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Wei-Song Lin [a]
[a] Department of Electrical Engineering, and Jing Ling Industrial Research Institute, National Taiwan University, Taipei, Taiwan, Republic of China.

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An algorithm for computing the minimum-time regulator of multivariable discrete linear systems

WEI-SONG LIN†

This paper constructs an algorithm in general form for computing time-invariant feedback laws which assume deadbeat regulation in minimum time of multivariable discrete-time linear systems. The algorithm simply assumes that the system is irreducible. The key idea is the division of the state space into sub-statespaces according to the time needed to reach the equilibrium state. Based on the spans of all sub-statespaces, iterative formulae for computing all the feedback gains are obtained. The formulae can be programmed easily, and the transient response can be shaped by choosing different sets of feedback gains. An example is worked out to demonstrate the feasibility of the proposed method.

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1. Introduction

The deadbeat strategy is unique to discrete-time systems: there is no corresponding feature for continuous-time systems. Several works have shown the possibility of obtaining deadbeat performance in discrete-time linear systems and the properties of deadbeat controllers have also been investigated (Cadzow 1968, Farison and Fu 1970, Kučera 1971, Leden 1977, Pachter 1977, Sebakhy and Abdel-Moneim 1979). By assuming the state matrix to be non-singular, a direct method of computing the deadbeat controller for multivariable discrete-time linear systems was obtained (Leden 1977). Without the minimum-time limitation, the deadbeat problems were solved by Pachter (1977). On the assumption that the control matrix is of full rank, a method for computing generalized control form of any controllable pair was achieved by Sebakhy (1979).

The purpose of this paper is to present an iterative algorithm for computing the feedback laws which assure deadbeat regulation in minimum finitetime for multivariable discrete-time linear systems. The only assumption is that the state-space description is irreducible. In § 2 a statement of deadbeat regulation by time-invariant feedback gain matrix is given. The formulae for computing the feedback law is derived in § 3. An example is worked out in § 4 to illustrate the feasibility of the proposed method.

2. Statement of the problem

Consider a multivariable discrete-time linear system governed by the vector difference equation

$$X(k+1) = AX(k) + BU(k) \tag{1}$$

$$U(k) = HX(k) \tag{2}$$

where $X(k)$ is an n-dimensional state vector at the kth iteration, $U(k)$ is an

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† Department of Electrical Engineering, and Jing Ling Industrial Research Institute, National Taiwan University, Taipei, Taiwan, Republic of China.

r-dimensional control vector at the kth iteration, A is the constant $(n \times n)$ system matrix, B is the constant $(n \times r)$ control matrix, and H is the $(r \times n)$ constant feedback gain matrix. The only assumption will be that system (1) is an irreducible n-dimensional system (Swisher 1983). Thus the system is controllable and the following properties hold for the pair (A, B):

$$\operatorname{rank}\,[B\ AB\ \dots\ A^{n-1}\ B] = n \tag{3}$$

and

$$\operatorname{rank}\,[A\ B] = n \tag{4}$$

The controllability index is defined as the minimum number, m, such that

$$\operatorname{rank}\,[B\ AB\ \dots\ A^{m-1}\ B] = n \tag{5}$$

Obviously, for multi-input systems, m is no larger than the dimension, n, of the system.

The problem of deadbeat regulation in minimum time is the following. For the multivariable system (1) and the feedback law (2) compute the constant feedback gain matrix H such that the system can be driven by unbounded control inputs from any initial state, $X(0)$, to the equilibrium state X_e in deadbeat mode, or

$$(A + BH)^m X(0) = X_e \tag{6}$$

with the number of control periods m being the controllability index of the system. Without loss of generality, the equilibrium state will be assumed to be the zero state in the following sections.

3. Derivation of the algorithm

Let $R_{(i)}$, $i = 0, 1, 2, \dots, m$ represent sub-statespaces such that the minimum time required to bring any state X_i, $X_i \in R_{(i)}$ to the equilibrium, $R_{(0)} = X_e$, is i control periods. Let S_i be an $(n \times n)$ matrix such that the columns of S_i span $R_{(i)}$, or $R_{(i)} = \operatorname{span}(S_i)$. Then any state X_i, $X_i \in R_{(i)}$ can be represented as a linear combination of the columns of S_i or $X_i = S_i Y_i$ where Y_i is an $(n \times 1)$ matrix.

Since, for completely state-controllable systems, any initial state can be brought to the equilibrium in no more than m control periods, therefore the whole n-dimensional state space, R^n, is the union of the sub-statespaces $R_{(i)}$, $i = 0, 1, \dots, m$. Consequently, we obtain $R^n = \operatorname{span}(S_0, S_1, \dots, S_m)$ or

$$\operatorname{rank}\,[S_1\ S_2\ \dots\ S_m] = n \tag{7}$$

Furthermore, for the closed-loop system, any $X_i = S_i Y_i$, $i = 1, 2, \dots, m$ can be driven to the zero state in no more than i control periods, therefore we obtain

$$(A + BH)^i X_i = (A + BH)^i S_i Y_i = \mathbf{0}, \quad \text{for } i = 1, 2, \dots, m \tag{8}$$

Since (8) can be satisfied by any Y_i, $i = 1, 2, \dots, m$ it follows that

$$(A + BH)^i S_i = \mathbf{0}, \quad \text{for } i = 1, 2, \dots, m \tag{9}$$

For a specified value of i, the above equation can be written for i and $i - 1$ as

$$(A + BH)^i S_i = (A + BH)^{i-1}\{(A + BH)S_i\} = \mathbf{0} \tag{10}$$

and

$$(A + BH)^{i-1} S_{i-1} = \mathbf{0} \tag{11}$$

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Comparing the second term of (10) to (11) we may form reasonably the relationships

$$(A+BH)S_i = S_{i-1}, \quad \text{for } i = 1, 2, \ldots, m \tag{12}$$

where S_0 is an $(n \times n)$ zero matrix. In accordance with matrix operations, (12) can be rewritten as

$$\begin{aligned}(A+BH)S_i &= [A\ B]\begin{bmatrix} I_n \\ H \end{bmatrix} S_i = [A\ B]\begin{bmatrix} S_i \\ HS_i \end{bmatrix} \\ &= S_{i-1}, \quad \text{for } i = 1, 2, \ldots, m\end{aligned} \tag{13}$$

It has been assumed in (4) that the $n \times (n+r)$ matrix $[A\ B]$ is of full rank n. Therefore $[A\ B]$ has a right inverse matrix $[A\ B]^{RI}$ such that $[A\ B][A\ B]^{RI} = I_n$ (Cadzow and Martens 1970). In general, the right inverse of $[A\ B]$ is not unique; one of them is

$$[A\ B]^{RI} = [A\ B]^t\{[A\ B][A\ B]^t\}^{-1} \tag{14}$$

Then a general solution of (13) is found to be

$$\begin{aligned}\begin{bmatrix} S_i \\ HS_i \end{bmatrix} &= [A\ B]^t\{[A\ B][A\ B]^t\}^{-1}S_{i-1} \\ &\quad + \{I_{n+r} - [A\ B]^t\{[A\ B][A\ B]^t\}^{-1}[A\ B]\}Z_{i-1} \quad \text{for } i = 1, 2, \ldots, m\end{aligned} \tag{15}$$

where Z_{i-1} is any $(n+r) \times n$ matrix (Lin *et al.* 1983). Since $S_0 = \mathbf{0}$, a non-trivial solution of $\begin{bmatrix} S_i \\ HS_i \end{bmatrix}$, $i = 1, 2, \ldots, m$ can be found iteratively by choosing a Z_0 such that $\begin{bmatrix} S_1 \\ HS_1 \end{bmatrix}$ is a non-zero matrix. Then form the matrices S and Q as follows:

$$\left.\begin{aligned} S &= [S_1\ S_2 \ldots S_m] \\ Q &= [HS_1\ HS_2 \ldots HS_m] = HS \end{aligned}\right\} \tag{16}$$

It has been shown in (7) that S is of full rank n. Therefore, a right inverse of S is

$$S^{RI} = S^t\{SS^t\}^{-1} \tag{17}$$

The feedback gain matrix H for deadbeat regulation can thus be solved from (16) as

$$H = Q\{S^t\{SS^t\} + \{I_{nm} - S^t\{SS^t\}^{-1}S\}W \tag{18}$$

where W is any $(nm \times n)$ matrix. The second term on the right side of (18) is the homogeneous solution. The algorithm for determining an H can thus be summarized as follows.

Step 1. Compute iteratively the matrices S_i and HS_i which satisfy (15) until rank $([S_1\ S_2 \ldots S_i]) = n$ is obtained and then let the controllability index m be the last value of i.

Step 2. Form the matrices S and Q as shown in (16).

Step 3. Compute the feedback gain matrix H from (18).

The solution of the feedback gain matrix H of a multivariable system may not be unique. The designer can shape the transient response of the closed-loop system by

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choosing a proper H (Klein and Moore 1977). In the algorithm there are two sources of freedom for this choice. They are the selections of $Z_{i-1}, i = 1, 2, \ldots, m$ in (15) and W in (18). However, in order to obtain minimum-time regulation, Z_0 in (15) should be selected such that the columns of $[S_1 \ HS_1]^t$ span the null row space of $[A \ B]$.

4. Numerical example

Consider the discrete system (1) with system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that A is a singular matrix.

Step 1

Since Z_0 in (15) can be an arbitrary non-zero (5×3) matrix, we can take

$$Z_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For simplicity of calculations, we can also let

$$Z_j = \mathbf{0}_{5 \times 3}, \quad \text{for } j > 0$$

Substituting Z_0 and $S_0 = \mathbf{0}$ into (15), we obtain

$$\begin{bmatrix} S_1 \\ \cdots \\ HS_1 \end{bmatrix} = \begin{bmatrix} 0{\cdot}75 & -0{\cdot}25 & 0{\cdot}25 \\ -0{\cdot}25 & 0{\cdot}25 & 0{\cdot}25 \\ 0{\cdot}25 & 0{\cdot}25 & 0{\cdot}75 \\ \cdots & \cdots & \cdots \\ 0{\cdot}25 & -0{\cdot}25 & -0{\cdot}25 \\ 0 & 0 & 0 \end{bmatrix}$$

since rank $(S_1) = 2$ is less than the dimension $n = 3$ of the given system. We substitute S_1 and $Z_1 = \mathbf{0}$ into (15) and obtain

$$\begin{bmatrix} S_2 \\ \cdots \\ HS_2 \end{bmatrix} = \begin{bmatrix} -0{\cdot}0625 & -0{\cdot}0625 & -0{\cdot}0625 \\ 0{\cdot}3125 & 0{\cdot}3125 & 0{\cdot}3125 \\ 0{\cdot}0625 & 0{\cdot}0625 & 0{\cdot}0625 \\ \cdots & \cdots & \cdots \\ 0{\cdot}4375 & 0{\cdot}4375 & 0{\cdot}4375 \\ 1{\cdot}25 & 1{\cdot}25 & 1{\cdot}25 \end{bmatrix}$$

Rank $([S_1 \ S_2]) = 3$ shows that the controllability index is $m = 2$.

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Step 2

Form the matrices S and Q as follows:

$$S=[S_1 \vdots S_2]=\begin{bmatrix} 0\cdot75 & 0\cdot25 & 0\cdot25 & \vdots & -0\cdot0625 & 0\cdot0625 & -0\cdot0625 \\ -0\cdot25 & 0\cdot25 & 0\cdot25 & \vdots & 0\cdot3125 & 0\cdot3125 & 0\cdot3125 \\ 0\cdot25 & 0\cdot25 & 0\cdot75 & \vdots & 0\cdot0625 & 0\cdot0625 & 0\cdot0625 \end{bmatrix}$$

and

$$Q=[HS_1 \vdots HS_2]=\begin{bmatrix} 0\cdot25 & -0\cdot25 & -0\cdot25 & \vdots & 0\cdot4375 & 0\cdot4375 & 0\cdot4375 \\ 0 & 0 & 0 & \vdots & 1\cdot25 & 1\cdot25 & 1\cdot25 \end{bmatrix}$$

Step 3

Applying (18) and taking $W=\mathbf{0}$, we get

$$H=\begin{bmatrix} 1\cdot5 & 2 & -1\cdot5 \\ 2\cdot5 & 5 & -2\cdot5 \end{bmatrix}$$

As a check, the values of H are applied to the given system. Assuming the initial state of interest if $x(0)=[1\ 1\ 1]^t$. The control vectors and system responses are shown as follows:

k	$U(k)$	$X(k)$
0	$\begin{bmatrix} 2 \\ 5 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
1	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$
2	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

5. Conclusions

A general algorithm has been obtained for constructing time-invariant feedback laws which assure deadbeat response in minimum time of a multivariable discrete-time linear system. The algorithm puts no assumption on the singularity of the system matrix and the rank of the control matrix. The designer can easily convert the algorithm into a computer program. The transient response of the closed-loop system can also be shaped by choosing a different set of generalized eigenvectors.

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