \tilde{A}_e)^{-1}\tilde{B}_e + \tilde{D}_e\|_\infty < \mu_e \tag{28}$$

2.3 Two useful lemmas

Lemma 2 [10]: Let $I - \bar{J}^T\bar{J} > 0$ and define the set

$$\Upsilon = \{\Delta(I - \bar{J}\Delta)^{-1}, \quad \Delta^T\Delta \le I\}$$

Then, $\Upsilon = \{\bar{J}^T(I - \bar{J}\bar{J}^T)^{-1} + \Pi^T(I - \bar{J}\bar{J}^T)^{-1/2}$, $\Pi^T\Pi \le (I - \bar{J}^T\bar{J})^{-1}\}$.

Lemma 3 [11]: Let Ω, H, F and $R > 0$ be real matrices with appropriate dimensions, and the matrix Π satisfies $\Pi^T\Pi \le R$. Then, for all $\Pi^T\Pi \le R$, the matrix inequality

$$\Omega + H\Pi F + F^T\Pi^T H^T < 0$$

holds if and only if there exists a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \Omega & H \\ H^T & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} F^T R F & 0 \\ 0 & -I \end{bmatrix} < 0$$

3 Robust filter design

Two kinds of robust filters are discussed here: the singular and the normal filters. For the sake of brevity, only the results for the case with $N_2 = 0$ [6] will be presented.

3.1 Singular filter design

The filter Σ_f is called a singular filter when rank $E_f < n_f$ in (12). In the literature, for example, Xu *et al.* [1] and Yue and Han [2], the choice $E_f = E$ is the most discussed, as it is often dictated by the derivation process. This case is studied here for comparison with the normal filters. Theorem 1 is developed as an extension of Theorem 1 in Xu *et al.* [1], which concerns nominal singular systems.

Theorem 1: If and only if there exist feasible solutions ρ, ε, $Y \in \mathcal{R}^{n\times n}$, $Z \in \mathcal{R}^{n\times p}$, $\bar{M} \in \mathcal{R}^{n\times n}$, $\bar{N} \in \mathcal{R}^{q\times n}$, $P_c \in \mathcal{R}^{n\times n}$ and $D_f \in \mathcal{R}^{q\times p}$ to the LMIs

$$E_{0r}^T P_c = P_c^T E_{0r} \ge 0, \quad E_{0r}^T Y = Y^T E_{0r} \ge 0$$

$$E_{0r}^T(P_c - Y) \ge 0 \tag{29}$$

$$\left[\begin{array}{cc} A_{0r}^T\Theta^T Y + Y^T\Theta A_{0r} & * \\ \begin{pmatrix} A_{0r}^T\Theta^T Y + P_c^T\Theta A_{0r} \\ + ZC_{0r} + \overline{M} \end{pmatrix} & \begin{pmatrix} P_c^T\Theta A_{0r} + A_{0r}^T\Theta^T P_c \\ + ZC_{0r} + C_{0r}^T Z^T \end{pmatrix} \\ B_{0r}^T\Theta^T Y & B_{0r}^T\Theta^T P_c + D_{0r}^T Z^T \\ L_{0r} - \overline{N} & L_{0r} \\ \frac{1}{\varepsilon}\Xi_2 N_{0r}A_{0r} & \frac{1}{\varepsilon}\Xi_2 N_{0r}A_{0r} \\ M_{0r}^T Y & M_{0r}^T P_c \\ M_{xr}^T\Theta^T Y & M_{xr}^T\Theta^T P_c + M_{yr}^T Z^T \\ \rho N_{xr} & \rho N_{xr} \end{array}\right.$$

$$\begin{array}{ccc} * & * & * \\ * & * & * \\ -\mu_e^2 I & * & * \\ J_{0r} - D_f D_{0r} & -I & * \\ \frac{1}{\varepsilon}\Xi_2 N_{0r}B_{0r} & 0 & \frac{-1}{\varepsilon}I \\ 0 & 0 & 0 \\ 0 & M_{zr}^T - M_{yr}^T D_f^T & \frac{1}{\varepsilon}M_{xr}^T N_{0r}^T\Xi_2^T \\ \rho N_{ur} & 0 & 0 \end{array}$$

$$\begin{matrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ \dfrac{-1}{\varepsilon}\boldsymbol{\Xi}_3 & * & * \\ \boldsymbol{0} & -\rho \boldsymbol{I} & * \\ \boldsymbol{0} & \boldsymbol{0} & -\rho \boldsymbol{I} \end{matrix} \Bigg] < \boldsymbol{0} \tag{30}$$

where

$$\boldsymbol{\Theta} = \boldsymbol{I} + \boldsymbol{M}_{0\mathrm{r}}\boldsymbol{\Xi}_1\boldsymbol{N}_{0\mathrm{r}}, \quad \boldsymbol{\Xi}_1 = \boldsymbol{M}_1^{\mathrm{T}}\boldsymbol{N}_1^{\mathrm{T}}(\boldsymbol{I} - \boldsymbol{N}_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}}\boldsymbol{N}_1^{\mathrm{T}})^{-1}$$
$$\boldsymbol{\Xi}_2 = (\boldsymbol{I} - \boldsymbol{N}_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}}\boldsymbol{N}_1^{\mathrm{T}})^{-1/2}, \quad \boldsymbol{\Xi}_3 = \boldsymbol{I} - \boldsymbol{M}_1^{\mathrm{T}}\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1\boldsymbol{M}_1 \tag{31}$$

then the filter Σ_{f} in (12) is admissible with the filter gains

$$\begin{aligned} \boldsymbol{E}_{\mathrm{f}} &= \boldsymbol{E}_{0\mathrm{r}}, \quad \boldsymbol{A}_{\mathrm{f}} = \boldsymbol{U}^{-\mathrm{T}}\overline{\boldsymbol{M}}\boldsymbol{Y}^{-1}\boldsymbol{W}^{-1} \\ \boldsymbol{B}_{\mathrm{f}} &= \boldsymbol{U}^{-\mathrm{T}}\boldsymbol{Z}, \quad \boldsymbol{C}_{\mathrm{f}} = \overline{\boldsymbol{N}}\boldsymbol{Y}^{-1}\boldsymbol{W}^{-1} \end{aligned} \tag{32}$$

where $\boldsymbol{U}$ and $\boldsymbol{W}$ are non-singular matrices satisfying

$$\begin{aligned} \boldsymbol{E}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{V} &= \boldsymbol{U}^{\mathrm{T}}\boldsymbol{E}_{0\mathrm{r}}, \quad \boldsymbol{E}_{0\mathrm{r}}\boldsymbol{W} = \tilde{\boldsymbol{W}}^{\mathrm{T}}\boldsymbol{E}_{0\mathrm{r}}^{\mathrm{T}} \\ \boldsymbol{P}_{\mathrm{c}}\boldsymbol{Y}^{-1} &= \boldsymbol{I} - \boldsymbol{V}\boldsymbol{W}, \quad \boldsymbol{Y}^{-1}\boldsymbol{P}_{\mathrm{c}} = \boldsymbol{I} - \tilde{\boldsymbol{W}}\boldsymbol{U} \end{aligned} \tag{33}$$

for any non-singular matrices $\boldsymbol{V}$ and $\tilde{\boldsymbol{W}}$ and the filtering error dynamics $\tilde{\Sigma}_{\mathrm{e}}$ in (19) is quadratically admissible as well as satisfying (28).

Proof: (Sufficiency) The proof may be divided into three parts. The first part is to show that there exist non-singular matrices $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ and $\tilde{\boldsymbol{W}}$ such that (33) holds. The second part involves showing that

$$\boldsymbol{P}_{\mathrm{e}} = \begin{bmatrix} \boldsymbol{P}_{\mathrm{c}} & \boldsymbol{V} \\ \boldsymbol{U} & -\boldsymbol{U}\boldsymbol{Y}^{-1}\boldsymbol{W}^{-1} \end{bmatrix} \tag{34}$$

is non-singular, and for the error dynamics $\tilde{\Sigma}_{\mathrm{e}}$ in (19) with $\tilde{\boldsymbol{E}}_e = \mathrm{diag}(\boldsymbol{E}_{0\mathrm{r}}, \boldsymbol{E}_{0\mathrm{r}})$ in (20)

$$\tilde{\boldsymbol{E}}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{e}} = \boldsymbol{P}_{\mathrm{e}}^{\mathrm{T}}\tilde{\boldsymbol{E}}_{\mathrm{e}} \geq \boldsymbol{0} \tag{35}$$

is true. These two parts may be proved by exactly following the corresponding parts of the proofs for Theorem 1 in Xu *et al.* [1].

The third and only part that must be proved here is that if the LMI (30) holds, then the error dynamics $\tilde{\Sigma}_{\mathrm{e}}$ in (19) satisfies the constraint (8). To start, denote

$$\begin{aligned} \boldsymbol{H}_{\varepsilon}^{\mathrm{T}} &= [\boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\boldsymbol{A}_{0\mathrm{r}} \quad \boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\boldsymbol{A}_{0\mathrm{r}} \quad \boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\boldsymbol{B}_{0\mathrm{r}} \quad \boldsymbol{0}] \\ \boldsymbol{F}_{\varepsilon} &= [\boldsymbol{M}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{Y} \quad \boldsymbol{M}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} \quad \boldsymbol{0} \quad \boldsymbol{0}] \\ \boldsymbol{H}_{\rho}^{\mathrm{T}} &= [\boldsymbol{M}_{x\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} \quad \boldsymbol{M}_{x\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} + \boldsymbol{M}_{y\mathrm{r}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} \quad \boldsymbol{0} \\ &\qquad \boldsymbol{M}_{z\mathrm{r}}^{\mathrm{T}} - \boldsymbol{M}_{y\mathrm{r}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{f}}^{\mathrm{T}} \quad \varepsilon^{-1}\boldsymbol{M}_{x\mathrm{r}}^{\mathrm{T}}\boldsymbol{N}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Xi}_2^{\mathrm{T}} \quad \boldsymbol{0}] \\ \boldsymbol{F}_{\rho} &= [\boldsymbol{N}_{x\mathrm{r}} \quad \boldsymbol{N}_{x\mathrm{r}} \quad \boldsymbol{N}_{u\mathrm{r}} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0}] \end{aligned}$$

Inequality (30) can be rewritten as

$$\boldsymbol{\Omega}_{\rho} = \begin{bmatrix} \boldsymbol{\Omega}_{\varepsilon} & * & * \\ \boldsymbol{H}_{\rho}^{\mathrm{T}} & -\rho\boldsymbol{I} & * \\ \rho\boldsymbol{F}_{\rho} & \boldsymbol{0} & -\rho\boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{36}$$

where

$$\boldsymbol{\Omega}_{\varepsilon} = \begin{bmatrix} \boldsymbol{\Omega}_0 & * & * \\ \varepsilon^{-1}\boldsymbol{H}_{\varepsilon}^{\mathrm{T}} & -\varepsilon^{-1}\boldsymbol{I} & * \\ \boldsymbol{F}_{\varepsilon} & \boldsymbol{0} & -\varepsilon^{-1}\boldsymbol{\Xi}_3 \end{bmatrix}$$

and

$$\boldsymbol{\Omega}_0 = \left[\begin{matrix} \boldsymbol{A}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{A}_{0\mathrm{r}} \\ \begin{pmatrix} \boldsymbol{A}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} + \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{A}_{0\mathrm{r}} \\ +\boldsymbol{Z}\boldsymbol{C}_{0\mathrm{r}} + \overline{\boldsymbol{M}} \end{pmatrix} \\ \boldsymbol{B}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} \\ \boldsymbol{L}_{0\mathrm{r}} - \overline{\boldsymbol{N}} \end{matrix}\right.$$
$$\left.\begin{matrix} * & * & * \\ \begin{pmatrix} \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{A}_{0\mathrm{r}} + \boldsymbol{A}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} \\ +\boldsymbol{Z}\boldsymbol{C}_{0\mathrm{r}} + \boldsymbol{C}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} \end{pmatrix} & * & * \\ \boldsymbol{B}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} + \boldsymbol{D}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} & -\mu_{\mathrm{e}}^2\boldsymbol{I} & * \\ \boldsymbol{L}_{0\mathrm{r}} & \boldsymbol{J}_{0\mathrm{r}} - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{D}_{0\mathrm{r}} & -\boldsymbol{I} \end{matrix}\right]$$

By the Schur complement [12] and Lemma 3, (36) with a $\rho > 0$ is equivalent to

$$\tilde{\boldsymbol{\Omega}}_{\varepsilon} = \boldsymbol{\Omega}_{\varepsilon} + \boldsymbol{H}_{\rho}\boldsymbol{\Delta}\boldsymbol{F}_{\rho} + \boldsymbol{F}_{\rho}^{\mathrm{T}}\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{H}_{\rho}^{\mathrm{T}} < \boldsymbol{0} \tag{37}$$

for all $\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{\Delta} \leq \boldsymbol{I}$. The inequality (37) may be more explicitly written as

$$\left[\begin{matrix} \bar{\boldsymbol{A}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{\Theta}\bar{\boldsymbol{A}}_{0\mathrm{r}} & * \\ \begin{pmatrix} \bar{\boldsymbol{A}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} + \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Theta}\bar{\boldsymbol{A}}_{0\mathrm{r}} \\ +\boldsymbol{Z}\bar{\boldsymbol{C}}_{0\mathrm{r}} + \overline{\boldsymbol{M}} \end{pmatrix} & \begin{pmatrix} \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Theta}\bar{\boldsymbol{A}}_{0\mathrm{r}} + \bar{\boldsymbol{A}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} \\ +\boldsymbol{Z}\bar{\boldsymbol{C}}_{0\mathrm{r}} + \bar{\boldsymbol{C}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} \end{pmatrix} \\ \bar{\boldsymbol{B}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{Y} & \bar{\boldsymbol{B}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} + \bar{\boldsymbol{D}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}} \\ \bar{\boldsymbol{L}}_{0\mathrm{r}} - \overline{\boldsymbol{N}} & \bar{\boldsymbol{L}}_{0\mathrm{r}} \\ \dfrac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\bar{\boldsymbol{A}}_{0\mathrm{r}} & \dfrac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\bar{\boldsymbol{A}}_{0\mathrm{r}} \\ \boldsymbol{M}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{Y} & \boldsymbol{M}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} \end{matrix}\right.$$
$$\left.\begin{matrix} * & * & * & * \\ * & * & * & * \\ -\mu_{\mathrm{e}}^2\boldsymbol{I} & * & * & * \\ \bar{\boldsymbol{J}}_{0\mathrm{r}} - \boldsymbol{D}_{\mathrm{f}}\bar{\boldsymbol{D}}_{0\mathrm{r}} & -\boldsymbol{I} & * & * \\ \dfrac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\bar{\boldsymbol{B}}_{0\mathrm{r}} & \boldsymbol{0} & -\dfrac{1}{\varepsilon}\boldsymbol{I} & * \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\dfrac{1}{\varepsilon}\boldsymbol{\Xi}_3 \end{matrix}\right] < \boldsymbol{0} \tag{38}$$

with

$$\begin{bmatrix} \bar{\boldsymbol{A}}_{0\mathrm{r}} & \bar{\boldsymbol{B}}_{0\mathrm{r}} \\ \bar{\boldsymbol{C}}_{0\mathrm{r}} & \bar{\boldsymbol{D}}_{0\mathrm{r}} \\ \bar{\boldsymbol{L}}_{0\mathrm{r}} & \bar{\boldsymbol{J}}_{0\mathrm{r}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{0\mathrm{r}} & \boldsymbol{B}_{0\mathrm{r}} \\ \boldsymbol{C}_{0\mathrm{r}} & \boldsymbol{D}_{0\mathrm{r}} \\ \boldsymbol{L}_{0\mathrm{r}} & \boldsymbol{J}_{0\mathrm{r}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{M}_{x\mathrm{r}} \\ \boldsymbol{M}_{y\mathrm{r}} \\ \boldsymbol{M}_{z\mathrm{r}} \end{bmatrix}\boldsymbol{\Delta}[\boldsymbol{N}_{x\mathrm{r}} \quad \boldsymbol{N}_{u\mathrm{r}}] \tag{39}$$

Letting

$$\bar{\Omega}_0 = \begin{bmatrix} \bar{A}_{0r}^{\mathrm{T}}\Theta^{\mathrm{T}}Y + Y^{\mathrm{T}}\Theta\bar{A}_{0r} & * & * & * \\ \begin{pmatrix} \bar{A}_{0r}^{\mathrm{T}}\Theta^{\mathrm{T}}Y + P_c^{\mathrm{T}}\Theta\bar{A}_{0r} \\ +Z\bar{C}_{0r} + \overline{M} \end{pmatrix} & \begin{pmatrix} P_c^{\mathrm{T}}\Theta\bar{A}_{0r} + \bar{A}_{0r}^{\mathrm{T}}\Theta^{\mathrm{T}}P_c \\ +ZC_{0r} + \bar{C}_{0r}^{\mathrm{T}}Z^{\mathrm{T}} \end{pmatrix} & * & * \\ \bar{B}_{0r}^{\mathrm{T}}\Theta^{\mathrm{T}}Y & \bar{B}_{0r}^{\mathrm{T}}\Theta^{\mathrm{T}}P_c + \bar{D}_{0r}^{\mathrm{T}}Z^{\mathrm{T}} & -\mu_e^2 I & * \\ \bar{L}_{0r} - \overline{N} & \bar{L}_{0r} & \bar{J}_{0r} - D_f\bar{D}_{0r} & -I \end{bmatrix}$$

and

$$\bar{H}_\varepsilon^{\mathrm{T}} = [-\Xi_2 N_{0r}\bar{A}_{0r} \quad -\Xi_2 N_{0r}A_{0r} \quad -\Xi_2 N_{0r}\bar{B}_{0r} \quad 0]$$

one sees that (38) with an $\varepsilon^{-1} > 0$ is equivalent to

$$\bar{\Omega}_0 + \bar{H}_\varepsilon \Pi F_\varepsilon + F_\varepsilon^{\mathrm{T}}\Pi^{\mathrm{T}}\bar{H}_\varepsilon^{\mathrm{T}} < 0 \tag{40}$$

for all $\Pi^{\mathrm{T}}\Pi \le \Xi_3^{-1}$ by the Schur complement and Lemma 3. With $P_\Delta = I_n - M_{0r}(-\Xi_1 + \Pi^{\mathrm{T}}\Xi_2)N_{0r}$, the inequality (40) may be rewritten as

$$\begin{bmatrix} \bar{A}_{0r}^{\mathrm{T}}P_\Delta^{\mathrm{T}}Y + Y^{\mathrm{T}}P_\Delta\bar{A}_{0r} & * & * & * \\ \begin{pmatrix} \bar{A}_{0r}^{\mathrm{T}}P_\Delta^{\mathrm{T}}Y + P_c^{\mathrm{T}}P_\Delta\bar{A}_{0r} \\ +Z\bar{C}_{0r} + \overline{M} \end{pmatrix} & \begin{pmatrix} P_c^{\mathrm{T}}P_\Delta\bar{A}_{0r} + \bar{A}_{0r}^{\mathrm{T}}P_\Delta^{\mathrm{T}}P_c \\ +ZC_{0r} + \bar{C}_{0r}^{\mathrm{T}}Z^{\mathrm{T}} \end{pmatrix} & * & * \\ \bar{B}_{0r}^{\mathrm{T}}P_\Delta^{\mathrm{T}}Y & \bar{B}_{0r}^{\mathrm{T}}P_\Delta^{\mathrm{T}}P_c + \bar{D}_{0r}^{\mathrm{T}}Z^{\mathrm{T}} & -\mu_e^2 I & * \\ \bar{L}_{0r} - \overline{N} & \bar{L}_{0r} & \bar{J}_{0r} - D_f\bar{D}_{0r} & -I \end{bmatrix} < 0 \tag{41}$$

Note that $P_\Delta = I_n - M_{0r}\Delta(I + N_1M_1\Delta)^{-1}N_{0r}$ for $\Delta^{\mathrm{T}}\Delta \le I$ by Lemma 2 and the assumption $\|N_1M_1\| < 1$. Pre- and post-multiplying (41) by diag($Y^{-\mathrm{T}}$, I, I, I) and diag(Y^{-1}, I, I, I), respectively, show that (41) is equivalent to

$$\begin{bmatrix} P_1^{\mathrm{T}}\tilde{A}_e^{\mathrm{T}}P_eP_1 + P_1^{\mathrm{T}}P_e^{\mathrm{T}}\tilde{A}_eP_1 & * & * \\ \tilde{B}_e^{\mathrm{T}}P_eP_1 & -\mu_e^2 I & * \\ \tilde{C}_eP_1 & \tilde{D}_e & -I \end{bmatrix} < 0 \tag{42}$$

where P_e is defined in (34)

$$P_1 = \begin{bmatrix} Y^{-1} & I \\ W & 0 \end{bmatrix}$$

are non-singular, and $\tilde{A}_e$, $\tilde{B}_e$, $\tilde{C}_e$ and $\tilde{D}_e$ are given in (20) with A_f, B_f and C_f from (32). Pre- and post-multiplying (42) by diag($P_1^{-\mathrm{T}}$, I, I) and diag(P_1^{-1}, I, I), respectively, one finally obtains

$$\begin{bmatrix} \tilde{A}_e^{\mathrm{T}}P_e + P_e^{\mathrm{T}}\tilde{A}_e & * & * \\ \tilde{B}_e^{\mathrm{T}}P_e & -\mu_e^2 I & * \\ \tilde{C}_e & \tilde{D}_e & -I \end{bmatrix} < 0 \tag{43}$$

By Definition 3, the conditions in (35) and (43) with the P_e in (34) imply that $\tilde{\Sigma}_e$ in (19) is quadratically admissible with disturbance attenuation μ_e. The filter Σ_f in (12) is therefore admissible.

(Necessity) As the filtering error dynamics $\tilde{\Sigma}_e$ in (19) and (20) is quadratically admissible and has a transfer function matrix satisfying (28), there exists a matrix X_e such that

$$\tilde{E}_e^{\mathrm{T}}X_e = X_e^{\mathrm{T}}\tilde{E}_e \ge 0 \tag{44}$$

$$\begin{bmatrix} \tilde{A}_e^{\mathrm{T}}X_e + X_e^{\mathrm{T}}\tilde{A}_e & * & * \\ \tilde{B}_e^{\mathrm{T}}X_e & -\mu_e^2 I & * \\ \tilde{C}_e & \tilde{D}_e & -I \end{bmatrix} < 0 \tag{45}$$

It is easy to see that (45) implies

$$\tilde{A}_e^{\mathrm{T}}X_e + X_e^{\mathrm{T}}\tilde{A}_e < 0 \tag{46}$$

which implies X_e is non-singular. Partition X_e and X_e^{-1} as

$$X_e = \begin{bmatrix} X_{e1} & X_{e2} \\ U & X_{e3} \end{bmatrix}, \quad X_e^{-1} = \begin{bmatrix} Y_{e1} & Y_{e2} \\ W & Y_{e3} \end{bmatrix} \tag{47}$$

in accordance with the partition of $\tilde{A}_e$. The 2–2 block of (46) gives $A_f^{\mathrm{T}}X_{e3} + X_{e3}^{\mathrm{T}}A_f < 0$, which implies X_{e3} is non-singular. By (46)

$$X_e^{-\mathrm{T}}\tilde{A}_e^{\mathrm{T}} + \tilde{A}_eX_e^{-1} < 0 \tag{48}$$

The 1–1 block of (48) gives $\tilde{A}_rY_{e1} + Y_{e1}^{\mathrm{T}}\tilde{A}_r^{\mathrm{T}} < 0$, which implies Y_{e1} is also non-singular. Note that from (44)

$$E_{0r}^{\mathrm{T}}X_{e1} = X_{e1}^{\mathrm{T}}E_{0r} \ge 0, \quad E_f^{\mathrm{T}}X_{e3} = X_{e3}^{\mathrm{T}}E_f \ge 0 \tag{49}$$

and $X_e^{-\mathrm{T}}\tilde{E}_e^{\mathrm{T}} = \tilde{E}_eX_e^{-1} \ge 0$, whose 1–1 block implies

$$E_{0r}^{\mathrm{T}}Y_{e1}^{-1} = Y_{e1}^{-\mathrm{T}}E_{0r} \ge 0 \tag{50}$$

As U and W can be chosen to have full-column rank [1], it is possible to define two full-column-rank matrices

$$\hat{T} = \begin{bmatrix} Y_{e1} & I \\ W & 0 \end{bmatrix}, \quad \check{T} = \begin{bmatrix} I & X_{e1} \\ 0 & U \end{bmatrix} \tag{51}$$

so that $X_e\hat{T} = \check{T}$. Pre- and post-multiplying (45) by diag($\hat{T}^{\mathrm{T}}$, I, I) and diag($\hat{T}$, I, I), respectively, and then pre- and post-multiplying the resulting inequality by diag($Y_{e1}^{-\mathrm{T}}$, I, I, I) and diag(Y_{e1}^{-1}, I, I, I), respectively. Then, (30) is obtained by Lemmas 2 and 3 when

$$P_c = X_{e1}, \quad Y = Y_{e1}^{-1}, \quad A_f = U^{-\mathrm{T}}\overline{M}Y_{e1}W^{-1},$$
$$B_f = U^{-\mathrm{T}}Z, \quad C_f = \overline{N}Y_{e1}W^{-1} \tag{52}$$

are substituted. Also, the first inequality of (49) provides the first inequality of (29) and (50) provides the second inequality of (29). As $Y_{e1} = (X_{e1} - X_{e2}X_{e3}^{-1}U)^{-1}$, by (44) and the second inequality of (49), one has

$$E_{0r}^{\mathrm{T}}(X_{e1} - Y_{e1}^{-1}) = E_{0r}^{\mathrm{T}}X_{e2}X_{e3}^{-1}U = U^{\mathrm{T}}E_fX_{e3}^{-1}U \ge 0 \tag{53}$$

which provides the last inequality of (29). □

Remark 1: On the basis of Theorem 1, the following convex optimisation problem may be formulated to find the H_∞ optimal filter of the form (12) such that (28) is satisfied with the minimal μ_e^2

$$\min_{\mu_e^2, \rho, \varepsilon^{-1}, Y, Z, \overline{M}, \overline{N}, P_c, D_f} \mu_e^2 \tag{54}$$

subject to the LMIs (29) and (30). Note that by Theorem 1, the choice of the non-singular matrices U, V, W and $\tilde{W}$ does not affect the result of the optimal μ_e.

3.2 Normal filter design method I

The filter Σ_f in (12) is called a normal filter if $\boldsymbol{E}_f = \boldsymbol{I}_{n_f}$ in (12) [3]. In some cases, the realisation of singular filters is not easy [3, 4]. It may even be necessary to convert the derived singular filters into the equivalent normal filters by using additional algorithms [5]. For such cases, it is reasonable to find the normal filter directly. In this section, the first of two normal filter design methods is introduced.

Theorem 2: If there exist feasible solutions ε, ρ, $\boldsymbol{P}_c \in \mathcal{R}^{n\times n}$, $\boldsymbol{\Phi} \in \mathcal{R}^{n\times p}$, $\boldsymbol{\Psi} \in \mathcal{R}^{q\times n}$, $\boldsymbol{\Gamma} \in \mathcal{R}^{n\times n}$, $\hat{\boldsymbol{Q}} \in \mathcal{R}^{n\times n}$ and $\boldsymbol{D}_f \in \mathcal{R}^{q\times p}$ to the LMIs

$$\begin{bmatrix} \boldsymbol{P}_c + \boldsymbol{P}_c^T & (\boldsymbol{E}_{0r}^T + \boldsymbol{I})\boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}(\boldsymbol{E}_{0r} + \boldsymbol{I}) & 2\boldsymbol{\Gamma} \end{bmatrix} > \boldsymbol{0}, \quad \begin{bmatrix} \boldsymbol{E}_{0r}^T\boldsymbol{P}_c & \boldsymbol{E}_{0r}^T\boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}\boldsymbol{E}_{0r} & \boldsymbol{\Gamma} \end{bmatrix} \geq \boldsymbol{0} \tag{55}$$

$$\begin{bmatrix}
\begin{pmatrix} \boldsymbol{A}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \boldsymbol{P}_c^T\boldsymbol{\Theta}\boldsymbol{A}_{0r} \\ + \boldsymbol{C}_{0r}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} + \boldsymbol{E}_{0r}^T\boldsymbol{\Phi}\boldsymbol{C}_{0r} \end{pmatrix} & * & * & * & * & * & * & * \\
\boldsymbol{\Gamma}\boldsymbol{\Theta}\boldsymbol{A}_{0r} + \boldsymbol{\Phi}\boldsymbol{C}_{0r} + \hat{\boldsymbol{Q}}^T\boldsymbol{E}_{0r} & \hat{\boldsymbol{Q}} + \hat{\boldsymbol{Q}}^T & * & * & * & * & * & * \\
\boldsymbol{B}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \boldsymbol{D}_{0r}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} & \boldsymbol{B}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{\Gamma} + \boldsymbol{D}_{0r}^T\boldsymbol{\Phi}^T & -\mu_e^2\boldsymbol{I} & * & * & * & * & * \\
\boldsymbol{L}_{0r} - \boldsymbol{D}_f\boldsymbol{C}_{0r} & -\boldsymbol{\Psi} & \boldsymbol{J}_{0r} - \boldsymbol{D}_f\boldsymbol{D}_{0r} & -\boldsymbol{I} & * & * & * & * \\
\frac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\boldsymbol{A}_{0r} & \boldsymbol{0} & \frac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\boldsymbol{B}_{0r} & \boldsymbol{0} & \frac{-1}{\varepsilon}\boldsymbol{I} & * & * & * \\
\boldsymbol{M}_{0r}^T\boldsymbol{P}_c & \boldsymbol{M}_{0r}^T\boldsymbol{\Gamma} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{-1}{\varepsilon}\boldsymbol{\Xi}_3 & * & * \\
\boldsymbol{M}_{xr}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \boldsymbol{M}_{yr}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} & \boldsymbol{M}_{xr}^T\boldsymbol{\Theta}^T\boldsymbol{\Gamma} + \boldsymbol{M}_{yr}^T\boldsymbol{\Phi}^T & \boldsymbol{0} & \boldsymbol{M}_{zr}^T - \boldsymbol{M}_{yr}^T\boldsymbol{D}_f^T & \frac{1}{\varepsilon}\boldsymbol{M}_{xr}^T\boldsymbol{N}_{0r}^T\boldsymbol{\Xi}_2^T & \boldsymbol{0} & -\rho\boldsymbol{I} & * \\
\rho\boldsymbol{N}_{xr} & \boldsymbol{0} & \rho\boldsymbol{N}_{ur} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\rho\boldsymbol{I}
\end{bmatrix} < \boldsymbol{0} \tag{56}$$

where $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ and $\boldsymbol{\Xi}_3$ are given in (31), then any normal filter Σ_f with $\boldsymbol{E}_f = \boldsymbol{I}_n$ and

$$\boldsymbol{A}_f = \hat{\boldsymbol{Q}}\boldsymbol{\Gamma}^{-1}, \quad \boldsymbol{B}_f = \boldsymbol{\Phi}, \quad \boldsymbol{C}_f = \boldsymbol{\Psi}\boldsymbol{\Gamma}^{-1}, \quad \boldsymbol{D}_f \tag{57}$$

is stable and makes (28) hold for $\tilde{\Sigma}_e$ in (19).

Proof: Let

$$\boldsymbol{P}_e = \begin{bmatrix} \boldsymbol{P}_c & \boldsymbol{I} \\ \boldsymbol{E}_{0r} & \boldsymbol{\Gamma}^{-1} \end{bmatrix} \tag{58}$$

With the normal filter, $\tilde{\boldsymbol{E}}_e$ in (20) is $\mathrm{diag}(\boldsymbol{E}_{0r}, \boldsymbol{I}_n)$, so inequalities in (55) guarantee that $\boldsymbol{P}_e$ is non-singular and $\boldsymbol{P}_e$ satisfies

$$\tilde{\boldsymbol{E}}_e^T\boldsymbol{P}_e = \boldsymbol{P}_e^T\tilde{\boldsymbol{E}}_e \geq \boldsymbol{0} \tag{59}$$

Next, define

$$\boldsymbol{H}_1 = \begin{bmatrix} \boldsymbol{P}_c^T\boldsymbol{\Theta}\boldsymbol{M}_{xr} + \boldsymbol{E}_{0r}^T\boldsymbol{\Phi}\boldsymbol{M}_{yr} \\ \boldsymbol{\Gamma}\boldsymbol{\Theta}\boldsymbol{M}_{xr} + \boldsymbol{\Phi}\boldsymbol{M}_{yr} \\ \boldsymbol{0} \\ \boldsymbol{M}_{zr} - \boldsymbol{D}_f\boldsymbol{M}_{yr} \\ \varepsilon^{-1}\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\boldsymbol{M}_{xr} \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{F}_1^T = \begin{bmatrix} \boldsymbol{N}_{xr}^T \\ \boldsymbol{0} \\ \boldsymbol{N}_{ur}^T \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{60}$$

$$\boldsymbol{\Omega}_1 = \begin{bmatrix}
\begin{pmatrix} \boldsymbol{A}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \boldsymbol{P}_c^T\boldsymbol{\Theta}\boldsymbol{A}_{0r} \\ + \boldsymbol{C}_{0r}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} + \boldsymbol{E}_{0r}^T\boldsymbol{\Phi}\boldsymbol{C}_{0r} \end{pmatrix} & * & * & * & * & * \\
\boldsymbol{\Gamma}\boldsymbol{\Theta}\boldsymbol{A}_{0r} + \boldsymbol{\Phi}\boldsymbol{C}_{0r} + \hat{\boldsymbol{Q}}^T\boldsymbol{E}_{0r} & \hat{\boldsymbol{Q}} + \hat{\boldsymbol{Q}}^T & * & * & * & * \\
\boldsymbol{B}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \boldsymbol{D}_{0r}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} & \boldsymbol{B}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{\Gamma} + \boldsymbol{D}_{0r}^T\boldsymbol{\Phi}^T & -\mu_e^2\boldsymbol{I} & * & * & * \\
\boldsymbol{L}_{0r} - \boldsymbol{D}_f\boldsymbol{C}_{0r} & -\boldsymbol{\Psi} & \boldsymbol{J}_{0r} - \boldsymbol{D}_f\boldsymbol{D}_{0r} & -\boldsymbol{I} & * & * \\
\frac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\boldsymbol{A}_{0r} & \boldsymbol{0} & \frac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\boldsymbol{B}_{0r} & \boldsymbol{0} & \frac{-1}{\varepsilon}\boldsymbol{I} & * \\
\boldsymbol{M}_{0r}^T\boldsymbol{P}_c & \boldsymbol{M}_{0r}^T\boldsymbol{\Gamma} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{-1}{\varepsilon}\boldsymbol{\Xi}_3
\end{bmatrix} \tag{61}$$

By Lemma 3, (56) with a $\rho > 0$ is equivalent to

$$\boldsymbol{\Omega}_1 + \boldsymbol{H}_1\boldsymbol{\Delta}\boldsymbol{F}_1 + \boldsymbol{F}_1^T\boldsymbol{\Delta}^T\boldsymbol{H}_1^T < \boldsymbol{0} \tag{62}$$

for all $\boldsymbol{\Delta}^T\boldsymbol{\Delta} \leq \boldsymbol{I}$. Denote

$$\boldsymbol{H}_2^T = [-\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\bar{\boldsymbol{A}}_{0r} \quad \boldsymbol{0} \quad -\boldsymbol{\Xi}_2\boldsymbol{N}_{0r}\bar{\boldsymbol{B}}_{0r} \quad \boldsymbol{0}]$$
$$\boldsymbol{F}_2 = [\boldsymbol{M}_{0r}^T\boldsymbol{P}_c \quad \boldsymbol{M}_{0r}^T\boldsymbol{\Gamma} \quad \boldsymbol{0} \quad \boldsymbol{0}]$$

and

$$\boldsymbol{\Omega}_2 = \begin{bmatrix}
\begin{pmatrix} \bar{\boldsymbol{A}}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \boldsymbol{P}_c^T\boldsymbol{\Theta}\bar{\boldsymbol{A}}_{0r} \\ + \bar{\boldsymbol{C}}_{0r}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} + \boldsymbol{E}_{0r}^T\boldsymbol{\Phi}\bar{\boldsymbol{C}}_{0r} \end{pmatrix} & * & * & * \\
\boldsymbol{\Gamma}\Theta\bar{\boldsymbol{A}}_{0r} + \boldsymbol{\Phi}\bar{\boldsymbol{C}}_{0r} + \hat{\boldsymbol{Q}}^T\boldsymbol{E}_{0r} & \hat{\boldsymbol{Q}} + \hat{\boldsymbol{Q}}^T & * & * \\
\bar{\boldsymbol{B}}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{P}_c + \bar{\boldsymbol{D}}_{0r}^T\boldsymbol{\Phi}^T\boldsymbol{E}_{0r} & \bar{\boldsymbol{B}}_{0r}^T\boldsymbol{\Theta}^T\boldsymbol{\Gamma} + \bar{\boldsymbol{D}}_{0r}^T\boldsymbol{\Phi}^T & -\mu_e^2\boldsymbol{I} & * \\
\bar{\boldsymbol{L}}_{0r} - \boldsymbol{D}_f\bar{\boldsymbol{C}}_{0r} & -\boldsymbol{\Psi} & \bar{\boldsymbol{J}}_{0r} - \boldsymbol{D}_f\bar{\boldsymbol{D}}_{0r} & -\boldsymbol{I}
\end{bmatrix}$$

in the left side of (62), where $\bar{\boldsymbol{A}}_{0r}$, $\bar{\boldsymbol{B}}_{0r}$, $\boldsymbol{C}_{0r}$, $\boldsymbol{D}_{0r}$, $\bar{\boldsymbol{L}}_{0r}$ and $\bar{\boldsymbol{J}}_{0r}$ are defined in (39). By Lemma 3, (62) with an $\varepsilon^{-1} > 0$ is equivalent to

$$\boldsymbol{\Omega}_2 + \boldsymbol{H}_2\boldsymbol{\Pi}\boldsymbol{F}_2 + \boldsymbol{F}_2^T\boldsymbol{\Pi}^T\boldsymbol{H}_2^T < \boldsymbol{0} \tag{63}$$

for all $\boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{\Pi} \le \boldsymbol{\Xi}_3^{-1}$. With $\boldsymbol{P}_\Delta = \boldsymbol{I}_{\mathrm{n}} - \boldsymbol{M}_{0\mathrm{r}}(-\boldsymbol{\Xi}_1 + \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{\Xi}_2)\boldsymbol{N}_{0\mathrm{r}}$, the inequality (63) may be rewritten as

$$\begin{bmatrix} \begin{pmatrix} \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{P}_\Delta\bar{\boldsymbol{A}}_{0\mathrm{r}} + \boldsymbol{E}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Phi}\bar{\boldsymbol{C}}_{0\mathrm{r}} \\ +\bar{\boldsymbol{A}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_\Delta^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} + \bar{\boldsymbol{C}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{E}_{0\mathrm{r}} \end{pmatrix} & * & * & * \\ \boldsymbol{\Gamma}\boldsymbol{P}_\Delta\bar{\boldsymbol{A}}_{0\mathrm{r}} + \boldsymbol{\Phi}\bar{\boldsymbol{C}}_{0\mathrm{r}} + \hat{\boldsymbol{Q}}\boldsymbol{E}_{0\mathrm{r}} & \hat{\boldsymbol{Q}} + \hat{\boldsymbol{Q}}^{\mathrm{T}} & * & * \\ \bar{\boldsymbol{B}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_\Delta^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} + \bar{\boldsymbol{D}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{E}_{0\mathrm{r}} & \bar{\boldsymbol{B}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_\Delta^{\mathrm{T}}\boldsymbol{\Gamma} + \bar{\boldsymbol{D}}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}} & -\mu_{\mathrm{e}}^2\boldsymbol{I} & * \\ \bar{\boldsymbol{L}}_{0\mathrm{r}} - \boldsymbol{D}_{\mathrm{f}}\bar{\boldsymbol{C}}_{0\mathrm{r}} & -\boldsymbol{\Psi} & \bar{\boldsymbol{J}}_{0\mathrm{r}} - \boldsymbol{D}_{\mathrm{f}}\bar{\boldsymbol{D}}_{0\mathrm{r}} & -\boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{64}$$

Note that $\boldsymbol{P}_\Delta = \boldsymbol{I}_{\mathrm{n}} - \boldsymbol{M}_{0\mathrm{r}}\boldsymbol{\Delta}(\boldsymbol{I} + \boldsymbol{N}_1\boldsymbol{M}_1\boldsymbol{\Delta})^{-1}\boldsymbol{N}_{0\mathrm{r}}$ for $\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{\Delta} \le \boldsymbol{I}$ by Lemma 2 and the assumption $\|\boldsymbol{N}_1\boldsymbol{M}_1\| < 1$. Pre- and post-multiplying (64) by diag($\boldsymbol{I}$, $\boldsymbol{P}^{\mathrm{T}}$, $\boldsymbol{I}$, $\boldsymbol{I}$), one obtains

$$\begin{bmatrix} \tilde{\boldsymbol{A}}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{e}} + \boldsymbol{P}_{\mathrm{e}}^{\mathrm{T}}\tilde{\boldsymbol{A}}_{\mathrm{e}} & * & * \\ \tilde{\boldsymbol{B}}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{e}} & -\mu_{\mathrm{e}}^2\boldsymbol{I} & * \\ \tilde{\boldsymbol{C}}_{\mathrm{e}} & \tilde{\boldsymbol{D}}_{\mathrm{e}} & -\boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{65}$$

where $\boldsymbol{P}_{\mathrm{e}}$ is given in (58) and $\tilde{\boldsymbol{A}}_{\mathrm{e}}$, $\tilde{\boldsymbol{B}}_{\mathrm{e}}$, $\tilde{\boldsymbol{C}}_{\mathrm{e}}$ and $\tilde{\boldsymbol{D}}_{\mathrm{e}}$ are given in (20) with $\boldsymbol{A}_{\mathrm{f}}$, $\boldsymbol{B}_{\mathrm{f}}$ and $\boldsymbol{C}_{\mathrm{f}}$ from (57). Expressions (59) and (65) together imply that $\tilde{\Sigma}_{\mathrm{e}}$ in (19) is quadratically admissible with disturbance attenuation μ_{e}. The normal filter Σ_{f} in (12) thus is stable. □

Remark 2: On the basis of Theorem 2, the following convex optimisation problem may be formulated to find the H_∞ optimal normal filter of order n such that (28) is satisfied with the minimal μ_{e}^2

$$\min_{\mu_{\mathrm{e}}^2,\rho,\varepsilon^{-1},\boldsymbol{P}_{\mathrm{c}},\boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Gamma},\hat{\boldsymbol{Q}},\boldsymbol{D}_{\mathrm{f}}} \mu_{\mathrm{e}}^2 \tag{66}$$

subject to the LMIs (55) and (56).

3.3 Normal filter design method II

In this section, a method for designing normal filters of lower order than those obtainable by the first method is developed.

Theorem 3: If there exist feasible solutions ε, ρ, $\boldsymbol{P}_{\mathrm{c}} \in \mathcal{R}^{n\times n}$, $\boldsymbol{P}_{\mathrm{f}} \in \mathcal{R}^{r\times r}$, $\tilde{\boldsymbol{\Phi}} \in \mathcal{R}^{r\times p}$, $\tilde{\boldsymbol{Q}} \in \mathcal{R}^{r\times r}$, $\boldsymbol{C}_{\mathrm{f}} \in R^{q\times r}$ and $\boldsymbol{D}_{\mathrm{f}} \in \mathcal{R}^{q\times p}$ to the LMIs

$$\boldsymbol{P}_{\mathrm{f}} > \boldsymbol{0}, \quad \boldsymbol{E}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} = \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{E}_{0\mathrm{r}} \ge \boldsymbol{0} \tag{67}$$

$$\begin{bmatrix} \boldsymbol{A}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} + \boldsymbol{P}_{\mathrm{c}}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{A}_{0\mathrm{r}} & * & * & * & * & * & * & * \\ \tilde{\boldsymbol{\Phi}}\boldsymbol{C}_{0\mathrm{r}} & \tilde{\boldsymbol{Q}} + \tilde{\boldsymbol{Q}}^{\mathrm{T}} + 2\boldsymbol{P}_{\mathrm{f}} & * & * & * & * & * & * \\ \boldsymbol{B}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} & \boldsymbol{D}_{0\mathrm{r}}^{\mathrm{T}}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}} & -\mu_{\mathrm{e}}^2\boldsymbol{I} & * & * & * & * & * \\ \boldsymbol{L}_{0\mathrm{r}} - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{C}_{0\mathrm{r}} & -\boldsymbol{C}_{\mathrm{f}} & \boldsymbol{J}_{0\mathrm{r}} - \boldsymbol{D}_{\mathrm{f}}\boldsymbol{D}_{0\mathrm{r}} & -\boldsymbol{I} & * & * & * & * \\ \frac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\boldsymbol{A}_{0\mathrm{r}} & \boldsymbol{0} & \frac{1}{\varepsilon}\boldsymbol{\Xi}_2\boldsymbol{N}_{0\mathrm{r}}\boldsymbol{B}_{0\mathrm{r}} & \boldsymbol{0} & \frac{-1}{\varepsilon}\boldsymbol{I} & * & * & * \\ \boldsymbol{M}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{-1}{\varepsilon}\boldsymbol{\Xi}_3 & * & * \\ \boldsymbol{M}_{x\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{c}} & \boldsymbol{M}_{y\mathrm{r}}^{\mathrm{T}}\tilde{\boldsymbol{\Phi}}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{M}_{z\mathrm{r}}^{\mathrm{T}} - \boldsymbol{M}_{y\mathrm{r}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{f}}^{\mathrm{T}} & \frac{1}{\varepsilon}\boldsymbol{M}_{x\mathrm{r}}^{\mathrm{T}}\boldsymbol{N}_{0\mathrm{r}}^{\mathrm{T}}\boldsymbol{\Xi}_2^{\mathrm{T}} & \boldsymbol{0} & -\rho\boldsymbol{I} & * \\ \rho\boldsymbol{N}_{x\mathrm{r}} & \boldsymbol{0} & \rho\boldsymbol{N}_{u\mathrm{r}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\rho\boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{68}$$

where $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ and $\boldsymbol{\Xi}_3$ are given in (31), then any normal filter Σ_{f} with $\boldsymbol{E}_{\mathrm{f}} = \boldsymbol{I}_{\mathrm{r}}$ and

$$\boldsymbol{A}_{\mathrm{f}} = \boldsymbol{P}_{\mathrm{f}}^{-1}\tilde{\boldsymbol{Q}} + \boldsymbol{I}_{\mathrm{r}}, \quad \boldsymbol{B}_{\mathrm{f}} = \boldsymbol{P}_{\mathrm{f}}^{-1}\tilde{\boldsymbol{\Phi}}, \quad \boldsymbol{C}_{\mathrm{f}}, \quad \boldsymbol{D}_{\mathrm{f}} \tag{69}$$

is stable and makes (28) hold for $\tilde{\Sigma}_{\mathrm{e}}$ in (19).

Proof: The proof is similar to that for Theorem 2, except $\boldsymbol{P}_{\mathrm{e}}$ is set to diag($\boldsymbol{P}_{\mathrm{c}}$, $\boldsymbol{P}_{\mathrm{f}}$). The remaining steps are the same and omitted here for the sake of brevity. □

Remark 3: On the basis of Theorem 3, the following convex optimisation problem may be formulated to find the H_∞ optimal normal filter of order r such that (28) is satisfied with the minimal μ_{e}^2

$$\min_{\mu_{\mathrm{e}}^2,\rho,\varepsilon^{-1},\boldsymbol{P}_{\mathrm{c}},\boldsymbol{P}_{\mathrm{f}},\tilde{\boldsymbol{\Phi}},\tilde{\boldsymbol{Q}},\boldsymbol{C}_{\mathrm{f}},\boldsymbol{D}_{\mathrm{f}}} \mu_{\mathrm{e}}^2 \tag{70}$$

subject to the LMIs (67) and (68).

4 Numerical example

In this section, an example is worked out to illustrate the proposed filter design methods. Suppose that the system matrices of the system Σ in (9) are as follows

$$\boldsymbol{E}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_0 = \begin{bmatrix} -2 & 1 & 0 \\ 0.5 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{B}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{C}_0 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

$$\boldsymbol{L}_0 = \begin{bmatrix} -2 & 1 & 0.5 \end{bmatrix}, \quad \boldsymbol{D}_0 = 0.5, \quad \boldsymbol{J}_0 = 0 \tag{71}$$

The uncertainty matrices in (10) are assumed to be

$$\boldsymbol{M} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad \boldsymbol{M}_x = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}, \quad \boldsymbol{M}_y = -1, \quad \boldsymbol{M}_z = 2$$

$$\boldsymbol{N} = \begin{bmatrix} 0.1 & -0.5 & 0 \end{bmatrix}, \quad \boldsymbol{N}_x = \begin{bmatrix} 0.1 & 0 & 0.1 \end{bmatrix}, \quad \boldsymbol{N}_u = 1$$

and $|\boldsymbol{\Delta}| \le 1$. It is easy to verify that $(\boldsymbol{E}_0 + \boldsymbol{M}\boldsymbol{\Delta}\boldsymbol{N}, \boldsymbol{A}_0 + \boldsymbol{M}_x\boldsymbol{\Delta}\boldsymbol{N}_x)$ is an admissible pair. Let $\boldsymbol{P} = \boldsymbol{Q} = \boldsymbol{I}_3$ in (11) and $\boldsymbol{P}_\Delta = \boldsymbol{I}_3 - \boldsymbol{M}\boldsymbol{\Delta}(1 - 0.04\boldsymbol{\Delta})^{-1}\boldsymbol{N}$ as well as $\boldsymbol{Q}_\Delta = \boldsymbol{I}_3$. Then, the considered system Σ is r.s.e. to the system $\tilde{\Sigma}_{\mathrm{r}}$ in (16) with $\tilde{\boldsymbol{E}}_{\mathrm{r}} = \boldsymbol{E}_0$ in (71).

Three H_∞ optimal filters are designed by solving the convex optimisation problems listed in Remarks 1, 2 and 3, respectively, which are implemented by the MATLAB LMI Control Toolbox [13]. Because in this example $\boldsymbol{E}_{0\mathrm{r}} = \boldsymbol{P}\boldsymbol{E}_0\boldsymbol{Q} = \mathrm{diag}(1,\ 1,\ 0)$, the non-strict LMIs in (29), (55) and (67) are manually adjusted to strict ones when applying the MATLAB LMI Control Toolbox. The resulting optimal μ_e's are 2.7591, 2.1415 and 2.4739, respectively, for the singular, third-order normal and second-order normal filters. Obviously, in this case, normal filters have better μ_e's than the singular filter. Also, the third-order normal filter, having larger degrees of freedom, owns a better μ_e than that for the second-order normal filter.

As a check, for the H_∞ optimal singular filter, the pair $(\boldsymbol{E}_\mathrm{f}, \boldsymbol{A}_\mathrm{f})$ corresponding to $\boldsymbol{U} = \boldsymbol{V} = \boldsymbol{P}_\mathrm{c}$ in (33) is

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -20.3388 & 12.4829 & 20.3279 \\ 16.2319 & -15.9105 & -6.9030 \\ 0.6263 & 0.4655 & -3.4626 \end{bmatrix} \right)$$

which is admissible.

5 Conclusion

The H_∞ optimal filter design problem has been considered for uncertain singular systems, in which uncertainties appear in all system matrices. For designing singular filters, the method proposed in Yu *et al.* [1] was extended, but for designing normal filters, two new methods were proposed, one giving higher-order filters with more degrees of freedom and one giving lower-order filters. By the numerical example used to illustrate the applications of the proposed methods, it is seen that singular filters should not be the only choice when considering uncertain singular systems.

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