3

Adaptive variable structure control

C.-J. Chien and L.-C. Fu

3.1 Introduction

In the past two decades, model reference adaptive control (MRAC) using only input/output measurements has evolved as one of the most soundly developed adaptive control techniques. Not only has the stability property been rigorously established [17], [19] but also the robustness issue due to unmodelled dynamics and input/output disturbance has been successfully solved [15], [18]. However, several limitations on MRAC remain to be relaxed, especially the problem of unpredictable transient response and tracking performance which has recently become one of the challenging research topics in the field of MRAC. A considerable amount of effort has been made to improve these schemes to obtain better control effects [6], [9], [11], [22]. One effort out of several is to try to incorporate the variable structure design (VSD) [9], [11] concept into the traditional model reference adaptive controller structure. Notably, Hsu and Costa [11] have first successfully proposed a plausible scheme in this line, which was then followed by a series of more general results [12], [13], [14]. Aside from those, Fu [9], [10] has taken up a different approach in placing the variable structure design in the overall resulting adaptive controller. An offspring of the work [9] and part of the work [12] include various versions of results respectively applied to SISO [20], [23], MIMO [2], [5], time-varying [4], decentralized [24] and affine nonlinear [3] systems.

It is well known that a main difficulty for the design of the variable structure MRAC system is the so-called general case when relative degree of the plant is greater than one. In this chapter, we present a new algorithm to solve the variable structure model reference adaptive control for a single input single output system with unmodelled dynamics and output disturbances. The design concept will be first introduced for relative degree-one plants and then be extended to the general case. Compared with the previous works, which used adaptive variable structure design or traditional robust adaptive approaches for the MRAC problem, this algorithm has the following special features:

(1) This control algorithm successfully applies the variable structure adaptive controller for the general case under robustness consideration.
(2) The control strategy using the concept of 'average control' rather than that of 'equivalent control' is thoroughly analysed.
(3) A systematic design approach is proposed and a new adaptation mechanism is developed so that the prior upper bounds on some appropriately defined but unavailable system parameters are not needed. It is shown that without any persistent excitation the global stability and robustness with asymptotic tracking performance can be guaranteed. The output tracking error can be driven to zero for relative degree-one plants and to a small residual set (whose size depends on the level of magnitude of some design parameter) for plants with any higher relative degree. Both results are achieved even when the unmodelled dynamic and output disturbance are present.
(4) If the aforementioned bounds on the system parameters are available by some means before controller design, then with a suitable choice of initial control parameters, the output tracking error can even be driven to zero in finite time for relative degree-one plants and to a small residual set exponentially for plants with any higher relative degree. It is noted that these bounds are usually assumed to be known before the construction of the variable structure controller or the robust adaptation law.

In order to make a comparison between the proposed adaptive variable structure scheme and the traditional approaches, some computer simulations are made to illustrate the differences of the tracking performance. The simulations will clearly demonstrate the excellent transient responses as well as tracking performance, which are almost never possible to achieve when traditional MRAC schemes are employed [19].

The theoretical framework in this chapter is developed based on Filippov's solution concept for a differential equation with discontinuous right-hand side [8]. In the subsequent discussions, the following notations will be used:

(1) $P(s)[u](t)$: denotes the filtered version of $u(t)$ with any proper or strictly proper transfer function $P(s)$.
(2) $|\cdot|$: denotes the absolute value of any scalar or the Euclidean norm of any vector or matrix.
(3) $\|(\cdot)_t\|_\infty = \sup_{\tau \le t} |(\cdot)(\tau)|$: denotes the truncated L_∞ norm of the argument function or vector.
(4) $\|P(s)\|_\infty$: denotes the H_∞ norm of the transfer function $P(s)$.

The chapter is organized as follows. In Section 3.2, we give the plant description, control objective and then derive the MRAC based error model. In Section 3.3, the adaptive variable structure controller for relative degree-one plants is proposed with stability and performance analysis. The extension to plants with relative degree greater than one is presented in Section 3.4. Section 3.5 gives simulation results to demonstrate the effectiveness of the adaptive variable structure controller. Finally, a conclusion is made in Section 3.6.

3.2 Problem formulation

3.2.1 Plant description and control objective

In this chapter, we consider the following SISO linear time-invariant plant described by the equation:

$$y_p(t) = P(s)\big(1 + \mu P_u(s)\big)[u_p](t) + d_o(t) \tag{3.1}$$

where $u_p(t)$ and $y_p(t)$ are plant input and plant output respectively, $\mu P_u(s)$ is the multiplicative unmodelled dynamics with some $\mu \in R^+$, and d_o is the output disturbance. Here, $P(s)$ represents the strictly proper rational transfer function of the nominal plant which is described by

$$P(s) = k_p \frac{n_p(s)}{d_p(s)} \tag{3.2}$$

where $n_p(s)$ and $d_p(s)$ are some monic coprime polynomials and k_p is the high frequency gain. Now suppose that the plant (3.1) is not precisely known but some prior knowledge about the transfer function may be available. The control objective is to design an adaptive variable structure control scheme such that the output $y_p(t)$ of the plant will track the output $y_m(t)$ of a linear time-invariant reference model described by

$$y_m(t) = M(s)[r_m](t) = k_m \frac{n_m(s)}{d_m(s)}[r_m](t) \tag{3.3}$$

where $M(s)$ is a stable transfer function and $r_m(t)$ is a uniformly bounded reference input. In order to achieve such an objective, we need some assumptions on the modelled part of the plant and the reference model as well as the unmodelled part of the plant. These assumptions are made in the following.

For the modelled part of the plant and reference model:

(A1) All the coefficients of $n_p(s)$ and $d_p(s)$ are unknown a priori, but the order of $P(s)$ and its relative degree are known to be n and ρ, respectively. Without loss of generality, we will assume that the order of $M(s)$ and its relative degree are also n and ρ, respectively.

(A2) The value of high frequency gain k_p is unknown, but its sign should be known. Without loss of generality, we will assume k_p, and hence k_m, are positive.

(A3) $P(s)$ is minimum phase, i.e. all its zeros lie in the open left half complex plane.

For the unmodelled part of the plant:

(A4) The unmodelled dynamics $P_u(s-k_1)$ is a strictly proper and stable transfer matrix such that $|D| < a_1, \|(P_u(s-k_1)s - D)(s+a_2)\|_\infty < a_1$, for some constants $a_1, a_2 > 0$, where $D = \lim_{s\to\infty} P_u(s)s$ and $\|X(s)\|_\infty \equiv \sup_{w\in R} |X(jw)|$ [15].

(A5) The output disturbance is differentiable and the upper bounds on $|d_o(t)|, \left|\frac{d}{dt} d_o(t)\right|$ exist.

Remark 3.1

- Minimum-phase assumption (A3) on the nominal plant $P(s)$ is to guarantee the internal stability since the model reference control involves the cancellation of the plant zeros. However, as commented by [15], this assumption does not imply that the overall plant (3.1) possesses the minimum-phase property.
- The latter part of assumption (A4) is simply to emphasize the fact that $P_u(s)$ are uncorrelated with μ in any case [16]. The reasons for assumption (A5) will be clear in the proof of Theorem 3.1 and that of Theorem 4.1.

3.2.2 MRAC based error model

Since the plant parameters are assumed to be unknown, a basic strategy from the traditional MRAC [17], [19] is now used to construct the error model between $y_p(t)$ and $y_m(t)$. Instead of applying the traditional MRAC technique, a new adaptive variable structure control will be given here in order to pursue better robustness and tracking performance. Let (3.1) be rewritten as

$$y_p(t) - d_o(t) = P(s)\Big[u_p + \mu P_u(s)[u_p]\Big](t) \triangleq P(s)[u_p + \bar{u}](t) \tag{3.4}$$

then from the traditional model reference control strategy [19], it can be shown that there exists $\Theta^* = [\theta_1^*, \ldots, \theta_{2n}^*]^\top \in R^{2n}$ such that if

$$D_b^*(s) = [\theta_1^*, \theta_2^*, \ldots, \theta_{n-1}^*]\frac{a(s)}{\lambda(s)}$$

$$D_f^*(s) = [\theta_n^*, \theta_{n+1}^*, \ldots, \theta_{2n-2}^*]\frac{a(s)}{\lambda(s)} + \theta_{2n-1}^*$$

where $a(s) = [1, s, \ldots, s^{n-2}]^\top$ and $\lambda(s)$ is an nth order monic Hurwitz polynomial, we have

$$1 - D_b^*(s) - D_f^*(s)P(s) = \theta_{2n}^* M^{-1}(s)P(s) \tag{3.5}$$

Applying both sides of (3.5) on $u_p + \bar{u}$, we have

$$u_p(t)+\bar{u}(t)-D_b^*(s)[u_p+\bar{u}](t)-D_f^*(s)[y_p-d_o](t)=\theta_{2n}^* M^{-1}(s)[y_p-d_o](t) \tag{3.6}$$

so that

$$y_p(t) - d_o(t) = M(s)\theta_{2n}^{*-1}\Big[u_p + \bar{u} - D_b^*(s)[u_p+\bar{u}] - D_f^*(s)[y_p - d_o]\Big](t) \tag{3.7}$$

Since

$$D_b^*(s)[u_p+\bar{u}](t) + D_f^*(s)[y_p-d_o](t) + \theta_{2n}^* r_m(t)$$

$$= \Theta^{*\mathsf{T}} \begin{bmatrix} \dfrac{a(s)}{\lambda(s)}[u_p+\bar{u}](t) \\ \dfrac{a(s)}{\lambda(s)}[y_p-d_o](t) \\ y_p(t)-d_o(t) \\ r_m(t) \end{bmatrix}$$

$$= \Theta^{*\mathsf{T}} \begin{bmatrix} \dfrac{a(s)}{\lambda(s)}[u_p](t) \\ \dfrac{a(s)}{\lambda(s)}[y_p](t) \\ y_p(t) \\ r_m(t) \end{bmatrix} - \Theta^{*\mathsf{T}} \begin{bmatrix} 0 \\ \dfrac{a(s)}{\lambda(s)}[d_o](t) \\ d_o(t) \\ 0 \end{bmatrix} + D_b^*(s)[\bar{u}](t)$$

$$\triangleq \Theta^{*\mathsf{T}} w(t) - \Theta^{*\mathsf{T}} w_{d_o}(t) + D_b^*(s)[\bar{u}](t) \tag{3.8}$$

we have

$$\begin{aligned} y_p(t) - d_o(t) &= M(s)\theta_{2n}^{*-1}[u_p - \Theta^{*\mathsf{T}} w + \Theta^{*\mathsf{T}} w_{d_o} + (1 - D_b^*(s))[\bar{u}] + \theta_{2n}^* r_m](t) \\ &= M(s)\theta_{2n}^{*-1}[u_p - \Theta^{*\mathsf{T}} w + \Theta^{*\mathsf{T}} w_{d_o} + \mu\Delta(s)[u_p] + \theta_{2n}^* r_m](t) \end{aligned} \tag{3.9}$$

where $\Delta(s) = (1 - D_b^*(s))P_u(s) = \left(1 - \dfrac{\theta_1^* + \ldots + \theta_{n-1}^* s^{n-2}}{\lambda(s)}\right)P_u(s)$. If we define the tracking error $e_0(t)$ as $y_p(t) - y_m(t)$, then the error model due to the unknown parameters, unmodelled dynamics and output disturbances can be readily found from (3.3) and (3.9) as follows:

$$e_0(t) \;=\; M(s)\theta_{2n}^{*-1}\left[u_p - \Theta^{*\mathsf{T}}w + \Theta^{*\mathsf{T}}w_{d_o} + \mu\Delta(s)[u_p]\right](t) + d_o(t) \qquad (3.10)$$

In the following sections, the new adaptive variable structure scheme is proposed for plants with arbitrary relative degree. However, the control structure is much simpler for relative degree-one plant, and hence in Section 3.3 we will first give a discussion for this class of plants. Based on the analysis for relative degree-one plants, the general case can then be presented in a more straightforward manner in Section 3.4.

3.3 The case of relative degree one

When $P(s)$ is relative degree one, the reference model $M(s)$ can be chosen to be strictly positive real (SPR) (Narendra and Annaswamy, 1988). The error model (3.10) can now be rewritten as

$$e_0(t) \;=\; M(s)\theta_{2n}^{*-1}\left[u_p - \Theta^{*\mathsf{T}}w + \Theta^{*\mathsf{T}}w_{d_o} + \theta_{2n}^{*}M^{-1}(s)[d_o] + \mu\Delta(s)[u_p]\right](t) \qquad (3.11)$$

In the error model (3.11), the terms $\Theta^{*\mathsf{T}}w, \Theta^{*\mathsf{T}}w_{d_o} + \theta_{2n}^{*}M^{-1}(s)[d_o]$ and $\mu\Delta(s)[u_p]$ are the uncertainties due to the unknown plant parameters, output disturbance, and unmodelled dynamics, respectively. Let (A_m, B_m, C_m) be any minimal realization of $M(s)\theta_{2n}^{*-1}$ which is SPR, then we can get the following state space representation of (3.11) as:

$$\dot{e}(t) = A_m e(t) + B_m(u_p(t) - \Theta^{*\mathsf{T}}w(t) + \Theta^{*\mathsf{T}}w_{d_o}(t) + \theta_{2n}^{*}M^{-1}(s)[d_o](t) + \mu\Delta(s)[u_p](t))$$
$$e_0(t) = C_m e(t) \qquad (3.12)$$

where the triplet (A_m, B_m, C_m) satisfies

$$P_m A_m + A_m^{\mathsf{T}} P_m \;=\; -2Q_m; \quad P_m B_m \;=\; C_m^{\mathsf{T}} \qquad (3.13)$$

for some $P_m = P_m^{\mathsf{T}} > 0$ and $Q_m = Q_m^{\mathsf{T}} > 0$.

The adaptive variable structure controller for relative degree-one plants is now summarized as follows:

(1) Define the regressor signal

$$w(t) = \left[\frac{a(s)}{\lambda(s)}[u_p](t), \frac{a(s)}{\lambda(s)}[y_p](t), y_p(t), r_m(t)\right]^{\mathsf{T}} = [w_1(t), w_2(t), \ldots, w_{2n}(t)]^{\mathsf{T}} \qquad (3.14)$$

and construct the normalization signal $m(t)$ [15] as the state of the following system:

$$\dot{m}(t) = -\delta_0 m(t) + \delta_1(|u_p(t)| + 1), \; m(0) > \frac{\delta_1}{\delta_0} \tag{3.15}$$

where $\delta_0, \delta_1 > 0$ and $\delta_0 + \delta_2 < \min(k_1, k_2)$ for some $\delta_2 > 0$. The parameter $k_2 > 0$ is selected such that the roots of $\lambda(s - k_2)$ lie in the open left half complex plane, which is always achievable.

(2) Design the control signal $u_p(t)$ as

$$u_p(t) = \sum_{j=1}^{2n}(-\text{sgn}(e_0 w_j)\theta_j(t)w_j(t)) - \text{sgn}(e_0)\beta_1(t) - \text{sgn}(e_0)\beta_2(t)m(t) \tag{3.16}$$

$$\text{sgn}(e_0) = \begin{cases} 1 & \text{if } e_0 > 0 \\ 0 & \text{if } e_0 = 0 \\ -1 & \text{if } e_0 < 0 \end{cases}$$

(3) The adaptation law for the control parameters is given as

$$\begin{aligned} \dot{\theta}_j(t) &= \gamma_j|e_0(t)w_j(t)|, \quad j = 1, \ldots, 2n \\ \dot{\beta}_1(t) &= g_1|e_0(t)| \\ \dot{\beta}_2(t) &= g_2|e_0(t)|m(t) \end{aligned} \tag{3.17}$$

where $\gamma_j, g_1, g_2 > 0$ are the adaptation gains and $\theta_j(0), \beta_1(0), \beta_2(0) > 0$ (in general, as large as possible) $j = 1, \ldots, 2n$.

The design concept of the adaptive variable structure controller (3.15) and (3.16) is simply to construct some feedback signals to compensate for the uncertainties because of the following reasons:

- By assumption (A5), it can be easily found that $|\Theta^{*\text{T}} w_{d_o}(t) + \theta^*_{2n} M(s)^{-1}[d_o](t)| \leq \beta^*_1$ for some $\beta^*_1 > 0$.
- With the construction of m, it can be shown [15] that $\mu\Delta(s)[u_p](t) \leq \beta^*_2 m(t)$, $\forall t \geq 0$ and for some constant $\beta^*_2 > 0$.

Now, we are ready to state our results concerning the properties of global stability, robust property, and tracking performance of our new adaptive variable structure scheme with relative degree-one system.

Theorem 3.1 (Global Stability, Robustness and Asymptotic Zero Tracking Performance) Consider the system (3.1) satisfying assumptions (A1)–(A5) with relative degree being one. If the control input is designed as in (3.15), (3.16) and the adaptation law is chosen as in (3.17), then there exists $\mu^* > 0$ such that for $\mu \in [0, \mu^*]$ all signals inside the closed loop system are bounded and the tracking error will converge to zero asymptotically.

Proof: Consider the Lyapunov function

$$V_a = \tfrac{1}{2}e^{\mathrm{T}}P_m e + \sum_{j=1}^{2n}\frac{1}{2\gamma_j}(\theta_j - |\theta_j^*|)^2 + \sum_{j=1}^{2}\frac{1}{2g_j}(\beta_j - \beta_j^*)^2$$

where P_m satisfies (3.13). Then, the time derivative of V_a along the trajectory (3.12) (3.17) will be

$$\begin{aligned}\dot{V}_a &= -e^{\mathrm{T}}Q_m e + e_0\left(u_p - \Theta^{*\mathrm{T}}w + \Theta^{*\mathrm{T}}w_{d_o} + \theta_{2n}^* M^{-1}(s)[d_o] + \mu\Delta(s)[u_p]\right)\\ &\quad + \sum_{j=1}^{2n}\frac{1}{\gamma_j}(\theta_j - |\theta_j^*|)\dot{\theta}_j + \sum_{j=1}^{2}\frac{1}{g_j}(\beta_j - \beta_j^*)\dot{\beta}_j\\ &\le -e^{\mathrm{T}}Q_m e - \sum_{j=1}^{2n}|e_0 w_j|(\theta_j - |\theta_j^*|) - |e_0|(\beta_1 - \beta_1^*) - |e_0|(\beta_2 - \beta_2^*)m\\ &\quad + \sum_{j=1}^{2n}\frac{1}{\gamma_j}(\theta_j - |\theta_j^*|)\dot{\theta}_j + \sum_{j=1}^{2}\frac{1}{g_j}(\beta_j - \beta_j^*)\dot{\beta}_j\\ &\le -q_m|e|^2\end{aligned}$$

for some constant $q_m > 0$. This implies that $e \in L_2 \cap L_\infty$ and $\theta_j, j = 1, \ldots, 2n, \beta_1, \beta_2, e_0 \in L_\infty$ and, hence, all signals inside the closed loop system are bounded owing to Lemma A in the Appendix. On the other hand, it can be concluded that $\dot{e} \in L_\infty$ by (3.12). Hence, $e \in L_2 \cap L_\infty$ and $\dot{e} \in L_\infty$ readily imply that e and e_0 will at least converge to zero asymptotically by Barbalat's lemma [19]. Q.E.D.

In Theorem 3.1, suitable integral adaptation laws are given to compensate for the unavailable knowledge of the bounds on $|\theta_j^*|$ and β_j^*. Theoretically, the adaptive variable structure controller will stabilize the closed loop system with guaranteed robustness and asymptotic zero tracking performance no matter what $\theta_j(0)$'s and $\beta_j(0)$'s are. However, according to the following Theorem 3.2, we will expect that positive and large values of $\theta_j(0), \beta_j(0)$ should result in better transient response and tracking performance, especially when $\theta_j(0) > |\theta_j^*|, \beta_j(0) > \beta_j^*$.

Theorem 3.2 (Finite-Time Zero Tracking Performance with High Gain Design) Consider the system set-up in Theorem 3.1. If $\theta_j(0) \ge |\theta_j^*|, \beta_j(0) \ge \beta_j^*$, then the output tracking error will converge to zero in finite time with all signals inside the closed loop system remaining bounded.

Proof Consider the Lyapunov function $V_b = \tfrac{1}{2}e^{\mathrm{T}}P_m e$ where P_m satisfies

(3.13). The time derivative of V_b along the trajectory (3.12) becomes

$$\begin{aligned}\dot{V}_b &= -e^{\mathrm{T}} Q_m e - \sum_{j=1}^{2n} |e_0 w_j|(\theta_j - |\theta_j^*|) - |e_0|(\beta_1 - \beta_1^*) - |e_0|(\beta_2 - \beta_2^*)m \\ &\leq -e^{\mathrm{T}} Q_m e \\ &\leq -k_3 V_b\end{aligned}$$

for some $k_3 > 0$ since $\theta_j(t) \geq |\theta_j^*|, \beta_j(t) \geq \beta_j^*, \forall t \geq 0$. This implies that e approaches zero at least exponentially fast. Furthermore, by the fact that

$$\begin{aligned}e_0\dot{e}_0 &= e_0\{C_m A_m e + C_m B_m(u_p - \Theta^{*\mathrm{T}} w + \Theta^{*\mathrm{T}} w_{d_o} + \theta_{2n}^* M^{-1}(s)[d_o] + \mu\Delta(s)[u_p])\} \\ &\leq k_4|e_0||e| - \sum_{j=1}^{2n} |e_0 w_j|(\theta_j - |\theta_j^*|) - |e_0|(\beta_1 - \beta_1^*) - |e_0|(\beta_2 - \beta_2^*)m \\ &\leq k_4|e_0||e| - |e_0|\left(\sum_{j=1}^{2n} |w_j|(\theta_j - |\theta_j^*|) + (\beta_1 - \beta_1^*) + (\beta_2 - \beta_2^*)m\right)\end{aligned}$$

where $k_4 = |C_m A_m|$, and that $|e|$ approaches zero at least exponentially fast, there exists a finite time $T_1 > 0$ such that $e_0\dot{e}_0 \leq -k_5|e_0|$ for all $t > T_1$ and for some $k_5 > 0$. This implies that the sliding surface $e_0 = 0$ is guaranteed to be reached in some finite time $T_2 > T_1 > 0$. Q.E.D.

Remark 3.2: Although theoretically only asymptotic zero tracking performance is achieved when the initial control parameters are arbitrarily chosen, it is encouraged to set the adaptation gains γ_j and g_j in (3.17) as large as possible. This is because the large adaptation gains will provide high adaptation speed and, hence, increase the control parameters to a suitable level of magnitude so as to achieve a satisfactory performance as quickly as possible. These expected results can be observed in the simulation examples.

3.4 The case of arbitrary relative degree

When the relative degree of (3.1) is greater than one, the controller design becomes more complicated than that given in Section 3.3. The main difference between the controller design of a relative degree-one system and a system with relative degree greater than one can be described as follows. When (3.1) is relative degree-one, the reference model can be chosen to be strictly positive real (SPR) [19]. Moreover, the control structure and its subsequent analysis of global stability, robustness and tracking performance are much simpler. On the contrary, when the relative degree of (3.1) is greater than one, the reference model $M(s)$ is no longer SPR so that the controller and the analysis technique in relative degree-one systems cannot be directly applied. In order to use the similar techniques given in Section 3.3, the adaptive variable structure controller is now designed systematically as follows:

(1) Choose an operator $L_1(s) = \ell_1(s)\ldots\ell_{\rho-1}(s) = (s+\alpha_1)\ldots(s+\alpha_{\rho-1})$ such that $M(s)L_1(s)$ is SPR and denote $L_i(s) = \ell_i(s)\ldots\ell_{\rho-1}(s), i = 2,\ldots,\rho-1, L_\rho(s) = 1$.
(2) Define augmented signal

$$y_a(t) = M(s)L_1(s)\left[u_1 - \frac{1}{L_1(s)}[u_p]\right](t)$$

and auxiliary errors

$$e_{a1}(t) = e_0(t) + y_a(t) \tag{3.18}$$

$$e_{a2}(t) = \frac{1}{\ell_1(s)}[u_2](t) - \frac{1}{F(\tau s)}[u_1](t) \tag{3.19}$$

$$e_{a3}(t) = \frac{1}{\ell_2(s)}[u_3](t) - \frac{1}{F(\tau s)}[u_2](t) \tag{3.20}$$

$$\vdots$$

$$e_{a\rho}(t) = \frac{1}{\ell_{\rho-1}(s)}[u_p](t) - \frac{1}{F(\tau s)}[u_{\rho-1}](t) \tag{3.21}$$

where $\frac{1}{F(\tau s)}[u_i](t)$ is the average control of $u_i(t)$ with $F(\tau s) = (\tau s+1)^2$, τ being small enough. In fact, $F(\tau s)$ can be any Hurwitz polynomial in τs with degree at least two and $F(0) = 1$. In the literature, $\frac{1}{F(\tau s)}$ is referred to as an *averaging filter*, which is obviously a low-pass filter whose bandwidth can be arbitrarily enlarged as $\tau \to 0$. In other words, if τ is smaller and smaller, the filter $\frac{1}{F(\tau s)}$ is flatter and flatter.

(3) Design the control signals u_p, u_i, and the bounding function m as follows:

$$u_1(t) = \sum_{j=1}^{2n}(-\text{sgn}\,(e_{a1}\xi_j)\theta_j(t)\xi_j(t)) - \text{sgn}\,(e_{a1})\beta_1(t) - \text{sgn}\,(e_{a1})\beta_2(t)m(t) \tag{3.22}$$

$$u_i(t) = -\text{sgn}\,(e_{ai})\left(\left|\frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}](t)\right| + \eta\right),\quad i = 2,\ldots,\rho \tag{3.23}$$

$$u_p(t) = u_\rho(t) \tag{3.24}$$

with $\eta > 0$ and

$$\xi(t) = \frac{1}{\ell_1(s)} \cdots \frac{1}{\ell_{\rho-1}(s)}[w](t) = \frac{1}{L_1(s)}[w](t)$$

The bounding function $m(t)$ is designed as the state of the system

$$\dot{m}(t) = -\delta_0 m(t) + \delta_1(|u_p(t)| + 1), \; m(0) > \frac{\delta_1}{\delta_0} \tag{3.25}$$

with $\delta_0, \delta_1 > 0$ and $\delta_0 + \delta_2 < \min(k_1, k_2, \alpha_1, \ldots, \alpha_{\rho-1})$ for some $\delta_2 > 0$.

(4) Finally, the adaptation law for the control parameters $\theta_j, j = 1, \ldots, 2n$ and β_1, β_2 are given as follows:

$$\dot{\theta}_j(t) = \gamma_j |e_{a1}(t)\xi_j(t)|, \quad j = 1, \ldots, 2n \tag{3.26}$$

$$\dot{\beta}_1(t) = g_1 |e_{a1}(t)| \tag{3.27}$$

$$\dot{\beta}_2(t) = g_2 |e_{a1}(t)| m(t) \tag{3.28}$$

with $\theta_j(0) > 0, \beta_j(0) > 0$ and $\gamma_j > 0, g_j > 0$.

In the following discussions, the construction of feedback signals $\xi(t), m(t)$ and the controller (3.22) (3.23) will be clear.

In order to analyse the proposed adaptive variable structure controller, we first rewrite the error model (3.10) as follows:

$$\begin{aligned} e_0(t) &= M(s)[u_p - \theta_{2n}^{*-1}\Theta^{*\mathrm{T}} w + \theta_{2n}^{*-1}\Theta^{*\mathrm{T}} w_{d_o} + \theta_{2n}^{*-1}\mu\Delta(s)[u_p] \\ &\quad + (\theta_{2n}^{*-1} - 1)u_p](t) + d_o(t) \\ &= M(s)L_1(s)\left[\frac{1}{L_1(s)}[u_p] - -\theta_{2n}^{*-1}\Theta^{*\mathrm{T}}\xi + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\Theta^{*\mathrm{T}} w_{do} + \theta_{2n}^{*} M^{-1}(s)[d_o]] \right. \\ &\quad \left. + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\mu\Delta(s)[u_p] + (1 - \theta_{2n}^{*})u_p]\right](t) \end{aligned} \tag{3.29}$$

Now, according to the design of the above auxiliary error (3.18) and error model (3.29), we can readily find that e_{a1} always satisfies

$$\begin{aligned} e_{a1}(t) &= M(s)L_1(s)\left[u_1 - \theta_{2n}^{*-1}\Theta^{*\mathrm{T}}\xi + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\Theta^{*\mathrm{T}} w_{do} + \theta_{2n}^{*} M^{-1}(s)[d_o]] \right. \\ &\quad \left. + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\mu\Delta(s)[u_p] + (1 - \theta_{2n}^{*})u_p]\right](t) \end{aligned} \tag{3.30}$$

It is noted that the auxiliary error e_{a1} is now explicitly expressed as the output term of a linear system with SPR transfer function $M(s)L_1(s)$ driven by some uncertain signals due to unknown parameters, output disturbances, unmodelled dynamics and unknown high frequency gain sign.

Remark 4.1 The construction of the adaptive variable structure controller (3.22) is now clear since the following facts hold:

- Since $\frac{\theta_{2n}^{*-1}}{L_1(s)}\left[\Theta^{*\mathrm{T}} w_{do} + \theta_{2n}^* M^{-1}(s)[d_o]\right](t)$ is uniformly bounded due to (A5), we have

$$\left|\frac{\theta_{2n}^{*-1}}{L_1(s)}\left[\Theta^{*\mathrm{T}} w_{do} + \theta_{2n}^* M^{-1}(s)[d_o]\right](t)\right| \le \beta_1^* \tag{3.31}$$

  for some β_1^*.
- With the design of the bounding function $m(t)$ (3.25), it can be shown that

$$\left|\frac{\theta_{2n}^{*-1}}{L_1(s)}\left[\mu\Delta(s)[u_p] + (1-\theta_{2n}^*)u_p\right](t)\right| \le \beta_2^* m(t) \tag{3.32}$$

  for some $\beta_2^* > 0$.

The results described in Remark 4.1 show that the similar techniques for the controller design of a relative degree-one system can now be used for auxiliary error e_{a1}. But what happens to the other auxiliary errors $e_{a2}, \dots, e_{ap}$, especially the real output error e_0 as concerned? In Theorem 4.1, we summarize the main results of the systematically designed adaptive variable structure controller for plants with relative degree greater than one.

Theorem 4.1 (Global Stability, Robustness and Asymptotic Tracking Performance) Consider the nonlinear time-varying system (3.1) with relative degree $\rho > 1$ satisfying (A1)–(A5). If the controller is designed as in (3.18)–(3.25) and parameter update laws are chosen as in (3.26)–(3.28), then there exists $\tau^* > 0$ and $\mu^* > 0$ such that for all $\tau \in (0, \tau^*)$ and $\mu \in (0, \mu^*)$, the following facts hold:

(i) all signals inside the closed-loop system remain uniformly bounded;
(ii) the auxiliary error e_{a1} converges to zero asymptotically;
(iii) the auxiliary errors $e_{ai}, i = 2, \dots, \rho$, converge to zero at some finite time;
(iv) the output tracking error e_0 will converge to a residual set asymptotically whose size is a class K function of the design parameter τ.

Proof The proof consists of three parts.

Part I Prove the boundedness of e_{ai} and $\theta_1, \dots, \theta_{2n}, \beta_1, \beta_2$.

<u>Step 1</u> First, consider the auxiliary error e_{a1} which satisfies (3.30). Since

$M(s)L_1(s)$ is SPR, we have the following realization of (3.20)

$$\dot{e}_1 = A_1 e_1 + B_1\left(u_1 - \theta_{2n}^{*-1}\Theta^{*\mathrm{T}}\xi + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\Theta^{*\mathrm{T}} w_{d_o} + \theta_{2n}^{*} M^{-1}(s)[d_o]] + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\mu\Delta(s)[u_p] + (1-\theta_{2n}^{*})u_p]\right)$$

$$e_{a1} = C_1 e_1 \tag{3.33}$$

with $P_1A_1 + A_1^{\mathrm{T}}P_1 = -2Q_1$, $P_1B_1 = C_1^{\mathrm{T}}$ for some $P_1 = P_1^{\mathrm{T}} > 0$ and $Q_1 = Q_1^{\mathrm{T}} > 0$. Given a Lyapunov function as follows:

$$V_1 = \tfrac{1}{2}e_1^{\mathrm{T}}P_1e_1 + \sum_{j=1}^{2n}\frac{1}{2\gamma_j}\left(\theta_j - \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|\right)^2 + \sum_{j=1}^{2}\frac{1}{2g_j}(\beta_j - \beta_j^*)^2 \tag{3.34}$$

it can be shown by using (3.32) and (3.31) that

$$\begin{aligned}
\dot{V}_1 &= -e_1^{\mathrm{T}}Q_1e_1 + e_{a1}\left(u_1 - \theta_{2n}^{*-1}\Theta^{*\mathrm{T}}\xi + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\Theta^{*\mathrm{T}} w_{d_o} + \theta_{2n}^{*} M^{-1}(s)[d_o]] + \frac{\theta_{2n}^{*-1}}{L_1(s)}[\mu\Delta(s)[u_p] + (1-\theta_{2n}^{*})u_p]\right) \\
&\quad + \sum_{j=1}^{2n}\frac{1}{\gamma_j}\left(\theta_j - \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|\right)\dot{\theta}_j + \sum_{j=1}^{2}\frac{1}{g_j}(\beta_j - \beta_j^*)\dot{\beta}_j \\
&\le -e_1^{\mathrm{T}}Q_1e_1 - \sum_{j=1}^{2n}|e_{a1}\xi_j|\left(\theta_j - \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|\right) - |e_{a1}|(\beta_1 - \beta_1^*) - |e_{a1}|(\beta_2 - \beta_2^*)m \\
&\quad + \sum_{j=1}^{2n}\frac{1}{\gamma_j}\left(\theta_j - \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|\right)\dot{\theta}_j + \sum_{j=1}^{2}\frac{1}{g_j}(\beta_j - \beta_j^*)\dot{\beta}_j \\
&= -e_1^{\mathrm{T}}Q_1e_1 \\
&\le -q_1|e_1|^2
\end{aligned}$$

for some $q_1 > 0$ if the controller in (3.22) and update laws in (3.26)–(3.28) are given. This implies that $e_1, \theta_1, \ldots, \theta_{2n}, \beta_1, \beta_2 \in L_\infty$ and $e_{a1} \in L_2 \cap L_\infty$.

<u>Step 2</u> From (3.19)–(3.21), we can find that $e_{a2}, \ldots, e_{a\rho}$ satisfy

$$\dot{e}_{a2} = -\alpha_1 e_{a2} + u_2 - \frac{\ell_1(s)}{F(\tau s)}[u_1]$$

$$\vdots$$

$$\dot{e}_{a\rho} = -\alpha_{\rho-1} e_{a\rho} + u_\rho - \frac{\ell_{\rho-1}(s)}{F(\tau s)}[u_{\rho-1}]$$

Now by the following facts that for $i = 2, \ldots, \rho$:

$$\begin{aligned}
\frac{d}{dt}\left(\tfrac{1}{2}e_{ai}^2\right) &= e_{ai}\dot{e}_{ai} \\
&= e_{ai}\left(-\alpha_{i-1}e_{ai} + u_i - \frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}]\right) \\
&= -\alpha_{i-1}e_{ai}^2 + e_{ai}\left\{-sgn(e_{ai})\left(\left|\frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}]\right| + \eta\right) - \frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}]\right\}
\end{aligned}$$

or

$$\frac{d}{dt}|e_{ai}| \le -\alpha_{i-1}|e_{ai}| - \eta \tag{3.35}$$

when $|e_{ai}| \neq 0$. This implies that e_{ai} will converge to zero after some finite time $T > 0$.

Part II Prove the boundedness of all signals inside the closed loop system.

Define $\bar{e}_{ai} = M(s)L_{i-1}(s)[e_{ai}], i = 2, \ldots, \rho$ and $E_a = e_{a1} + \bar{e}_{a2} + \cdots + \bar{e}_{a\rho}$ which is uniformly bounded due to the boundedness of e_{ai}. Then, from (3.18)–(3.21), we can derive that

$$\begin{aligned}
E_a &= e_0 + M(s)L_1(s)\left[u_1 - \frac{1}{L_1(s)}[u_\rho]\right] \\
&\quad + M(s)L_1(s)\left[\frac{1}{\ell_1(s)}[u_2] - \frac{1}{F(\tau s)}[u_1]\right] \\
&\quad + M(s)L_2(s)\left[\frac{1}{\ell_2(s)}[u_3] - \frac{1}{F(\tau s)}[u_2]\right] \\
&\quad \vdots \\
&\quad + M(s)L_{\rho-1}(s)\left[\frac{1}{\ell_{\rho-1}(s)}[u_\rho] - \frac{1}{F(\tau s)}[u_{\rho-1}]\right] \\
&= e_0 + \left(1 - \frac{1}{F(\tau s)}\right)M(s)L_1(s)\left[u_1 + \frac{1}{\ell_1(s)}[u_2] + \cdots + \frac{1}{\ell_1(s)\ldots\ell_{\rho-2}(s)}[u_{\rho-1}]\right] \\
&\triangleq e_0 + R
\end{aligned} \tag{3.36}$$

Now, since $\|(u_i)_t\|_\infty \le k_6\|(e_0)_t\|_\infty + k_6, i = 1, \ldots, \rho - 1$ for some $k_6 > 0$ by Lemma A in the appendix, it can be easily found that

$$\left\|\left(u_1 + \frac{1}{\ell_1(s)}[u_2] + \cdots + \frac{1}{\ell_1(s)\ldots\ell_{\rho-2}(s)}[u_{\rho-1}]\right)_t\right\|_\infty \le k_7\|(e_0)_t\|_\infty + k_7$$

for some $k_7 > 0$. Furthermore, since the H_∞ norm of $\|\frac{1}{s}(1 - \frac{1}{F(\tau s)})\|_\infty = 2\tau$ and $\|sM(s)L_1(s)\|_\infty = k_8$ for some $k_8 > 0$, we can conclude that

$$\|(R)_t\|_\infty \le \left\|\frac{1}{s}\left(1 - \frac{1}{F(\tau s)}\right)\right\|_\infty \left\|sM(s)L_1(s)\right\|_\infty (k_7\|(e_0)_t\|_\infty + k_7)$$
$$\le \tau(k_9\|(e_0)_t\|_\infty + k_9)$$

for some $k_9 > 0$. Now from (3.36) we have

$$\|(e_0)_t\|_\infty \le \|(E_a)_t\|_\infty + \|(R)_t\|_\infty \le \|(E_a)_t\|_\infty + \tau(k_9\|(e_0)_t\|_\infty + k_9)$$

which implies that there exists a $\tau^* > 0$ such that $1 - \tau^* k_9 > 0$ and for all $\tau \in (0, \tau^*)$:

$$\|(e_0)_t\|_\infty \le \frac{\|(E_a)_t\|_\infty + \tau k_9}{1 - \tau k_9} \tag{3.37}$$

Combining Lemma A and (3.37), we readily conclude that all signals inside the closed loop system remain uniformly bounded.

Part III: Investigate the tracking performance of e_{a1} and e_0.

Since all signals inside the closed loop system are uniformly bounded, we have

$$e_{a1} \in L_2 \cap L_\infty, \quad \dot{e}_{a1} \in L_\infty$$

Hence, by Barbalat's lemma, e_{a1} approaches zero asymptotically and $E_a = e_{a1} + \bar{e}_{a2} + \cdots + \bar{e}_{ap}$ also approaches zero asymptotically. Now, from the fact of (3.37) and E_a approaching zero, it is clear that e_0 will converge to a small residual set whose size depends on the design parameter τ. Q.E.D.

As discussed in Theorem 3.2, if the initial choices of control parameters $\theta_j(0), \beta_j(0)$ satisfy the high gain conditions $\theta_j(0) \ge \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|$ and $\beta_j(0) \ge \beta_j^*$, then, by using the same argument given in the proof of Theorem 3.2, we can guarantee the exponential convergent behaviour and finite-time tracking performance of all the auxiliary errors e_{ai}. Since e_{ai} reaches zero in some finite time and $E_a = e_{a1} + \bar{e}_{a2} + \cdots + \bar{e}_{ap}$, it can be concluded that E_a converges to zero exponentially and e_0 converges to a small residual set whose size depends on the design parameter τ. We now summarize the results in the following Theorem 4.2.

Theorem 4.2: (Exponential Tracking Performance with High Gain Design) Consider the system set-up in Theorem 4.1. If the initial value of control parameters satisfy the high gain conditions $\theta_j(0) \ge \left|\frac{\theta_j^*}{\theta_{2n}^*}\right|$ and $\beta_j(0) \ge \beta_j^*$, then there exists a τ^* and μ^* such that for all $\tau \in (0, \tau^*]$ and $\mu \in (0, \mu^*]$, the following facts hold:

(i) all signals inside the closed loop system remain bounded;

(ii) the auxiliary errors $e_{ai}, i = 1, \ldots, \rho$, converge to zero in finite time;
(iii) the output tracking errors e_0 will converge to a residual set exponentially whose size depends on the design parameter τ.

Remark 4.2: It is well known that the chattering behaviour will be observed in the input channel due to variable structure control, which causes the implementation problem in practical design. A remedy to the undesirable phenomenon is to introduce the boundary layer concept. Take the case of relative degree one, for example, the practical redesign of the proposed adaptive variable structure controller by using boundary layer design is now stated as follows:

$$u_p(t) = \sum_{j=1}^{2n}\left(-\pi(e_0 w_j)\theta_j(t)w_j(t)\right) - \pi(e_0)\beta_1(t) - \pi(e_0)\beta_2(t)m(t) \tag{3.38}$$

$$\pi(e_0) = \begin{cases} \text{sgn}\,(e_0) & \text{if } |e_0| > \varepsilon \\ \dfrac{e_0}{\varepsilon} & \text{if } |e_0| \le \varepsilon \end{cases}$$

for some small $\epsilon > 0$. Note that $\pi(e_0)$ is now a continuous function. However, one can expect that the boundary layer design will result in bounded tracking error, i.e. e_0 cannot be guaranteed to converge to zero. This causes the parameter drift in parameter adaptation law. Hence, a leakage term is added into the adaptation law as follows:

$$\begin{aligned} \dot{\theta}_j(t) &= \gamma_j|e_0(t)w_j(t)| - \sigma\theta_j(t),\ j = 1, \ldots, 2n \\ \dot{\beta}_1(t) &= g_1|e_0(t)| - \sigma\beta_1(t) \\ \dot{\beta}_2(t) &= g_2|e_0(t)|m(t) - \sigma\beta_2(t) \end{aligned} \tag{3.39}$$

for some $\sigma > 0$.

3.5 Computer simulations

The adaptive variable structure scheme is now applied to the following unstable plant with unmodelled dynamics and output disturbances:

$$y_p(t) = \frac{8}{s^3 + s^2 + s - 2}\left(1 + 0.01\frac{1}{s+10}\right)[u_p](t) + 0.05\sin(5t)$$

Since the nominal plant is relative degree three, we choose the following steps to design the adaptive variable structure controller:

- reference model and reference input:

$$M(s) = \frac{8}{(s+2)^3}$$

$$r_m(t) = \begin{cases} 2 & \text{if } t < 5 \\ -2 & \text{if } 5 \leq t < 10 \end{cases}$$

- design parameters:

$$L_1(s) = \ell_1(s)\ell_2(s), \ell_1(s) = s+1, \ell_2(s) = s+2$$

$$\lambda(s) = (s+1)^2$$

$$F(\tau s) = \left(\frac{1}{60}s + 1\right)^2$$

- augmented signal and auxiliary errors:

$$y_a(t) = M(s)L_1(s)\left[u_1 - \frac{1}{L_1(s)}[u_p]\right](t)$$

$$e_{a1}(t) = e_0(t) + y_a(t)$$

$$e_{a2}(t) = \frac{1}{\ell_1(s)}[u_2](t) - \frac{1}{F(\tau s)}[u_1](t)$$

$$e_{a3}(t) = \frac{1}{\ell_2(s)}[u_3](t) - \frac{1}{F(\tau s)}[u_2](t)$$

- controller:

$$u_1(t) = \sum_{j=1}^{6}\left(-\text{sgn}\,(e_{a1}\xi_j)\theta_j(t)\xi_j(t)\right) - \text{sgn}\,(e_{a1})\beta_1(t) - \text{sgn}\,(e_{a1})\beta_2(t)m(t)$$

$$u_i(t) = -\text{sgn}\,(e_{ai})\left(\left|\frac{\ell_{i-1}(s)}{F(\tau s)}[u_{i-1}](t)\right| + 1\right), \quad i = 2, 3$$

$$u_p(t) = u_\rho(t)$$

$$\dot{m}(t) = -m(t) + 0.005(|u_p(t)| + 1), \quad m(0) = 0.2$$

- adaptation law:

$$\dot{\theta}_j(t) = \gamma_j|e_{a1}(t)\xi_j(t)|, \quad j = 1, \ldots, 6$$

$$\dot{\beta}_1(t) = g_1|e_{a1}(t)|$$

$$\dot{\beta}_2(t) = g_2|e_{a1}(t)|m(t)$$

Three simulation cases are studied extensively in this example in order to verify

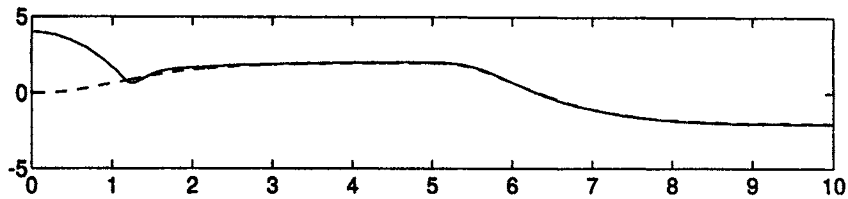

Figure 3.1 $y_p(-), y_m(--)$, *time (sec)*

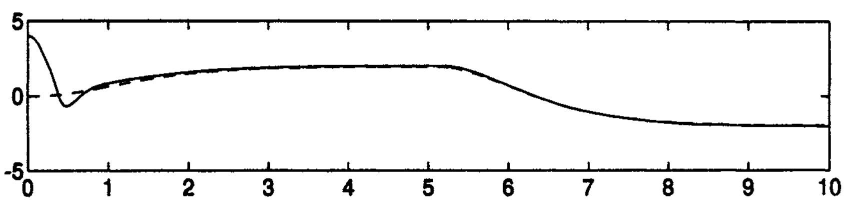

Figure 3.2 $y_p\ (-), ym\ (--)$, *time (sec)*

all the theoretical results and corresponding claims. All the cases will assume that there are initial output error $y_p(0) - y_m(0) = 4$.

(1) In the first case, we arbitrarily choose the initial control parameters as

$$\theta_j(0) = 0.1, \quad j = 1, \ldots, 6$$
$$\beta_j(0) = 0.1, \quad j = 1, 2$$

and set all the adaptation gains $\gamma_j = g_j = 0.1$. As shown in Figure 3.1 (the time trajectories of y_p and y_m), the global stability, robustness, and asymptotic tracking performance are achieved.

(2) In the second case, we want to demonstrate the effectiveness of a proper choice of $\theta_j(0)$ and $\beta_j(0)$ and repeat the previous simulation case by increasing the values of the controller parameters to be

$$\theta_j(0) = 1, \quad j = 1, \ldots, 6$$
$$\beta_j(0) = 1, \quad j = 1, 2$$

The better transient and tracking performance between y_p and y_m can now be observed in Figure 3.2.

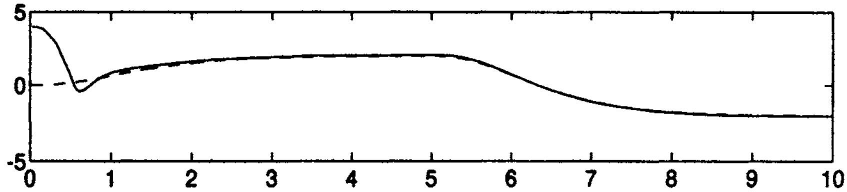

Figure 3.3 y_p (–), y_m; (– –), *time (sec)*

(3) As commented in Remark 3.2, if there is no easy way to estimate the suitable initial control parameters $\theta_j(0)$ and $\beta_j(0)$ like those in the second simulation case, it is suggested to use large adaptation gains in order to increase the adaptation rate of control parameters such that the nice transient and tracking performance as described in case 2 can be retained to some extent. Hence, in this case, we use the initial control parameters as in case 1 but set all the adaptation gains to $\gamma_j = g_j = 1$. The expected results are now shown in Figure 3.3, where rapid increase of control parameters do lead to satisfactory transient and tracking performance.

3.6 Conclusion

In this chapter, a new adaptive variable structure scheme is proposed for model reference adaptive control problems for plants with unmodelled dynamic and output disturbance. The main contribution of the chapter is the complete version of adaptive variable structure design for solving the robustness and performance of the traditional MRAC problem with arbitrary relative degree. A detailed analysis of the closed-loop stability and tracking performance is given. It is shown that without any persistent excitation the output tracking error can be driven to zero for relative degree-one plants and driven to a small residual set asymptotically for plants with any higher relative degree. Furthermore, under suitable choice of initial conditions on control parameters, the tracking performance can be improved, which are hardly achievable by the traditional MRAC schemes, especially for plants with uncertainties.

Appendix

Lemma A Consider the controller design in Theorem 3.1 or 4.1. If the control parameters $\theta_j(t), j = 1, \ldots, 2n, \beta_1(t)$ and $\beta_2(t)$ are uniformly bounded $\forall t$, then there exists $\mu^* > 0$ such that $u_p(t)$ satisfies

$$\|(u_p)_t\|_\infty \le \kappa\|(e_0)_t\|_\infty + \kappa \tag{A.1}$$

with some positive constant $\kappa > 0$.

Proof Consider the plant (3.1) which is rewritten as follows:

$$y(t) - d_o(t) = P(s)(1 + \mu P_u(s))[u_p](t) \tag{A.2}$$

Let $f(s)$ be the Hurwitz polynomial with degree $n - \rho$ such that $f(s)P(s)$ is proper, and hence, $f^{-1}(s)P^{-1}(s)$ is proper stable since $P(s)$ is minimum phase by assumption (A3). Then

$$y(t) - d_o(t) = P(s)f(s)f^{-1}(s)(1 + \mu P_u(s))[u_p](t) \tag{A.3}$$

which implies that

$$f^{-1}(s)P^{-1}(s)[y - d_o](t) - \mu f^{-1}(s)P_u(s)[u_p](t) = f^{-1}(s)[u_p](t) \triangleq u^*(t) \tag{A.4}$$

Since $f^{-1}(s)P^{-1}(s)$ and $f^{-1}(s)P_u(s)$ are proper or strictly proper stable, we can find by small gain theorem [7] that there exists $\mu^* > 0$ such that

$$\|(u^*)_t\|_\infty \le \kappa\|(y_p)_t\|_\infty + \kappa \le \kappa\|(e_0)_t\|_\infty + \kappa \tag{A.5}$$

for some suitably defined $\kappa > 0$ and for all $\mu \in [0, \mu^*]$. Now if we can show that

$$\|(u_p)_t\|_\infty \le \kappa\|(u^*)_t\|_\infty + \kappa \tag{A.6}$$

for some $\kappa > 0$, then (A.1) is achieved. By using Lemma 2.8 in [19], the key point to show the boundedness between u_p and u^* in (A.6) is the growing behaviour of signal u_p. The above statement can be stated more precisely as follows: if u_p satisfies the following requirement

$$|u_p(t_1)| \ge c|u_p(t_1 + T)| \tag{A.7}$$

where t_1 and $t_1 + T$ are the time instants defined as

$$[t_1, t_1 + T] \subset \Omega = \{t \mid |u_p| = \|(u_p)_t\|_\infty\} \tag{A.8}$$

and c is a constant $\in (0, 1)$, then u_p will be bounded by u^*, i.e. (A.6) is achieved. Now in order to establish (A.7) and (A.8), let (A_p, B_p, C_p) and (Λ, B) be the state space realizations of $P(s)(1 + \mu P_u(s))$ and $\dfrac{a(s)}{\lambda(s)}$ respectively. Also define $S = [x_p^{\mathrm{T}}, w_1^{\mathrm{T}}, w_2^{\mathrm{T}}, m]^{\mathrm{T}}$. Then, using the augmented system

$$\begin{bmatrix} \dot{x}_p \\ \dot{w}_1 \\ \dot{w}_2 \\ \dot{m} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ BC_p & 0 & \Lambda & 0 \\ 0 & 0 & 0 & -\delta_0 \end{bmatrix} \begin{bmatrix} x_p \\ w_1 \\ w_2 \\ m \end{bmatrix} + \begin{bmatrix} B_p u_p \\ Bu_p \\ Bd_o \\ \delta_1 |u_p| + 1 \end{bmatrix}$$

Since d_o is uniformly bounded, we can easily show according to the control design (3.16) or (3.24) that there exists κ such that

$$|\dot{S}| \leq \kappa \|(S)_t\|_\infty + \kappa$$

This means that S is regular [21] so that x_p, w_1, w_2, m, y_p and u_p will grow at most exponentially fast (if unbounded), which in turn guarantees (A.7) and (A.8) by Lemma 2.8 in [19]. This completes our proof. Q.E.D.

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